Álgebra Multilineal

Tercer y Cuarto Exámenes Parciales

Noviembre y Diciembre del 2003

1. En R^2 considere el campo vectorial

$$V \quad x \quad y \, \frac{}{x} \quad x \quad y \, \frac{}{y}$$

Encuentre la familia de curvas a las cuales es tangente V. **¿Cuáles son las componentes de** V **en coordenadas polares**? **Las transformaciones entre coordenadas polares y cartesianas son**:

$$x \quad r\cos \qquad\qquad y \quad r\sin$$

$$r \quad \sqrt{x^2 \quad y^2} \qquad\qquad \arctan\left(\frac{y}{x}\right)$$

Tenemos:

$$\frac{dx}{x \quad y} \quad \frac{dy}{x \quad y}$$

$$\frac{d\,x \quad y}{x \quad y \quad x \quad y} \quad \frac{d\,x \quad y}{2x} \quad \frac{dy}{x \quad y}$$

Entonces:

$$x \quad y\; d\,x \quad y \qquad 2xdy$$

$$\frac{x \quad y^{\;2}}{2} \quad 2xy$$

$$x^2 \quad 2xy \quad y^2 \quad x \quad y^{\;2} \quad C_1$$

También:

$$\frac{d\,x \quad y}{x \quad y \quad x \quad y} \quad \frac{d\,x \quad y}{2y} \quad \frac{dx}{x \quad y}$$

Entonces:

$$x \quad y\; d\,x \quad y \qquad 2ydx$$

$$\frac{x \quad y^{\;2}}{2} \quad 2xy$$

$$x \quad y^{\;2} \quad 4xy \quad x^2 \quad 2xy \quad y^2 \quad x \quad y^{\;2} \quad C_2$$

Luego, la solución es de la forma $\boxed{F\left(x \quad y^{\;2},\; x \quad y^{\;2}\right).}$

Expresando V en coordenadas polares obtenemos:

$$V \quad x \quad y \, \frac{}{x} \quad x \quad y \, \frac{}{y}$$

$$V \quad r\cos \quad r\sin \, \frac{}{x} \quad r\cos \quad r\sin \, \frac{}{y}$$

donde:

$$\frac{}{x} \quad \frac{r}{x}\frac{}{r} \quad \frac{}{x}\frac{}{}$$

con

$$\frac{r}{x} \quad \frac{x}{\sqrt{x^2 \quad y^2}} \quad \frac{r\cos}{r} \quad \cos$$

$$\frac{}{x} \quad \frac{y}{x^2 \quad y^2} \quad \frac{r\sin}{r^2} \quad \frac{\sin}{r}$$

y

$$\frac{}{y} \quad \frac{r}{y}\frac{}{r} \quad \frac{}{y}\frac{}{}$$

con

$$\frac{r}{y} \quad \frac{y}{\sqrt{x^2 \quad y^2}} \quad \frac{r\sin}{r} \quad \sin$$

$$\frac{}{y} \quad \frac{x}{x^2 \quad y^2} \quad \frac{r\cos}{r^2} \quad \frac{\cos}{r}$$

Entonces V es:

$$V \quad r\cos \quad r\sin \left(\cos \frac{}{r} \quad \frac{\sin}{r}\frac{}{}\right) \quad r\cos \quad r\sin \left(\sin \frac{}{r} \quad \frac{\cos}{r}\frac{}{}\right)$$

que reagrupando da:

$$V \quad r \; \cos \quad \sin \quad \cos \quad \cos \quad \sin \quad \sin \; \frac{}{r} \quad (\cos \quad \sin \quad \cos \quad r \cos \quad \sin$$

$$V \quad r\left(\cos^2 \quad \cos \; \sin \quad \cos \; \sin \quad \sin^2\right)\frac{}{r}$$

$$\left(\cos^2 \quad \sin \; \cos \quad \left(\sin \; \cos \quad \sin^2\right)\right)\frac{}{}$$

$$V \quad r\left(\cos^2 \quad \sin^2 \quad \cos \; \sin \quad \cos \; \sin\right)\frac{}{r}$$

$$\left(\cos^2 \quad \sin^2 \quad \sin \; \cos \quad \sin \; \cos\right)\frac{}{}$$

$$\boxed{V \quad r\frac{}{r} \quad \frac{}{}}$$

2. Para la métrica bidimensional

$$ds^2 \quad \frac{1}{t^2} \; dx^2 \quad dt^2$$

Encuentre todos los coeficientes de conexión y todas las geodésicas con vectores tangentes nulos, **i.e.**, **si** v^i **son las componentes del vector tangente**, **entonces** $g_{ij} v^i v^j \quad 0.$

Podemos encontrar la matriz de componentes del tensor métrico a partir del elemento de línea:

$$g_{ij} \quad \begin{pmatrix} \frac{1}{t^2} & 0 \\ 0 & \frac{1}{t^2} \end{pmatrix}$$

ya que el elemento de línea para este caso es:

$$ds^2 \quad g_{ij} dx^i dx^j \quad g_{11} dxdx \quad g_{12} dxdt \quad g_{21} dtdx \quad g_{22} dtdt$$
$$\frac{1}{t^2} dx^2 \quad 0 \quad 0 \quad \frac{1}{t^2} dt^2 \quad \frac{1}{t^2} \; dx^2 \quad dt^2$$

La matriz inversa a g_{ij} describe g^{ij} :

$$g^{ij} \quad \begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix}$$

Los coeficientes de conexión están dados por:

$$^{m}_{ij} \quad \frac{g^{mk}}{2} \left(\frac{}{x^i} g_{jk} \quad \frac{}{x^j} g_{ki} \quad \frac{}{x^k} g_{ij} \right)$$

Con $m \quad 1$ esto da:

$$^{1}_{11} \quad \frac{t^2}{2} \; 0 \quad 0 \quad 0 \quad 0$$
$$^{1}_{12} \quad \frac{t^2}{2} \; 0 \quad 2t^{\;\;3} \quad 0 \quad \frac{1}{t}$$
$$^{1}_{21} \quad \frac{t^2}{2} \; 2t^{\;\;3} \quad 0 \quad 0 \quad \frac{1}{t}$$
$$^{1}_{22} \quad \frac{t^2}{2} \; 0 \quad 0 \quad 0 \quad 0$$

Con $m \quad 2$:

$$^{2}_{11} \quad \frac{t^2}{2} \; 0 \quad 0 \quad 2t^{\;\;3} \quad \frac{1}{t}$$
$$^{2}_{11} \quad \frac{t^2}{2} \; 0 \quad 0 \quad 0 \quad 0$$
$$^{2}_{11} \quad \frac{t^2}{2} \; 0 \quad 0 \quad 0 \quad 0$$
$$^{2}_{11} \quad \frac{t^2}{2} \; 2t^{\;\;3} \quad 0 \quad 0 \quad \frac{1}{t}$$

En resumen,

$$ {}^{1}_{ij} \begin{pmatrix} 0 & \frac{1}{t} \\ \frac{1}{t} & 0 \end{pmatrix} \qquad {}^{2}_{ij} \begin{pmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{t} \end{pmatrix} $$

Las geodésicas están dadas por la ecuación diferencial:

$$ \frac{d^2x^k}{dt^2} \qquad {}^{k}_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \qquad 0 $$

Las soluciones a esta ecuación son curvas de la forma

$$ x^1 \qquad x^2 \; w $$
$$ x^2 \qquad x^2 \; w $$

Sus vectores tangentes estan dados por:

$$ \frac{dx^k}{dw} $$

pero por enunciado tienen que cumplir:

$$ g_{ij} v^i v^j \qquad \left(\frac{dx^1}{dw} \right)^2 \quad \left(\frac{dx^2}{dw} \right)^2 \qquad 0 $$

Lo cual sólo es posible si:

$$ \frac{dx^k}{dw} \qquad 0 $$

Entonces las geodésicas tienen que ser curvas cuyos vectores tangentes sean nulos, es decir, funciones constantes del paramétro:

$$ x^k \qquad cte $$

Ya que esto implica que:

$$ \frac{d^2x^k}{dw^2} \qquad 0 \qquad \frac{dx^k}{dw} \qquad 0 $$

Entonces es fácil ver que se cumple la ecuación de geodésicas para todos los casos:

$$ \frac{d^2x^k}{dt^2} \qquad {}^{k}_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \qquad 0 $$

Luego, las geodésicas con vectores tangentes nulos son de la forma:

$$ \boxed{x^k \qquad cte} $$

3. Para variedades con dimensión menor que 4 sin torsión, las componentes del tensor de curvatura de Riemann puede expresarse de forma simplificada. ¿Cuáles son las componentes de ese tensor para una variedad unidimensional? A continuación considere un caso bidimensional y exprese dichas componentes en términos del tensor métrico y escalar de curvatura.

El tensor de curvatura de Riemann está dado por:

$$R^l_{ijk} \quad e_j \quad {}^{l}_{ik} \quad e_k \quad {}^{l}_{ij} \qquad {}^{r}_{ik}\,{}^{l}_{rj} \quad {}^{r}_{ij}\,{}^{l}_{rk} \quad C^s_{jk}\,{}^{l}_{is}$$

Pero como tenemos una sola dimensión, los índices valen todos 1:

$$R^1_{111} \quad e_1 \quad {}^1_{11} \quad e_1 \quad {}^1_{11} \qquad {}^1_{11}\,{}^1_{11} \quad {}^1_{11}\,{}^1_{11} \quad C^1_{11}\,{}^1_{11}$$

Eliminando términos:

$$R^1_{111} \quad C^1_{11}\,{}^1_{11}$$

Pero

$$C^1_{11} \quad e_1, e_1 \quad 0$$

Entonces el tensor de curvatura da:

$$\boxed{R^1_{111} \quad 0}$$

4. Pruebe que para una métrica diagonal, se cumple:

$${}^i_{jk} \quad 0, \qquad {}^i_{jj} \quad \frac{1}{2g_{ii}} \frac{}{x^i} g_{jj}$$

$${}^i_{ij} \quad \frac{}{x^j} \ln\sqrt{g_{ii}}, \qquad {}^i_{ii} \quad \frac{}{x^i} \ln\sqrt{g_{ii}}$$

donde $i \quad j \quad k$ **y no hay suma sobre índices repetidos**

Primero,

$${}^i_{jk} \quad \frac{g^{im}}{2}\left(\frac{}{x^j} g_{km} \quad \frac{}{x^k} g_{mj} \quad \frac{}{x^m} g_{jk} \right)$$

pero como la métrica es diagonal, $g_{km} \quad g_{mj} \quad g_{jk} \quad 0$, así que:

$$\boxed{{}^i_{jk} \quad 0}$$

Segundo,

$$ {}^{i}_{jj} \quad \frac{g^{im}}{2}\left(\frac{}{x^{j}} g_{jm} \quad \frac{}{x^{j}} g_{mj} \quad \frac{}{x^{m}} g_{jj} \right) $$

donde $g_{jm} \quad g_{mj} \quad 0$ y $g^{im} \quad \frac{1}{g_{im}}$, pero $i \quad m$, así que:

$$ \boxed{ {}^{i}_{jj} \quad \frac{1}{2g_{ii}} \frac{}{x^{i}} g_{jj} } $$

Tercero,

$$ {}^{i}_{ij} \quad \frac{g^{im}}{2}\left(\frac{}{x^{i}} g_{jm} \quad \frac{}{x^{j}} g_{mi} \quad \frac{}{x^{m}} g_{ij} \right) $$

donde $g_{jm} \quad g_{ij} \quad 0$, así que:

$$ {}^{i}_{ij} \quad \frac{1}{2g_{ii}} \frac{}{x^{j}} g_{ii} $$

pero esto último se puede reescribir como:

$$ \frac{1}{2g_{ii}} \frac{}{x^{j}} g_{ii} \quad \frac{}{x^{j}} \left(\ln \sqrt{g_{ii}} \right) $$

ya que:

$$ \frac{}{x^{j}} \left(\ln \sqrt{g_{ii}} \right) \quad \frac{1}{2} \frac{}{x^{j}} \ \ln g_{ii} \quad \frac{1}{2} \frac{1}{g_{ii}} \frac{}{x^{j}} \ g_{ii} \quad \frac{1}{2g_{ii}} \frac{}{x^{j}} g_{ii} $$

Entonces:

$$ \boxed{ {}^{i}_{ij} \quad \frac{}{x^{j}} \left(\ln \sqrt{g_{ii}} \right) } $$

Cuarto,

$$ {}^{i}_{ii} \quad \frac{g^{im}}{2}\left(\frac{}{x^{i}} g_{im} \quad \frac{}{x^{i}} g_{mi} \quad \frac{}{x^{m}} g_{ii} \right) \quad \frac{1}{2g_{ii}} \frac{}{x^{i}} g_{ii} $$

lo cual se puede reescribir como:

$$ \frac{1}{2g_{ii}} \frac{}{x^{i}} g_{ii} \quad \frac{}{x^{i}} \left(\ln \sqrt{g_{ii}} \right) $$

ya que:

$$ \frac{}{x^{i}} \left(\ln \sqrt{g_{ii}} \right) \quad \frac{1}{2} \frac{}{x^{i}} \ \ln g_{ii} \quad \frac{1}{2} \frac{1}{g_{ii}} \frac{}{x^{i}} g_{ii} $$

Entonces:

$$\boxed{\;{}^{i}_{ii} \quad \frac{}{x^i}\left(\ln\sqrt{g_{ii}}\right)}$$

5. El análisis vectorial en tres dimensiones, con coordenadas curvilineas ortogonales, es un caso particular del análisis tensorial con $g_{ij} \quad h_i^2 \; _{ij}$ **(no hay suma sobre** i**). Las** h_i**'s son funciones de las coordenadas, llamadas "factores de escala". Las componentes de los vectores se toman respecto a la base ortonormal** $w^i \quad h_i dx^i$ **(no hay suma sobre** i**). Obtenga expresiones para** S, **V**, **V y** ^2S **donde** S **es un campo escalar y V un campo vectorial**.

6. Tome $A \quad \det A_{ij}$, **con** A_{ij} **componentes de un tensor de segundo rango. Muestre que** A **no es un escalar, o sea que no permanece invariante ante transformaciones de coordenadas. Dado que** A **no es escalar, ¿como calcularía su derivada covariante**?

La posición de los índices indican que A_{ij} son los componentes de un tensor covariante de segundo rango, el cual se transforma como:

$$\tilde{A}_{ij} \quad \frac{x^m}{x^i}\frac{x^n}{x^j}A_{ij}$$

Como $A \quad \det A_{ij}$, podemos definir que después del cambio de coordenadas esto queda como:

$$\tilde{A} \quad \det\left(\tilde{A}_{ij}\right)$$

Pero por la forma en que se transforma $\tilde{A}_{ij}$ tenemos:

$$\tilde{A} \quad \det\left(\tilde{A}_{ij}\right) \quad \det\left(\frac{x^m}{x^i}\frac{x^n}{x^j}A_{ij}\right)$$

Puesto que $\frac{x^m}{x^i}, \frac{x^m}{x^i}$ y A_{ij} se pueden ver como matrices de 2x2, entonces podemos escribir:

$$\tilde{A} \quad \det\left(\frac{x^m}{x^i}\frac{x^n}{x^j}A_{ij}\right) \quad \det\left(\frac{x^m}{x^i}\frac{x^n}{x^j}\right)\det A_{ij} \quad \det\left(\frac{x^m}{x^i}\right)\det\left(\frac{x^n}{x^j}\right)\det A_{ij}$$

Pero $\det\left(\frac{x^m}{x^i}\right)\det\left(\frac{x^n}{x^j}\right)$ es el Jacobiano cuadrado $J^2 \; x, x$, y $\det A_{ij} \quad A$ por definición. Entonces:

$$\boxed{\tilde{A} = J^{2}\left(x,\bar{x}\right) A}$$

Y se ve que A no es un escalar ya que no es invariante ante transformaciones de coordenadas.

7. Si $F_{ij} = A_{j;i} - A_{i;j}$ **muestre que** $F_{ij;k} + F_{ki;j} + F_{jk;i} = 0$

Comencemos por establecer:

$$F_{ij;k} = \left(A_{j;i} - A_{i;j}\right)_{;k} = A_{j;ik} - A_{i;jk} \qquad (1)$$
$$F_{ki;j} = \left(A_{i;k} - A_{k;i}\right)_{;j} = A_{i;kj} - A_{k;ij} \qquad (2)$$
$$F_{jk;i} = \left(A_{k;j} - A_{j;k}\right)_{;i} = A_{k;ji} - A_{j;ki} \qquad (3)$$

Ahora, expandiendo los términos:

$$A_{j;ik} = \frac{\partial^2 A_j}{\partial x^i \partial x^k} - \frac{\partial}{\partial x^k}\left(\Gamma^n_{ji} A_n\right) = \frac{\partial^2 A_j}{\partial x^i \partial x^k} - \Gamma^n_{ji}\frac{\partial A_n}{\partial x^k} - A_n \frac{\partial \Gamma^n_{ji}}{\partial x^k}$$
$$A_{i;jk} = \frac{\partial^2 A_i}{\partial x^j \partial x^k} - \frac{\partial}{\partial x^k}\left(\Gamma^n_{ij} A_n\right) = \frac{\partial^2 A_i}{\partial x^j \partial x^k} - \Gamma^n_{ij}\frac{\partial A_n}{\partial x^k} - A_n \frac{\partial \Gamma^n_{ij}}{\partial x^k}$$
$$A_{i;kj} = \frac{\partial^2 A_i}{\partial x^k \partial x^j} - \frac{\partial}{\partial x^j}\left(\Gamma^n_{ik} A_n\right) = \frac{\partial^2 A_i}{\partial x^k \partial x^j} - \Gamma^n_{ik}\frac{\partial A_n}{\partial x^j} - A_n \frac{\partial \Gamma^n_{ik}}{\partial x^j}$$
$$A_{k;ij} = \frac{\partial^2 A_k}{\partial x^i \partial x^j} - \frac{\partial}{\partial x^j}\left(\Gamma^n_{ki} A_n\right) = \frac{\partial^2 A_k}{\partial x^i \partial x^j} - \Gamma^n_{ki}\frac{\partial A_n}{\partial x^j} - A_n \frac{\partial \Gamma^n_{ki}}{\partial x^j}$$
$$A_{k;ji} = \frac{\partial^2 A_k}{\partial x^j \partial x^i} - \frac{\partial}{\partial x^i}\left(\Gamma^n_{kj} A_n\right) = \frac{\partial^2 A_k}{\partial x^j \partial x^i} - \Gamma^n_{kj}\frac{\partial A_n}{\partial x^i} - A_n \frac{\partial \Gamma^n_{kj}}{\partial x^i}$$
$$A_{j;ki} = \frac{\partial^2 A_j}{\partial x^k \partial x^i} - \frac{\partial}{\partial x^i}\left(\Gamma^n_{jk} A_n\right) = \frac{\partial^2 A_j}{\partial x^k \partial x^i} - \Gamma^n_{jk}\frac{\partial A_n}{\partial x^i} - A_n \frac{\partial \Gamma^n_{jk}}{\partial x^i}$$

Entonces $F_{ij;k} + F_{ki;j} + F_{jk;i}$ se vuelve $(1) + (2) + (3)$, que con los términos expandidos da:

$$\begin{aligned}
F_{ij;k} + F_{ki;j} + F_{jk;i} &= A_{j;ik} - A_{i;jk} + A_{i;kj} - A_{k;ij} + A_{k;ji} - A_{j;ki} \\
&= \frac{\partial^2 A_j}{\partial x^i \partial x^k} - \Gamma^n_{ji}\frac{\partial A_n}{\partial x^k} - A_n \frac{\partial \Gamma^n_{ji}}{\partial x^k} - \frac{\partial^2 A_i}{\partial x^j \partial x^k} + \Gamma^n_{ij}\frac{\partial A_n}{\partial x^k} + A_n \frac{\partial \Gamma^n_{ij}}{\partial x^k} \\
&+ \frac{\partial^2 A_i}{\partial x^k \partial x^j} - \Gamma^n_{ik}\frac{\partial A_n}{\partial x^j} - A_n \frac{\partial \Gamma^n_{ik}}{\partial x^j} - \frac{\partial^2 A_k}{\partial x^i \partial x^j} + \Gamma^n_{ki}\frac{\partial A_n}{\partial x^j} + A_n \frac{\partial \Gamma^n_{ki}}{\partial x^j} \\
&+ \frac{\partial^2 A_k}{\partial x^j \partial x^i} - \Gamma^n_{kj}\frac{\partial A_n}{\partial x^i} - A_n \frac{\partial \Gamma^n_{kj}}{\partial x^i} - \frac{\partial^2 A_j}{\partial x^k \partial x^i} + \Gamma^n_{jk}\frac{\partial A_n}{\partial x^i} + A_n \frac{\partial \Gamma^n_{jk}}{\partial x^i}
\end{aligned}$$

Eliminando términos, esto es:

$$\begin{aligned}
F_{ij;k} + F_{ki;j} + F_{jk;i} &= -\Gamma^n_{ji}\frac{\partial A_n}{\partial x^k} - A_n \frac{\partial \Gamma^n_{ji}}{\partial x^k} + \Gamma^n_{ij}\frac{\partial A_n}{\partial x^k} + A_n \frac{\partial \Gamma^n_{ij}}{\partial x^k} \\
&- \Gamma^n_{ik}\frac{\partial A_n}{\partial x^j} - A_n \frac{\partial \Gamma^n_{ik}}{\partial x^j} + \Gamma^n_{ki}\frac{\partial A_n}{\partial x^j} + A_n \frac{\partial \Gamma^n_{ki}}{\partial x^j}
\end{aligned}$$

$$
\begin{aligned}
&{}^{n}_{kj}\frac{A_n}{x^i} \quad A_n\frac{{}^{n}_{kj}}{x^i} \quad {}^{n}_{jk}\frac{A_n}{x^i} \quad A_n\frac{{}^{n}_{jk}}{x^i}\\
&{}^{n}_{ij}\frac{A_n}{x^k} \quad {}^{n}_{ji}\frac{A_n}{x^k} \quad {}^{n}_{ki}\frac{A_n}{x^j} \quad {}^{n}_{ik}\frac{A_n}{x^j} \quad {}^{n}_{jk}\frac{A_n}{x^i} \quad {}^{n}_{kj}\frac{A_n}{x^i}\\
&A_n\frac{{}^{n}_{ij}}{x^k} \quad A_n\frac{{}^{n}_{ji}}{x^k} \quad A_n\frac{{}^{n}_{ki}}{x^j} \quad A_n\frac{{}^{n}_{ik}}{x^j} \quad A_n\frac{{}^{n}_{jk}}{x^i} \quad A_n\frac{{}^{n}_{kj}}{x^i}\\
&{}^{n}_{ij} \quad {}^{n}_{ji} \quad \frac{A_n}{x^k} \quad {}^{n}_{ki} \quad {}^{n}_{ik} \quad \frac{A_n}{x^j} \quad {}^{n}_{jk} \quad {}^{n}_{kj} \quad \frac{A_n}{x^i}\\
&A_n\left(\frac{{}^{n}_{ij}}{x^k} \quad \frac{{}^{n}_{ji}}{x^k} \quad \frac{{}^{n}_{ki}}{x^j} \quad \frac{{}^{n}_{ik}}{x^j} \quad \frac{{}^{n}_{jk}}{x^i} \quad \frac{{}^{n}_{kj}}{x^i} \right) \qquad 4
\end{aligned}
$$

De (4) es evidente que si ${}^{k}_{ij}$ ${}^{k}_{ji}$ (i.e., es simétrica) la expresión de anula. En el caso general, podemos expandir las ’s de la primera parte de (4):

$$
\begin{aligned}
{}^{n}_{ij} \quad &\frac{g^{nk}}{2}\left(\frac{}{x^i} g_{jk} \quad \frac{}{x^j} g_{ki} \quad \frac{}{x^k} g_{ij} \right)\\
{}^{n}_{ji} \quad &\frac{g^{nk}}{2}\left(\frac{}{x^j} g_{ik} \quad \frac{}{x^i} g_{kj} \quad \frac{}{x^k} g_{ji} \right)
\end{aligned}
$$

Como el tensor métrico es simétrico, entonces:

$$
\begin{aligned}
&{}^{n}_{ij} \quad {}^{n}_{ji} \quad \frac{g^{nk}}{2}\left(\frac{}{x^i} g_{jk} \quad \frac{}{x^j} g_{ki} \quad \frac{}{x^k} g_{ij} \right) \quad \frac{g^{nk}}{2}\left(\frac{}{x^j} g_{ik} \quad \frac{}{x^i} g_{kj} \quad \frac{}{x^k} g_{ji} \right)\\
&\frac{g^{nk}}{2}\left(\frac{}{x^i} g_{kj} \quad \frac{}{x^j} g_{ik} \quad \frac{}{x^k} g_{ji} \right) \quad \frac{g^{nk}}{2}\left(\frac{}{x^j} g_{ik} \quad \frac{}{x^i} g_{kj} \quad \frac{}{x^k} g_{ji} \right)\\
&0
\end{aligned}
$$

Esto se puede repetir para las otras dos parejas de la primera parte de (4). Entonces:

$$
F_{ij;k} \quad F_{ki;j} \quad F_{jk;i} \quad A_n\left(\frac{{}^{n}_{ij}}{x^k} \quad \frac{{}^{n}_{ji}}{x^k} \quad \frac{{}^{n}_{ki}}{x^j} \quad \frac{{}^{n}_{ik}}{x^j} \quad \frac{{}^{n}_{jk}}{x^i} \quad \frac{{}^{n}_{kj}}{x^i} \right) \qquad 5
$$

Ahora, para las derivadas de , tenemos:

$$
\frac{{}^{n}_{ij}}{x^k} \quad \frac{}{x^k}\left(\frac{g^{np}}{2}\left(\frac{}{x^i} g_{jp} \quad \frac{}{x^j} g_{pi} \quad \frac{}{x^p} g_{ij} \right)\right)
$$

Esto se puede expandir como producto en:

$$
\begin{aligned}
\frac{{}^{n}_{ij}}{x^k} \quad &\frac{1}{2}\left(\frac{g_{jp}}{x^i} \quad \frac{g_{pi}}{x^j} \quad \frac{g_{ij}}{x^p} \right)\frac{g^{np}}{x^k} \quad \frac{g^{np}}{2}\frac{}{x^k}\left(\frac{g_{jp}}{x^i} \quad \frac{g_{pi}}{x^j} \quad \frac{g_{ij}}{x^p} \right)\\
&\frac{1}{2}\left(\frac{g_{jp}}{x^i} \quad \frac{g_{pi}}{x^j} \quad \frac{g_{ij}}{x^p} \right)\frac{g^{np}}{x^k} \quad \frac{g^{np}}{2}\left(\frac{{}^{2}g_{jp}}{x^i \ x^k} \quad \frac{{}^{2}g_{pi}}{x^j \ x^k} \quad \frac{{}^{2}g_{ij}}{x^p \ x^k} \right)
\end{aligned}
$$

La derivada de la misma con índices inferiores invertidos y mismo índice superior es:

$$
\frac{{}^{n}_{ji}}{x^k} \quad \frac{1}{2}\left(\frac{g_{ip}}{x^j} \quad \frac{g_{pj}}{x^i} \quad \frac{g_{ji}}{x^p} \right)\frac{g^{np}}{x^k} \quad \frac{g^{np}}{2}\left(\frac{{}^{2}g_{ip}}{x^j \ x^k} \quad \frac{{}^{2}g_{pj}}{x^i \ x^k} \quad \frac{{}^{2}g_{ji}}{x^p \ x^k} \right)
$$

Al restar estas derivadas de , se obtiene:

$$\frac{{}^{n}_{ij}}{x^k} \quad \frac{{}^{n}_{ji}}{x^k}$$
$$\frac{1}{2}\left(\frac{g_{jp}}{x^i} \quad \frac{g_{pi}}{x^j} \quad \frac{g_{ij}}{x^p}\right)\frac{g^{np}}{x^k} \quad \frac{g^{np}}{2}\left(\frac{{}^2 g_{jp}}{x^i \; x^k} \quad \frac{{}^2 g_{pi}}{x^j \; x^k} \quad \frac{{}^2 g_{ij}}{x^p \; x^k}\right)$$
$$\frac{1}{2}\left(\frac{g_{ip}}{x^j} \quad \frac{g_{pj}}{x^i} \quad \frac{g_{ji}}{x^p}\right)\frac{g^{np}}{x^k} \quad \frac{g^{np}}{2}\left(\frac{{}^2 g_{ip}}{x^j \; x^k} \quad \frac{{}^2 g_{pj}}{x^i \; x^k} \quad \frac{{}^2 g_{ji}}{x^p \; x^k}\right)$$

De nuevo, por la propiedad de simetría del tensor métrico, esto es:

$$\frac{{}^{n}_{ij}}{x^k} \quad \frac{{}^{n}_{ji}}{x^k}$$
$$\frac{1}{2}\left(\frac{g_{pj}}{x^i} \quad \frac{g_{ip}}{x^j} \quad \frac{g_{ji}}{x^p}\right)\frac{g^{np}}{x^k} \quad \frac{g^{np}}{2}\left(\frac{{}^2 g_{pj}}{x^i \; x^k} \quad \frac{{}^2 g_{ip}}{x^j \; x^k} \quad \frac{{}^2 g_{ji}}{x^p \; x^k}\right)$$
$$\frac{1}{2}\left(\frac{g_{ip}}{x^j} \quad \frac{g_{pj}}{x^i} \quad \frac{g_{ji}}{x^p}\right)\frac{g^{np}}{x^k} \quad \frac{g^{np}}{2}\left(\frac{{}^2 g_{ip}}{x^j \; x^k} \quad \frac{{}^2 g_{pj}}{x^i \; x^k} \quad \frac{{}^2 g_{ji}}{x^p \; x^k}\right)$$
$$0$$

Regresando a (5), tenemos parejas de derivadas de ’s con índices inferiores invertidos y mismo índice superior, así que la expresión se anula también. Por lo tanto:

$$\boxed{F_{ij;k} \quad F_{ki;j} \quad F_{jk;i} \quad 0}$$

8. Pruebe que cualquier tensor de segundo rango satisface $A^{ij}_{\;;ij} \quad A^{ij}_{\;;ji}$

Primero, sea:

$$S^{ij} \quad A^{ij}_{\;;i} \quad \frac{A^{ij}}{x^i} \quad {}^{i}_{ai}A^{aj} \quad {}^{j}_{bi}A^{ib}$$

Entonces:

$$A^{ij}_{\;;ij} \quad S^{ij}_{\;;j} \quad \frac{S^{ij}}{x^j} \quad {}^{i}_{aj}S^{aj} \quad {}^{j}_{bj}S^{ib}$$

Usando la definición dada de S^{ij}:

$$A^{ij}_{\;;ij} \quad \frac{S^{ij}}{x^j} \quad {}^{i}_{aj}S^{aj} \quad {}^{j}_{bj}S^{ib}$$
$$\frac{}{x^j}\left(\frac{A^{ij}}{x^i} \quad {}^{i}_{ai}A^{aj} \quad {}^{j}_{bi}A^{ib}\right)$$
$${}^{i}_{aj}\left(\frac{A^{aj}}{x^i} \quad {}^{i}_{ai}A^{aj} \quad {}^{j}_{bi}A^{ab}\right)$$
$${}^{j}_{bj}\left(\frac{A^{ib}}{x^i} \quad {}^{i}_{ai}A^{ab} \quad {}^{j}_{bi}A^{ib}\right) \qquad 1$$

Segundo, sea

$$P^{ij} \quad A^{ij}_{\;;j} \quad \frac{A^{ij}}{x^j} \quad {}^{i}_{aj}A^{aj} \quad {}^{j}_{bj}A^{ib}$$

Entonces:

$$A^{ij}_{\;;ji} \quad P^{ij}_{\;;i} \quad \frac{P^{ij}}{x^i} \quad {}^{i}_{ai}P^{aj} \quad {}^{j}_{bi}P^{ib}$$

Usando la definición de P^{ij}:

$$\begin{aligned} A^{ij}_{\;;ji} \quad & \frac{P^{ij}}{x^i} \quad {}^{i}_{ai}P^{aj} \quad {}^{j}_{bi}P^{ib} \\ & \frac{}{x^i}\left(\frac{A^{ij}}{x^j} \quad {}^{i}_{aj}A^{aj} \quad {}^{j}_{bj}A^{ib} \right) \\ & {}^{i}_{ai}\left(\frac{A^{aj}}{x^j} \quad {}^{i}_{aj}A^{aj} \quad {}^{j}_{bj}A^{ab} \right) \\ & {}^{j}_{bi}\left(\frac{A^{ib}}{x^j} \quad {}^{i}_{aj}A^{ab} \quad {}^{j}_{bj}A^{ib} \right) \qquad 2 \end{aligned}$$

Restando (1) y (2), se tiene:

$$\begin{aligned} A^{ij}_{\;;ij} \quad A^{ij}_{\;;ji} \quad & \\ & \frac{}{x^j}\left(\frac{A^{ij}}{x^i} \quad {}^{i}_{ai}A^{aj} \quad {}^{j}_{bi}A^{ib} \right) \\ & {}^{i}_{aj}\left(\frac{A^{aj}}{x^i} \quad {}^{i}_{ai}A^{aj} \quad {}^{j}_{bi}A^{ab} \right) \\ & {}^{j}_{bj}\left(\frac{A^{ib}}{x^i} \quad {}^{i}_{ai}A^{ab} \quad {}^{j}_{bi}A^{ib} \right) \\ & \frac{}{x^i}\left(\frac{A^{ij}}{x^j} \quad {}^{i}_{aj}A^{aj} \quad {}^{j}_{bj}A^{ib} \right) \\ & {}^{i}_{ai}\left(\frac{A^{aj}}{x^j} \quad {}^{i}_{aj}A^{aj} \quad {}^{j}_{bj}A^{ab} \right) \\ & {}^{j}_{bi}\left(\frac{A^{ib}}{x^j} \quad {}^{i}_{aj}A^{ab} \quad {}^{j}_{bj}A^{ib} \right) \end{aligned}$$

Eliminando términos:

$$A^{ij}_{\;;ij} \quad A^{ij}_{\;;ji} \quad 0$$

Por lo tanto:

$$\boxed{A^{ij}_{\;;ij} \quad A^{ij}_{\;;ji}}$$

9. **Muestre la identidad de Bianchi**: $R^{l}_{ijk;m} \quad R^{l}_{imj;k} \quad R^{l}_{ikm;j} \quad 0$

Primero, tenemos:

$$R^{l}_{ijk;m} \quad \frac{}{x^{m}} R^{l}_{ijk} \quad R^{l}_{njk} \, {}^{i}_{nm} \quad R^{l}_{ipk} \, {}^{j}_{pm} \quad R^{l}_{ijq} \, {}^{k}_{qm}$$

Pero:

$$R^{l}_{ijk} \quad \frac{}{x^{j}} \, {}^{l}_{ik} \quad \frac{}{x^{k}} \, {}^{l}_{ij} \quad {}^{r}_{ik} \, {}^{l}_{rj} \quad {}^{r}_{ij} \, {}^{l}_{rk} \quad C^{s}_{jk} \, {}^{l}_{is}$$

Entonces:

$$\begin{aligned}
R^{l}_{ijk;m} \quad & \frac{}{x^{m}} \left(\frac{}{x^{j}} \, {}^{l}_{ik} \quad \frac{}{x^{k}} \, {}^{l}_{ij} \quad {}^{r}_{ik} \, {}^{l}_{rj} \quad {}^{r}_{ij} \, {}^{l}_{rk} \quad C^{s}_{jk} \, {}^{l}_{is} \right) \\
& {}^{i}_{nm} \left(\frac{}{x^{j}} \, {}^{l}_{nk} \quad \frac{}{x^{k}} \, {}^{l}_{nj} \quad {}^{r}_{nk} \, {}^{l}_{rj} \quad {}^{r}_{nj} \, {}^{l}_{rk} \quad C^{s}_{jk} \, {}^{l}_{ns} \right) \\
& {}^{j}_{pm} \left(\frac{}{x^{p}} \, {}^{l}_{ik} \quad \frac{}{x^{k}} \, {}^{l}_{ip} \quad {}^{r}_{ik} \, {}^{l}_{rp} \quad {}^{r}_{ip} \, {}^{l}_{rk} \quad C^{s}_{pk} \, {}^{l}_{is} \right) \\
& {}^{k}_{qm} \left(\frac{}{x^{j}} \, {}^{l}_{iq} \quad \frac{}{x^{q}} \, {}^{l}_{ij} \quad {}^{r}_{iq} \, {}^{l}_{rj} \quad {}^{r}_{ij} \, {}^{l}_{rq} \quad C^{s}_{jq} \, {}^{l}_{is} \right) \\
& \left(\frac{{}^{2} \, {}^{l}_{ik}}{x^{m} \; x^{j}} \quad \frac{{}^{2} \, {}^{l}_{ij}}{x^{m} \; x^{k}} \quad {}^{l}_{rj} \frac{{}^{r}_{ik}}{x^{m}} \quad {}^{r}_{ik} \frac{{}^{l}_{rj}}{x^{m}} \right) \\
& \left({}^{r}_{ij} \frac{{}^{l}_{rk}}{x^{m}} \quad {}^{l}_{rk} \frac{{}^{r}_{ij}}{x^{m}} \quad C^{s}_{jk} \frac{{}^{l}_{is}}{x^{m}} \quad {}^{l}_{is} \frac{C^{s}_{jk}}{x^{m}} \right) \\
& {}^{i}_{nm} \left(\frac{{}^{l}_{nk}}{x^{j}} \quad \frac{{}^{l}_{nj}}{x^{k}} \quad {}^{r}_{nk} \, {}^{l}_{rj} \quad {}^{r}_{nj} \, {}^{l}_{rk} \quad C^{s}_{jk} \, {}^{l}_{ns} \right) \\
& {}^{j}_{pm} \left(\frac{{}^{l}_{ik}}{x^{p}} \quad \frac{{}^{l}_{ip}}{x^{k}} \quad {}^{r}_{ik} \, {}^{l}_{rp} \quad {}^{r}_{ip} \, {}^{l}_{rk} \quad C^{s}_{pk} \, {}^{l}_{is} \right) \\
& {}^{k}_{qm} \left(\frac{{}^{l}_{iq}}{x^{j}} \quad \frac{{}^{l}_{ij}}{x^{q}} \quad {}^{r}_{iq} \, {}^{l}_{rj} \quad {}^{r}_{ij} \, {}^{l}_{rq} \quad C^{s}_{jq} \, {}^{l}_{is} \right) \qquad 1
\end{aligned}$$

Segundo,

$$R^{l}_{imj;k} \quad \frac{}{x^{k}} R^{l}_{imj} \quad R^{l}_{nmj} \, {}^{i}_{nk} \quad R^{l}_{ipj} \, {}^{m}_{pk} \quad R^{l}_{imq} \, {}^{j}_{qk}$$

Pero:

$$R^{l}_{imj} \quad \frac{}{x^{m}} \, {}^{l}_{ij} \quad \frac{}{x^{j}} \, {}^{l}_{im} \quad {}^{r}_{ij} \, {}^{l}_{rm} \quad {}^{r}_{im} \, {}^{l}_{rj} \quad C^{s}_{mj} \, {}^{l}_{is}$$

Entonces:

$$\begin{aligned}
R^{l}_{imj;k} \quad & \frac{}{x^{k}} \left(\frac{}{x^{m}} \, {}^{l}_{ij} \quad \frac{}{x^{j}} \, {}^{l}_{im} \quad {}^{r}_{ij} \, {}^{l}_{rm} \quad {}^{r}_{im} \, {}^{l}_{rj} \quad C^{s}_{mj} \, {}^{l}_{is} \right) \\
& {}^{i}_{nk} \left(\frac{}{x^{m}} \, {}^{l}_{nj} \quad \frac{}{x^{j}} \, {}^{l}_{nm} \quad {}^{r}_{nj} \, {}^{l}_{rm} \quad {}^{r}_{nm} \, {}^{l}_{rj} \quad C^{s}_{mj} \, {}^{l}_{ns} \right) \\
& {}^{m}_{pk} \left(\frac{}{x^{p}} \, {}^{l}_{ij} \quad \frac{}{x^{j}} \, {}^{l}_{ip} \quad {}^{r}_{ij} \, {}^{l}_{rp} \quad {}^{r}_{ip} \, {}^{l}_{rj} \quad C^{s}_{pj} \, {}^{l}_{is} \right)
\end{aligned}$$

$$
\begin{aligned}
&{}^{j}_{qk}\left(\frac{}{x^m}{}^{l}_{iq} \quad \frac{}{x^q}{}^{l}_{im} \quad {}^{r}_{iq}{}^{l}_{rm} \quad {}^{r}_{im}{}^{l}_{rq} \quad C^s_{mq}{}^{l}_{is}\right)\\
&\left(\frac{{}^{2}\,{}^{l}_{ij}}{x^k\, x^m} \quad \frac{{}^{2}\,{}^{l}_{im}}{x^k\, x^j} \quad {}^{l}_{rm}\frac{{}^{r}_{ij}}{x^k} \quad {}^{r}_{ij}\frac{{}^{l}_{rm}}{x^k}\right)\\
&\left({}^{l}_{rj}\frac{{}^{r}_{im}}{x^k} \quad {}^{r}_{im}\frac{{}^{l}_{rj}}{x^k} \quad {}^{l}_{is}\frac{C^s_{mj}}{x^k} \quad C^s_{mj}\frac{{}^{l}_{is}}{x^k}\right)\\
&{}^{i}_{nk}\left(\frac{{}^{l}_{nj}}{x^m} \quad \frac{{}^{l}_{nm}}{x^j} \quad {}^{r}_{nj}{}^{l}_{rm} \quad {}^{r}_{nm}{}^{l}_{rj} \quad C^s_{mj}{}^{l}_{ns}\right)\\
&{}^{m}_{pk}\left(\frac{{}^{l}_{ij}}{x^p} \quad \frac{{}^{l}_{ip}}{x^j} \quad {}^{r}_{ij}{}^{l}_{rp} \quad {}^{r}_{ip}{}^{l}_{rj} \quad C^s_{pj}{}^{l}_{is}\right)\\
&{}^{j}_{qk}\left(\frac{{}^{l}_{iq}}{x^m} \quad \frac{{}^{l}_{im}}{x^q} \quad {}^{r}_{iq}{}^{l}_{rm} \quad {}^{r}_{im}{}^{l}_{rq} \quad C^s_{mq}{}^{l}_{is}\right) \qquad 2
\end{aligned}
$$

Tercero:

$$R^l_{ikm;j} \quad \frac{}{x^j} R^l_{ikm} \quad R^l_{nkm}{}^{i}_{nj} \quad R^l_{ipm}{}^{k}_{pj} \quad R^l_{ikq}{}^{m}_{qj}$$

Pero:

$$R^l_{ikm} \quad \frac{}{x^k}{}^{l}_{im} \quad \frac{}{x^m}{}^{l}_{ik} \quad {}^{r}_{im}{}^{l}_{rk} \quad {}^{r}_{ik}{}^{l}_{rm} \quad C^s_{km}{}^{l}_{is}$$

Entonces:

$$
\begin{aligned}
R^l_{ikm;j} \quad &\frac{}{x^j}\left(\frac{}{x^k}{}^{l}_{im} \quad \frac{}{x^m}{}^{l}_{ik} \quad {}^{r}_{im}{}^{l}_{rk} \quad {}^{r}_{ik}{}^{l}_{rm} \quad C^s_{km}{}^{l}_{is}\right)\\
&{}^{i}_{nj}\left(\frac{}{x^k}{}^{l}_{nm} \quad \frac{}{x^m}{}^{l}_{nk} \quad {}^{r}_{nm}{}^{l}_{rk} \quad {}^{r}_{nk}{}^{l}_{rm} \quad C^s_{km}{}^{l}_{ns}\right)\\
&{}^{k}_{pj}\left(\frac{}{x^p}{}^{l}_{im} \quad \frac{}{x^m}{}^{l}_{ip} \quad {}^{r}_{im}{}^{l}_{rp} \quad {}^{r}_{ip}{}^{l}_{rm} \quad C^s_{pm}{}^{l}_{is}\right)\\
&{}^{m}_{qj}\left(\frac{}{x^k}{}^{l}_{iq} \quad \frac{}{x^q}{}^{l}_{ik} \quad {}^{r}_{iq}{}^{l}_{rk} \quad {}^{r}_{ik}{}^{l}_{rq} \quad C^s_{kq}{}^{l}_{is}\right)\\
&\left(\frac{{}^{2}\,{}^{l}_{im}}{x^j\, x^k} \quad \frac{{}^{2}\,{}^{l}_{ik}}{x^j\, x^m} \quad {}^{l}_{rk}\frac{{}^{r}_{im}}{x^j} \quad {}^{r}_{im}\frac{{}^{l}_{rk}}{x^j}\right)\\
&\left({}^{l}_{rm}\frac{{}^{r}_{ik}}{x^j} \quad {}^{r}_{ik}\frac{{}^{l}_{rm}}{x^j} \quad {}^{l}_{is}\frac{C^s_{km}}{x^j} \quad C^s_{km}\frac{{}^{l}_{is}}{x^j}\right)\\
&{}^{i}_{nj}\left(\frac{{}^{l}_{nm}}{x^k} \quad \frac{{}^{l}_{nk}}{x^m} \quad {}^{r}_{nm}{}^{l}_{rk} \quad {}^{r}_{nk}{}^{l}_{rm} \quad C^s_{km}{}^{l}_{ns}\right)\\
&{}^{k}_{pj}\left(\frac{{}^{l}_{im}}{x^p} \quad \frac{{}^{l}_{ip}}{x^m} \quad {}^{r}_{im}{}^{l}_{rp} \quad {}^{r}_{ip}{}^{l}_{rm} \quad C^s_{pm}{}^{l}_{is}\right)\\
&{}^{m}_{qj}\left(\frac{{}^{l}_{iq}}{x^k} \quad \frac{{}^{l}_{ik}}{x^q} \quad {}^{r}_{iq}{}^{l}_{rk} \quad {}^{r}_{ik}{}^{l}_{rq} \quad C^s_{kq}{}^{l}_{is}\right) \qquad 3
\end{aligned}
$$

Sumando (1), (2) y (3):

$$R^l_{ijk;m} \quad R^l_{imj;k} \quad R^l_{ikm;j}$$

$$\left(\frac{2\ {}^{l}_{ik}}{x^m\ x^j} \quad \frac{2\ {}^{l}_{ij}}{x^m\ x^k} \quad {}^{l}_{rj}\frac{{}^{r}_{ik}}{x^m} \quad {}^{r}_{ik}\frac{{}^{l}_{rj}}{x^m} \right)$$
$$\left({}^{r}_{ij}\frac{{}^{l}_{rk}}{x^m} \quad {}^{l}_{rk}\frac{{}^{r}_{ij}}{x^m} \quad C^s_{jk}\frac{{}^{l}_{is}}{x^m} \quad {}^{l}_{is}\frac{C^s_{jk}}{x^m} \right)$$
$${}^{i}_{nm}\left(\frac{{}^{l}_{nk}}{x^j} \quad \frac{{}^{l}_{nj}}{x^k} \quad {}^{r}_{nk}\,{}^{l}_{rj} \quad {}^{r}_{nj}\,{}^{l}_{rk} \quad C^s_{jk}\,{}^{l}_{ns} \right)$$
$${}^{j}_{pm}\left(\frac{{}^{l}_{ik}}{x^p} \quad \frac{{}^{l}_{ip}}{x^k} \quad {}^{r}_{ik}\,{}^{l}_{rp} \quad {}^{r}_{ip}\,{}^{l}_{rk} \quad C^s_{pk}\,{}^{l}_{is} \right)$$
$${}^{k}_{qm}\left(\frac{{}^{l}_{iq}}{x^j} \quad \frac{{}^{l}_{ij}}{x^q} \quad {}^{r}_{iq}\,{}^{l}_{rj} \quad {}^{r}_{ij}\,{}^{l}_{rq} \quad C^s_{jq}\,{}^{l}_{is} \right)$$
$$\left(\frac{2\ {}^{l}_{ij}}{x^k\ x^m} \quad \frac{2\ {}^{l}_{im}}{x^k\ x^j} \quad {}^{l}_{rm}\frac{{}^{r}_{ij}}{x^k} \quad {}^{r}_{ij}\frac{{}^{l}_{rm}}{x^k} \right)$$
$$\left({}^{l}_{rj}\frac{{}^{r}_{im}}{x^k} \quad {}^{r}_{im}\frac{{}^{l}_{rj}}{x^k} \quad {}^{l}_{is}\frac{C^s_{mj}}{x^k} \quad C^s_{mj}\frac{{}^{l}_{is}}{x^k} \right)$$
$${}^{i}_{nk}\left(\frac{{}^{l}_{nj}}{x^m} \quad \frac{{}^{l}_{nm}}{x^j} \quad {}^{r}_{nj}\,{}^{l}_{rm} \quad {}^{r}_{nm}\,{}^{l}_{rj} \quad C^s_{mj}\,{}^{l}_{ns} \right)$$
$${}^{m}_{pk}\left(\frac{{}^{l}_{ij}}{x^p} \quad \frac{{}^{l}_{ip}}{x^j} \quad {}^{r}_{ij}\,{}^{l}_{rp} \quad {}^{r}_{ip}\,{}^{l}_{rj} \quad C^s_{pj}\,{}^{l}_{is} \right)$$
$${}^{j}_{qk}\left(\frac{{}^{l}_{iq}}{x^m} \quad \frac{{}^{l}_{im}}{x^q} \quad {}^{r}_{iq}\,{}^{l}_{rm} \quad {}^{r}_{im}\,{}^{l}_{rq} \quad C^s_{mq}\,{}^{l}_{is} \right)$$
$$\left(\frac{2\ {}^{l}_{im}}{x^j\ x^k} \quad \frac{2\ {}^{l}_{ik}}{x^j\ x^m} \quad {}^{l}_{rk}\frac{{}^{r}_{im}}{x^j} \quad {}^{r}_{im}\frac{{}^{l}_{rk}}{x^j} \right)$$
$$\left({}^{l}_{rm}\frac{{}^{r}_{ik}}{x^j} \quad {}^{r}_{ik}\frac{{}^{l}_{rm}}{x^j} \quad {}^{l}_{is}\frac{C^s_{km}}{x^j} \quad C^s_{km}\frac{{}^{l}_{is}}{x^j} \right)$$
$${}^{i}_{nj}\left(\frac{{}^{l}_{nm}}{x^k} \quad \frac{{}^{l}_{nk}}{x^m} \quad {}^{r}_{nm}\,{}^{l}_{rk} \quad {}^{r}_{nk}\,{}^{l}_{rm} \quad C^s_{km}\,{}^{l}_{ns} \right)$$
$${}^{k}_{pj}\left(\frac{{}^{l}_{im}}{x^p} \quad \frac{{}^{l}_{ip}}{x^m} \quad {}^{r}_{im}\,{}^{l}_{rp} \quad {}^{r}_{ip}\,{}^{l}_{rm} \quad C^s_{pm}\,{}^{l}_{is} \right)$$
$${}^{m}_{qj}\left(\frac{{}^{l}_{iq}}{x^k} \quad \frac{{}^{l}_{ik}}{x^q} \quad {}^{r}_{iq}\,{}^{l}_{rk} \quad {}^{r}_{ik}\,{}^{l}_{rq} \quad C^s_{kq}\,{}^{l}_{is} \right)$$

Se ve que los términos con segunda derivada se anulan. Luego, si cambiamos índices, haciendo que *m* sea *k* y que *p* sea *q*, esto se reduce a:

$$R^l_{ijk;m} \quad R^l_{imj;k} \quad R^l_{ikm;j}$$
$${}^{i}_{nk}C^s_{jk}\,{}^{l}_{ns} \quad {}^{j}_{qk}C^s_{qk}\,{}^{l}_{is} \quad {}^{k}_{qk}C^s_{jq}\,{}^{l}_{is} \quad {}^{i}_{nk}C^s_{kj}\,{}^{l}_{ns} \quad {}^{k}_{qk}C^s_{qj}\,{}^{l}_{is}$$
$${}^{j}_{qk}C^s_{kq}\,{}^{l}_{is} \quad {}^{i}_{nj}C^s_{kk}\,{}^{l}_{ns} \quad {}^{k}_{qj}C^s_{qk}\,{}^{l}_{is} \quad {}^{k}_{qj}C^s_{kq}\,{}^{l}_{is} \quad {}^{l}_{is}\frac{C^s_{jk}}{x^k}$$
$${}^{l}_{is}\frac{C^s_{kj}}{x^k} \quad {}^{l}_{is}\frac{C^s_{kk}}{x^j} \quad C^s_{jk}\frac{{}^{l}_{is}}{x^k} \quad C^s_{kj}\frac{{}^{l}_{is}}{x^k} \quad C^s_{kk}\frac{{}^{l}_{is}}{x^j}$$

Pero,

$$C_{ij}^{k} \quad e_i, e_j \; e_k \quad e_j, e_i \; e_k \quad e_i e_j \quad e_j e_i \quad e_j e_i \quad e_i e_j \quad e_k \quad 0 \; e_k \quad 0$$

Como todos los términos restantes contienen un coeficiente de estructura, entonces:

$$\boxed{R^{l}_{ijk;m} \quad R^{l}_{imj;k} \quad R^{l}_{ikm;j} \quad 0}$$

10. **Muestre que**

$$A^{i}{}_{;i} \quad \frac{1}{\sqrt{|g|}} \frac{}{x^i}\left(\sqrt{|g|}\, A^i\right),$$

$$A^{ji}{}_{;i} \quad \frac{1}{\sqrt{|g|}} \frac{}{x^i}\left(\sqrt{|g|}\, A^{ji}\right), \qquad A^{ji} \qquad A^{ij}$$

$$A^{i}{}_{j;i} \quad \frac{1}{\sqrt{|g|}} \frac{}{x^i}\left(\sqrt{|g|}\, A^i_j\right) \quad \frac{1}{2}\frac{g_{mn}}{x^j} A^{mn}, \qquad A^{ji} \quad A^{ij}$$

Primero probaremos que

$$\,^{i}_{ki} \quad \frac{}{x^k} \ln \sqrt{|g|}$$

Dado que:

$$\,^{i}_{ki} \quad \frac{g^{im}}{2}\left(\frac{}{x^i} g_{km} \quad \frac{}{x^k} g_{im} \quad \frac{}{x^m} g_{ki}\right)$$

Entonces:

$$\,^{i}_{ki} \quad \frac{g^{im}}{2}\frac{g_{km}}{x^i} \quad \frac{g^{im}}{2}\frac{g_{im}}{x^k} \quad \frac{g^{im}}{2}\frac{g_{ki}}{x^m}$$

si $i \quad m$

$$\,^{i}_{ki} \quad \frac{g^{im}}{2}\frac{g_{im}}{x^k} \quad \frac{g^{im}}{2}\frac{g_{km}}{x^i} \quad \frac{g^{im}}{2}\frac{g_{ki}}{x^m} \quad \frac{g^{im}}{2}\frac{g_{im}}{x^k}$$

Esto se puede reescribir como:

$$\,^{i}_{ki} \quad \frac{g^{im}}{2}\frac{g_{im}}{x^k} \quad \frac{}{x^k} \ln \sqrt{|g|}$$

Ya que:

$$\frac{}{x^k} \ln \sqrt{|g|} \quad \frac{1}{2}\frac{}{x^k} \ln|g| \quad \frac{1}{2|g|}\frac{|g|}{x^k}$$

$$\frac{1}{2|g|}\frac{\det g_{ij}}{x^k} \quad \frac{1}{2|g|}\frac{}{x^k} \quad {}^{i_1, i_2, \ldots, i_n} g_{i_1 1} g_{i_2 2} \ldots g_{i_n n}$$

$$\frac{1}{2|g|} \; {}^{i_1,i_2,\ldots,i_n} \frac{}{x^k} \; g_{i_1 1} g_{i_2 2} \ldots g_{i_n n}$$

$$\frac{1}{2|g|} \; {}^{i_1,i_2,\ldots,i_n} \left(\frac{g_{i_1 1}}{x^k} g_{i_2 2}\ldots g_{i_n n} \quad g_{i_1 1}\frac{g_{i_2 2}}{x^k}\ldots g_{i_n n} \qquad g_{i_1 1}g_{i_2 2}\ldots\frac{g_{i_n n}}{x^k}\right)$$

$$\frac{g^{im}}{2}\frac{g_{im}}{x^k}$$

Con este resultado, ya es posible calcular:

$$A^i{}_{;i} \quad \frac{A^i}{x^i} \quad {}^{\;i}_{ki} A^k$$

$$\frac{A^i}{x^i} \quad A^k \frac{}{x^k} \ln\sqrt{|g|} \qquad \frac{A^i}{x^i} \quad \frac{A^k}{\sqrt{|g|}} \frac{}{x^k}\sqrt{|g|}$$

Pero esto se puede reescribir como:

$$\frac{1}{\sqrt{|g|}}\frac{}{x^i}\left(\sqrt{|g|}\,A^i\right) \quad \frac{1}{\sqrt{|g|}}\left(A^i\frac{\sqrt{|g|}}{x^i} \quad \sqrt{|g|}\,\frac{A^i}{x^i}\right) \quad \frac{A^i}{\sqrt{|g|}}\frac{\sqrt{|g|}}{x^i} \quad \frac{\sqrt{|g|}}{\sqrt{|g|}}\frac{A^i}{x^i}$$

$$\frac{A^i}{\sqrt{|g|}}\frac{\sqrt{|g|}}{x^i} \quad \frac{A^i}{x^i}$$

Entonces:

$$\boxed{A^i{}_{;i} \quad \frac{1}{\sqrt{|g|}}\frac{}{x^i}\left(\sqrt{|g|}\,A^i\right)}$$

Para $A^{ji}{}_{;i}$, tenemos:

$$A^{ji}{}_{;i} \quad \frac{A^{ji}}{x^i} \quad A^{li}\;{}^{\;j}_{li} \quad A^{jm}\;{}^{\;i}_{mi}$$

$$\frac{A^{ji}}{x^i} \quad A^{li}\;{}^{\;j}_{li} \quad A^{jm}\frac{}{x^m}\ln\sqrt{|g|}$$

Pero

$$A^{li}\;{}^{\;j}_{li} \qquad A^{il}\;{}^{\;j}_{li} \qquad A^{il}\;{}^{\;j}_{il} \qquad {}^{\;j}_{li}A^{li} \quad 0$$

Entonces:

$$A^{ji}{}_{;i} \quad \frac{A^{ji}}{x^i} \quad A^{jm}\frac{}{x^m}\ln\sqrt{|g|} \qquad \frac{A^{ji}}{x^i} \quad \frac{A^{ji}}{\sqrt{|g|}}\frac{}{x^i}\sqrt{|g|} \qquad \boxed{\frac{1}{\sqrt{|g|}}\frac{}{x^i}\left(\sqrt{|g|}\,A^{ji}\right)}$$

Finalmente, para $A^i{}_{j;i}$, tenemos:

$$A^{i}{}_{j;i} \quad \frac{A^i_j}{x^i} \quad A^l_j \; {}^{i}_{li} \quad A^i_m \; {}^{m}_{ji}$$

$$\frac{A^i_j}{x^i} \quad A^l_j \frac{}{x^l} \ln\sqrt{|g|} \quad A^i_m \frac{g^{mn}}{2}\left(\frac{g_{in}}{x^j} \quad \frac{g_{jn}}{x^i} \quad \frac{g}{x^n} \right)$$

$$\frac{A^i_j}{x^i} \quad A^l_j \frac{}{x^l} \ln\sqrt{|g|} \quad A^i_m \left(\frac{g^{mn}}{2} \frac{g_{in}}{x^j} \quad \frac{g^{mn}}{2} \frac{g_{jn}}{x^i} \quad \frac{g^{mn}}{2} \frac{g_{ji}}{x^n} \right)$$

si $n \quad i$

$$\frac{A^i_j}{x^i} \quad A^l_j \frac{}{x^l} \ln\sqrt{|g|} \quad A^i_m \left(\frac{g^{mn}}{2} \frac{g_{in}}{x^j} \right)$$

$$\frac{A^i_j}{x^i} \quad A^l_j \frac{}{x^l} \ln\sqrt{|g|} \quad \frac{A^{in}}{2} \frac{g_{ni}}{x^j}$$

$$\frac{A^i_j}{x^i} \quad \frac{A^l_j}{\sqrt{|g|}} \frac{\sqrt{|g|}}{x^l} \quad \frac{1}{2} \frac{g_{in}}{x^j} A^{in}$$

$$\boxed{\frac{1}{\sqrt{|g|}} \frac{}{x^l} \left(\sqrt{|g|} A^l_j \right) \quad \frac{1}{2} \frac{g_{in}}{x^j} A^{in}}$$

11. Considere la métrica de la esfera unitaria

$$ds^2 \quad d\;^2 \quad \sin^2 \; d\;^2 \qquad \textbf{donde } 0 \qquad \textbf{y } 0 \qquad 2$$

a) Muestre que esta métrica se reescribe como

$$ds^2 \quad \frac{d\;d}{P^2} \qquad \textbf{con} \qquad \cot\left(\frac{}{2}\right) e^{i} \;, P \quad \frac{1}{2} \; 1$$

b) Dados los vectores

$$m^i \quad \sqrt{2}\, P \;\; ^i, \qquad m^i \quad \sqrt{2}\, P \;\; ^i$$

muestre que $m^i m_i \quad 0 \quad m^i m_i$ **y** $m^i m_i \quad 1$

c) Calcule los coeficientes de conexión, tensor de curvatura de Riemann, tensor de Ricci y escalar de curvatura en la base natural, para la métrica en términos de y

a) Tenemos:

$$\cot\left(\frac{}{2}\right) e^{i} \qquad d \quad \frac{1}{2}\csc^2\left(\frac{}{2}\right) e^{i} \; d \quad i e^{i} \; \cot\left(\frac{}{2}\right) d$$

También:

$$\cot\left(\frac{}{2}\right) e^{\;i} \qquad d \quad \frac{1}{2}\csc^2\left(\frac{}{2}\right) e^{\;i} \; d \quad i e^{\;i} \; \cot\left(\frac{}{2}\right) d$$

Entonces:

$$d\;d \quad \left(\frac{1}{2}\csc^2\left(\frac{}{2}\right) e^{i} \; d \quad i e^{i} \; \cot\left(\frac{}{2}\right) d \right)\left(\frac{1}{2}\csc^2\left(\frac{}{2}\right) e^{\;i} \; d \quad i e^{\;i} \; \cot\left(\frac{}{2}\right) d \right)$$

$$\frac{1}{4}\csc^4\left(\frac{}{2}\right)d^{\ 2} \quad \frac{i}{2}\csc^2\left(\frac{}{2}\right)\cot\left(\frac{}{2}\right)d\ d \quad \frac{i}{2}\csc^2\left(\frac{}{2}\right)\cot\left(\frac{}{2}\right)d\ d \quad \cot^2\left(\frac{}{2}\right)d^{\ 2}$$

$$\frac{1}{4}\csc^4\left(\frac{}{2}\right)d^{\ 2} \quad \cot^2\left(\frac{}{2}\right)d^{\ 2}$$

También:

$$P^2 \quad \frac{1}{4}\ 1 \qquad ^2 \quad \frac{1}{4}\left(1 \quad 2 \qquad ^2\right) \quad \frac{1}{4}\left(1 \quad 2\cot^2\left(\frac{}{2}\right) \quad \cot^4\left(\frac{}{2}\right)\right)$$

Entonces:

$$ds^2 \quad \frac{d\ d}{P^2} \quad \frac{\frac{1}{4}\csc^4\left(\frac{}{2}\right)d^{\ 2} \quad \cot^2\left(\frac{}{2}\right)d^{\ 2}}{\frac{1}{4}\left(1 \quad 2\cot^2\left(\frac{}{2}\right) \quad \cot^4\left(\frac{}{2}\right)\right)} \quad \frac{\csc^4\left(\frac{}{2}\right)d^{\ 2} \quad 4\cot^2\left(\frac{}{2}\right)d^{\ 2}}{1 \quad 2\cot^2\left(\frac{}{2}\right) \quad \cot^4\left(\frac{}{2}\right)}$$

$$\frac{\csc^4\left(\frac{}{2}\right)d^{\ 2} \quad 4\cot^2\left(\frac{}{2}\right)d^{\ 2}}{\left(1 \quad \cot^2\left(\frac{}{2}\right)\right)^2} \quad \frac{\csc^4\left(\frac{}{2}\right)d^{\ 2}}{\csc^4\left(\frac{}{2}\right)} \quad \frac{4\cot^2\left(\frac{}{2}\right)d^{\ 2}}{\csc^4\left(\frac{}{2}\right)}$$

$$d^{\ 2} \quad \frac{4\cot^2\left(\frac{}{2}\right)d^{\ 2}}{\csc^4\left(\frac{}{2}\right)} \quad d^{\ 2} \quad 4\frac{\cos^2\left(\frac{}{2}\right)}{\sin^2\left(\frac{}{2}\right)}\sin^4\left(\frac{}{2}\right)d^{\ 2}$$

$$d^{\ 2} \quad 4\cos^2\left(\frac{}{2}\right)\sin^2\left(\frac{}{2}\right)d^{\ 2} \quad d^{\ 2} \quad \left(2\cos\left(\frac{}{2}\right)\sin\left(\frac{}{2}\right)\right)^2 d^{\ 2}$$

$$\boxed{ds^2 \quad d^{\ 2} \quad \sin^2\ d^{\ 2}}$$

c) El tensor métrico para la esfera unitaria en términos de y es:

$$g_{ij} \quad \begin{pmatrix} 0 & \frac{1}{2P^2} \\ \frac{1}{2P^2} & 0 \end{pmatrix} \qquad \text{con } P^2 \quad \frac{1}{4}\ 1 \qquad ^2$$

Como es diagonal, su inversa es:

$$g^{ij} \quad \begin{pmatrix} 0 & 2P^2 \\ 2P^2 & 0 \end{pmatrix}$$

Los coeficientes de conexión son entonces:

$$^{m}_{ij} \quad \frac{g^{mk}}{2}\left(\frac{}{x^i}g_{jk} \quad \frac{}{x^j}g_{ki} \quad \frac{}{x^k}g_{ij}\right)$$

Para $m \quad 1$

$$^{1}_{11}\quad \frac{g^{12}}{2}\left(\frac{}{x^1}g_{12}\quad \frac{}{x^1}g_{21}\quad \frac{}{x^2}g_{11}\right)\quad \frac{g^{12}}{2}\left(—\frac{1}{P^2}\right)$$
$$\frac{1}{4}\ 1\quad {}^{2}\left(—\frac{1}{P^2}\right)\quad \frac{1}{4}\ 1\quad {}^{2}\frac{8}{1\quad {}^{3}}\quad \frac{2}{1}$$
$$^{1}_{12}\quad ^{1}_{21}\quad \frac{g^{12}}{2}\left(\frac{}{x^1}g_{22}\quad \frac{}{x^2}g_{21}\quad \frac{}{x^2}g_{12}\right)\quad 0$$
$$^{1}_{22}\quad \frac{g^{12}}{2}\left(\frac{}{x^2}g_{22}\quad \frac{}{x^2}g_{22}\quad \frac{}{x^2}g_{22}\right)\quad 0$$

Para $m\quad 2$

$$^{2}_{11}\quad \frac{g^{21}}{2}\left(\frac{}{x^1}g_{11}\quad \frac{}{x^1}g_{11}\quad \frac{}{x^1}g_{11}\right)\quad 0$$
$$^{2}_{12}\quad ^{2}_{21}\quad \frac{g^{21}}{2}\left(\frac{}{x^1}g_{21}\quad \frac{}{x^2}g_{11}\quad \frac{}{x^1}g_{12}\right)\quad 0$$
$$^{2}_{22}\quad \frac{g^{21}}{2}\left(\frac{}{x^2}g_{21}\quad \frac{}{x^2}g_{12}\quad \frac{}{x^1}g_{22}\right)\quad g^{21}\left(\frac{}{x^2}g_{21}\right)$$
$$2P^2\left(—\frac{1}{2P^2}\right)\quad P^2\left(—\frac{1}{P^2}\right)\quad \frac{1}{4}\ 1\quad {}^{2}\frac{8}{1\quad {}^{3}}$$
$$\frac{2}{1}$$

En resumen:

$$^{1}_{ij}\quad \begin{pmatrix} \frac{2}{1} & 0 \\ 0 & 0 \end{pmatrix}\qquad ^{2}_{ij}\quad \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{1} \end{pmatrix}$$

El tensor de curvatura está dado por:

$$R^{l}_{ijk}\quad e_j\ {}^{l}_{ik}\quad e_k\ {}^{l}_{ij}\quad {}^{r}_{ik}\ {}^{l}_{rj}\quad {}^{r}_{ij}\ {}^{l}_{rk}\quad C^{s}_{jk}\ {}^{l}_{is}$$

Como la base es natural, $C^{s}_{jk}\quad 0$.
Para $l\quad 1$:

$$R^{1}_{111}\quad e_1\ {}^{1}_{11}\quad e_1\ {}^{1}_{11}\quad {}^{r}_{11}\ {}^{1}_{r1}\quad {}^{r}_{11}\quad 0$$
$$R^{1}_{112}\quad R^{1}_{121}\quad e_1\ {}^{1}_{12}\quad e_2\ {}^{1}_{11}\quad {}^{r}_{12}\ {}^{1}_{r1}\quad {}^{r}_{11}\ {}^{1}_{r2}\quad \frac{}{x^2}\left(\frac{2}{1}\right)$$
$$—\left(\frac{2}{1}\right)\quad \frac{2}{1\quad {}^{2}}$$
$$R^{1}_{122}\quad e_2\ {}^{1}_{1k}\quad e_2\ {}^{1}_{12}\quad {}^{r}_{12}\ {}^{1}_{r2}\quad {}^{r}_{12}\ {}^{1}_{r2}\quad 0$$
$$R^{1}_{222}\quad e_2\ {}^{1}_{22}\quad e_2\ {}^{1}_{22}\quad {}^{r}_{22}\ {}^{1}_{r2}\quad {}^{r}_{22}\ {}^{1}_{rk}\quad 0$$
$$R^{1}_{221}\quad e_2\ {}^{1}_{21}\quad e_1\ {}^{1}_{22}\quad {}^{r}_{21}\ {}^{1}_{r2}\quad {}^{r}_{22}\ {}^{1}_{r1}\quad 0$$
$$R^{1}_{211}\quad e_1\ {}^{1}_{21}\quad e_1\ {}^{1}_{21}\quad {}^{r}_{21}\ {}^{1}_{r1}\quad {}^{r}_{21}\ {}^{1}_{r1}\quad 0$$

Para $l \quad 2$:

$$R^2_{ijk} \quad e_j \ {}^{2}_{ik} \quad e_k \ {}^{2}_{ij} \quad {}^{r}_{ik}\ {}^{2}_{rj} \quad {}^{r}_{ij}\ {}^{2}_{rk}$$
$$R^2_{111} \quad e_1 \ {}^{2}_{11} \quad e_1 \ {}^{2}_{11} \quad {}^{r}_{11}\ {}^{2}_{r1} \quad {}^{r}_{11}\ {}^{2}_{r1} \quad 0$$
$$R^2_{112} \quad e_1 \ {}^{2}_{12} \quad e_2 \ {}^{2}_{11} \quad {}^{r}_{12}\ {}^{2}_{r1} \quad {}^{r}_{11}\ {}^{2}_{r2} \quad 0$$
$$R^2_{122} \quad e_2 \ {}^{2}_{12} \quad e_2 \ {}^{2}_{12} \quad {}^{r}_{12}\ {}^{2}_{r2} \quad {}^{r}_{12}\ {}^{2}_{r2} \quad 0$$
$$R^2_{222} \quad e_2 \ {}^{2}_{22} \quad e_2 \ {}^{2}_{22} \quad {}^{r}_{22}\ {}^{2}_{r2} \quad {}^{r}_{22}\ {}^{2}_{r2} \quad 0$$
$$R^2_{221} \quad R^2_{212} \quad e_2 \ {}^{2}_{21} \quad e_1 \ {}^{2}_{22} \quad {}^{r}_{21}\ {}^{2}_{r2} \quad {}^{r}_{22}\ {}^{2}_{r1} \quad \frac{}{x^1} \quad {}^{2}_{22}$$
$$\frac{}{x^1}\left(\frac{2}{1}\right) \quad \frac{}{}\left(\frac{2}{1}\right) \quad \frac{2}{1 \quad {}^2}$$
$$R^2_{211} \quad e_1 \ {}^{2}_{21} \quad e_1 \ {}^{2}_{21} \quad {}^{r}_{21}\ {}^{2}_{r1} \quad {}^{r}_{21}\ {}^{2}_{r1} \quad 0$$

En resumen:

$$R^1_{112} \quad R^1_{121} \quad R^2_{221} \quad R^2_{212} \quad \frac{2}{1 \quad {}^2}$$

El tensor de Ricci está dado por:

$$R_{ij} \quad R^l_{ilj}$$

$$R_{11} \quad R^l_{1l1} \quad 0$$
$$R_{12} \quad R^l_{1l2} \quad R^1_{112} \quad \frac{2}{1 \quad {}^2}$$
$$R_{21} \quad R^l_{2l1} \quad R^2_{221} \quad \frac{2}{1 \quad {}^2}$$
$$R_{22} \quad R^l_{2l2} \quad 0$$

Y el escalar de curvatura es:

$$R \quad R^i_i \quad g^{im}R_{mi}$$

$$R^1_1 \quad g^{1m}R_{m1} \quad g^{12}R_{21} \quad \frac{1}{2}\ 1 \quad {}^2\frac{2}{1 \quad {}^2} \quad 1$$
$$R^2_2 \quad g^{2m}R_{m2} \quad g^{21}R_{12} \quad \frac{1}{2}\ 1 \quad {}^2\frac{2}{1 \quad {}^2} \quad 1$$

$$R \quad R^1_1 \quad R^2_2 \quad 2$$

12. La métrica de las ondas gravitacionales con fronteras planas y rayos paralelos es

$$ds^2 \quad 2d \ \ d \quad 2dudv \quad 2h \ \ , \ \ ,u \ \ du^2$$

donde $\quad x \quad iy/\sqrt{2}$, $u \quad t \quad z$, $v \quad t \quad x$, h **es en principio una función arbitraria de** $\;,\;$ **y** u. **Calcule en la base natural los coeficientes de conexión y muestre que las correspondientes ecuaciones de geodésicas son**:

$$\frac{d^2u}{dt^2} \quad 0, \qquad \frac{d^2}{dt^2} \quad \frac{\ h}{}\left(\frac{du}{dt}\right)^2 \quad 0, \qquad \frac{d^2}{dt^2} \quad \frac{\ h}{}\left(\frac{du}{dt}\right)^2 \quad 0$$

$$\frac{d^2v}{dt^2} \quad 2\frac{du}{dt}\left(\frac{\ h}{}\frac{d}{dt} \quad \frac{\ h}{}\frac{d}{dt}\right) \quad \frac{\ h}{u}\left(\frac{du}{dt}\right)^2 \quad 0$$

Si hacemos $x^1 \quad , x^2 \quad , x^3 \quad u, x^4 \quad v$, las componentes del tensor métrico son:

$$g_{ij} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2h & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \text{y} \qquad g^{ij} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2h \end{pmatrix}$$

Los coeficientes de conexión están dados por:

$$\,^{m}_{ij} \quad \frac{g^{mk}}{2}\left(\frac{}{x^i}g_{jk} \quad \frac{}{x^j}g_{ki} \quad \frac{}{x^k}g_{ij}\right)$$

Para $m \quad 1$:

$$\,^{1}_{11} \quad \frac{g^{12}}{2}\left(\frac{}{x^1}g_{12} \quad \frac{}{x^1}g_{21} \quad \frac{}{x^2}g_{11}\right) \quad 0$$

$$\,^{1}_{12} \quad \,^{1}_{21} \quad \frac{g^{12}}{2}\left(\frac{}{x^1}g_{22} \quad \frac{}{x^2}g_{21} \quad \frac{}{x^2}g_{12}\right) \quad 0$$

$$\,^{1}_{13} \quad \,^{1}_{31} \quad \frac{g^{12}}{2}\left(\frac{}{x^1}g_{32} \quad \frac{}{x^3}g_{21} \quad \frac{}{x^3}g_{13}\right) \quad 0$$

$$\,^{1}_{14} \quad \,^{1}_{41} \quad \frac{g^{12}}{2}\left(\frac{}{x^1}g_{42} \quad \frac{}{x^4}g_{21} \quad \frac{}{x^2}g_{14}\right) \quad 0$$

$$\,^{1}_{22} \quad \frac{g^{12}}{2}\left(\frac{}{x^2}g_{22} \quad \frac{}{x^2}g_{22} \quad \frac{}{x^2}g_{22}\right) \quad 0$$

$$\,^{1}_{23} \quad \,^{1}_{32} \quad \frac{g^{12}}{2}\left(\frac{}{x^2}g_{32} \quad \frac{}{x^3}g_{22} \quad \frac{}{x^2}g_{23}\right) \quad 0$$

$$\,^{1}_{24} \quad \,^{1}_{42} \quad \frac{g^{12}}{2}\left(\frac{}{x^2}g_{42} \quad \frac{}{x^4}g_{22} \quad \frac{}{x^2}g_{24}\right) \quad 0$$

$$\,^{1}_{33} \quad \frac{g^{12}}{2}\left(\frac{}{x^3}g_{32} \quad \frac{}{x^3}g_{23} \quad \frac{}{x^2}g_{33}\right) \quad \frac{g^{12}}{2}\frac{g_{33}}{x^2}$$

$$\frac{1}{2}\frac{\ 2h}{} \quad \frac{\ h}{}$$

$$\,^{1}_{34} \quad \,^{1}_{43} \quad \frac{g^{12}}{2}\left(\frac{}{x^3}g_{42} \quad \frac{}{x^4}g_{23} \quad \frac{}{x^2}g_{34}\right) \quad 0$$

$$\,^{1}_{44} \quad \frac{g^{12}}{2}\left(\frac{}{x^4}g_{42} \quad \frac{}{x^4}g_{24} \quad \frac{}{x^2}g_{44}\right) \quad 0$$

Para $m \quad 2$

$$ {}^{2}_{11} \quad \frac{g^{21}}{2}\left(\frac{}{x^1} g_{11} \quad \frac{}{x^1} g_{11} \quad \frac{}{x^1} g_{11}\right) \quad 0 $$

$$ {}^{2}_{12} \quad {}^{2}_{21} \quad \frac{g^{21}}{2}\left(\frac{}{x^1} g_{21} \quad \frac{}{x^2} g_{11} \quad \frac{}{x^1} g_{12}\right) \quad 0 $$

$$ {}^{2}_{13} \quad {}^{2}_{31} \quad \frac{g^{21}}{2}\left(\frac{}{x^1} g_{31} \quad \frac{}{x^3} g_{11} \quad \frac{}{x^1} g_{13}\right) \quad 0 $$

$$ {}^{2}_{14} \quad {}^{2}_{41} \quad \frac{g^{21}}{2}\left(\frac{}{x^1} g_{41} \quad \frac{}{x^4} g_{11} \quad \frac{}{x^1} g_{14}\right) \quad 0 $$

$$ {}^{2}_{22} \quad \frac{g^{21}}{2}\left(\frac{}{x^2} g_{21} \quad \frac{}{x^2} g_{12} \quad \frac{}{x^1} g_{22}\right) \quad 0 $$

$$ {}^{2}_{23} \quad {}^{2}_{32} \quad \frac{g^{21}}{2}\left(\frac{}{x^2} g_{31} \quad \frac{}{x^3} g_{12} \quad \frac{}{x^1} g_{23}\right) \quad 0 $$

$$ {}^{2}_{24} \quad {}^{2}_{42} \quad \frac{g^{21}}{2}\left(\frac{}{x^2} g_{41} \quad \frac{}{x^4} g_{12} \quad \frac{}{x^1} g_{24}\right) \quad 0 $$

$$ {}^{2}_{33} \quad \frac{g^{21}}{2}\left(\frac{}{x^3} g_{31} \quad \frac{}{x^3} g_{13} \quad \frac{}{x^1} g_{33}\right) \quad \frac{g^{21}}{2}\frac{}{x^1} g_{33} $$

$$ \frac{1}{2}\frac{2h}{} \quad \frac{h}{} $$

$$ {}^{2}_{34} \quad {}^{2}_{43} \quad \frac{g^{21}}{2}\left(\frac{}{x^3} g_{41} \quad \frac{}{x^4} g_{13} \quad \frac{}{x^1} g_{34}\right) \quad 0 $$

$$ {}^{2}_{44} \quad \frac{g^{21}}{2}\left(\frac{}{x^4} g_{41} \quad \frac{}{x^4} g_{14} \quad \frac{}{x^1} g_{44}\right) \quad 0 $$

Para $m \quad 3$

$$ {}^{3}_{11} \quad \frac{g^{34}}{2}\left(\frac{}{x^1} g_{14} \quad \frac{}{x^1} g_{41} \quad \frac{}{x^4} g_{11}\right) \quad 0 $$

$$ {}^{3}_{12} \quad {}^{3}_{21} \quad \frac{g^{34}}{2}\left(\frac{}{x^1} g_{24} \quad \frac{}{x^2} g_{41} \quad \frac{}{x^4} g_{12}\right) \quad 0 $$

$$ {}^{3}_{13} \quad {}^{3}_{31} \quad \frac{g^{34}}{2}\left(\frac{}{x^1} g_{34} \quad \frac{}{x^3} g_{41} \quad \frac{}{x^4} g_{13}\right) \quad 0 $$

$$ {}^{3}_{14} \quad {}^{3}_{41} \quad \frac{g^{34}}{2}\left(\frac{}{x^1} g_{44} \quad \frac{}{x^1} g_{41} \quad \frac{}{x^4} g_{14}\right) \quad 0 $$

$$ {}^{3}_{22} \quad \frac{g^{34}}{2}\left(\frac{}{x^2} g_{24} \quad \frac{}{x^2} g_{42} \quad \frac{}{x^4} g_{22}\right) \quad 0 $$

$$ {}^{3}_{23} \quad {}^{3}_{32} \quad \frac{g^{34}}{2}\left(\frac{}{x^2} g_{34} \quad \frac{}{x^3} g_{42} \quad \frac{}{x^4} g_{23}\right) \quad 0 $$

$$ {}^{3}_{24} \quad {}^{3}_{42} \quad \frac{g^{34}}{2}\left(\frac{}{x^2} g_{44} \quad \frac{}{x^4} g_{42} \quad \frac{}{x^4} g_{24}\right) \quad 0 $$

$$ {}^{3}_{33} \quad \frac{g^{34}}{2}\left(\frac{}{x^3} g_{34} \quad \frac{}{x^3} g_{43} \quad \frac{}{x^4} g_{33}\right) \quad 0 $$

$$ {}^{3}_{34} \quad {}^{3}_{43} \quad \frac{g^{34}}{2}\left(\frac{}{x^3} g_{44} \quad \frac{}{x^4} g_{43} \quad \frac{}{x^4} g_{34}\right) \quad 0 $$

$$ {}^{3}_{44} \quad \frac{g^{34}}{2}\left(\frac{}{x^4} g_{44} \quad \frac{}{x^4} g_{44} \quad \frac{}{x^4} g_{44}\right) \quad 0 $$

Para $m \quad 4$

$$ {}^{4}_{ij} \quad \frac{g^{4k}}{2}\left(\frac{}{x^i} g_{jk} \quad \frac{}{x^j} g_{ki} \quad \frac{}{x^k} g_{ij}\right) \quad k \quad 3,4 $$

$$ {}^{4}_{11} \quad \frac{g^{4k}}{2}\left(\frac{}{x^1}g_{1k} \quad \frac{}{x^1}g_{k1} \quad \frac{}{x^k}g_{11}\right) \quad 0 $$

$$ {}^{4}_{12} \quad {}^{4}_{21} \quad \frac{g^{4k}}{2}\left(\frac{}{x^1}g_{2k} \quad \frac{}{x^2}g_{k1} \quad \frac{}{x^k}g_{12}\right) \quad 0 $$

$$ {}^{4}_{13} \quad {}^{4}_{31} \quad \frac{g^{4k}}{2}\left(\frac{}{x^1}g_{3k} \quad \frac{}{x^3}g_{k1} \quad \frac{}{x^k}g_{13}\right) \quad \frac{g^{43}}{2}\frac{g_{33}}{x^1} $$

$$ \frac{1}{2}\frac{2h}{} \quad \frac{h}{} $$

$$ {}^{4}_{14} \quad {}^{4}_{41} \quad \frac{g^{4k}}{2}\left(\frac{}{x^1}g_{4k} \quad \frac{}{x^4}g_{k1} \quad \frac{}{x^k}g_{14}\right) \quad 0 $$

$$ {}^{4}_{22} \quad \frac{g^{4k}}{2}\left(\frac{}{x^2}g_{2k} \quad \frac{}{x^2}g_{k2} \quad \frac{}{x^k}g_{22}\right) \quad 0 $$

$$ {}^{4}_{23} \quad {}^{4}_{32} \quad \frac{g^{4k}}{2}\left(\frac{}{x^2}g_{3k} \quad \frac{}{x^3}g_{k2} \quad \frac{}{x^k}g_{23}\right) \quad \frac{g^{43}}{2}\frac{g_{33}}{x^2} $$

$$ \frac{1}{2}\frac{2h}{x^2} \quad \frac{h}{} $$

$$ {}^{4}_{24} \quad {}^{4}_{42} \quad \frac{g^{4k}}{2}\left(\frac{}{x^2}g_{4k} \quad \frac{}{x^4}g_{k2} \quad \frac{}{x^k}g_{24}\right) \quad 0 $$

$$ {}^{4}_{33} \quad \frac{g^{4k}}{2}\left(\frac{}{x^3}g_{3k} \quad \frac{}{x^3}g_{k3} \quad \frac{}{x^k}g_{33}\right) \quad \frac{g^{43}}{2}\left(\frac{}{x^3}g_{33} \quad \frac{}{x^3}g_{33} \quad \frac{}{x^3}g_{33}\right) $$

$$ \frac{1}{2}\frac{2h}{x^3} \quad \frac{h}{u} $$

$$ {}^{4}_{34} \quad {}^{4}_{43} \quad \frac{g^{4k}}{2}\left(\frac{}{x^3}g_{4k} \quad \frac{}{x^4}g_{k3} \quad \frac{}{x^k}g_{34}\right) \quad 0 $$

$$ {}^{4}_{44} \quad \frac{g^{4k}}{2}\left(\frac{}{x^4}g_{4k} \quad \frac{}{x^4}g_{k4} \quad \frac{}{x^k}g_{44}\right) \quad 0 $$

En resumen:

$$ {}^{1}_{33} \quad \frac{h}{} \qquad {}^{2}_{33} \quad \frac{h}{} \qquad {}^{4}_{13} \quad {}^{4}_{31} \quad \frac{h}{} \qquad {}^{4}_{23} \quad {}^{4}_{32} \quad \frac{h}{} \qquad {}^{4}_{33} \quad \frac{h}{u} $$

Las geodésicas están dadas por:

$$ \frac{d^2x^k}{dt^2} \quad {}^{k}_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} \quad 0 $$

Se ve que si k 3, todas las ${}^{3}_{ij}$ 0,así que:

$$ \boxed{\frac{d^2u}{dt^2} \quad 0} $$

Si hacemos k 1, entonces sólo ${}^{1}_{33}$ $\frac{h}{}$ es no-nula, de forma que:

$$ \frac{d^2x^1}{dt^2} \quad {}^{1}_{33}\left(\frac{dx^3}{dt}\right)^2 \quad 0 $$

$$ \boxed{\frac{d^2}{dt^2} \quad \frac{h}{}\left(\frac{du}{dt}\right)^2 \quad 0} $$

Si hacemos k 2, entonces sólo ${}^{2}_{33}$ $\frac{h}{\ }$ es no-nula, y:

$$\frac{d^2x^2}{dt^2} \quad {}^{2}_{33}\left(\frac{dx^3}{dt}\right)^2 \quad 0$$

$$\boxed{\frac{d^2}{dt^2} \quad \frac{h}{\ }\left(\frac{du}{dt}\right)^2 \quad 0}$$

Por último, si hacemos k 4, entonces tenemos varias opciones:

$${}^{4}_{13} \quad {}^{4}_{31} \quad \frac{h}{\ } \qquad {}^{4}_{23} \quad {}^{4}_{32} \quad \frac{h}{\ } \qquad {}^{4}_{33} \quad \frac{h}{u}$$

Lo que lleva a las ecuaciones:

$$\frac{d^2x^4}{dt^2} \quad 2 \; {}^{4}_{13}\frac{dx^1}{dt}\frac{dx^3}{dt} \quad 0$$

$$\frac{d^2x^4}{dt^2} \quad 2 \; {}^{4}_{23}\frac{dx^2}{dt}\frac{dx^3}{dt} \quad 0$$

$$\frac{d^2x^4}{dt^2} \quad {}^{4}_{33}\left(\frac{dx^3}{dt}\right)^2 \quad 0$$

Que son:

$$\frac{d^2v}{dt^2} \quad 2\frac{h}{\ }\frac{d}{dt}\frac{du}{dt} \quad 0$$

$$\frac{d^2v}{dt^2} \quad 2\frac{h}{\ }\frac{d}{dt}\frac{du}{dt} \quad 0$$

$$\frac{d^2v}{dt^2} \quad \frac{h}{u}\left(\frac{du}{dt}\right)^2 \quad 0$$

Sumando las primeras dos ecuaciones, se obtiene:

$$\frac{d^2v}{dt^2} \quad 2\frac{h}{\ }\frac{d}{dt}\frac{du}{dt} \quad \frac{d^2v}{dt^2} \quad 2\frac{h}{\ }\frac{d}{dt}\frac{du}{dt} \quad 0$$

$$\frac{d^2v}{dt^2} \quad 2\frac{du}{dt}\left(\frac{h}{\ }\frac{d}{dt} \quad \frac{h}{\ }\frac{d}{dt}\right) \quad \frac{d^2v}{dt^2} \quad 0$$

Pero la tercera ecuación es:

$$\frac{d^2v}{dt^2} \quad \frac{h}{u}\left(\frac{du}{dt}\right)^2$$

Entonces:

$$\boxed{\frac{d^2v}{dt^2} \quad 2\frac{du}{dt}\left(\frac{h}{\ }\frac{d}{dt} \quad \frac{h}{\ }\frac{d}{dt}\right) \quad \frac{h}{u}\left(\frac{du}{dt}\right)^2 \quad 0}$$

13. **Muestre que los vectores**

$$e_1 \quad \frac{x}{\sqrt{x^2 \quad y^2}}\frac{}{x} \quad \frac{y}{\sqrt{x^2 \quad y^2}}\frac{}{y}$$

$$e_2 \quad \frac{y}{\sqrt{x^2 \quad y^2}}\frac{}{x} \quad \frac{x}{\sqrt{x^2 \quad y^2}}\frac{}{y}$$

satisfacen

$$e_1, e_2 \quad \frac{1}{\sqrt{x^2 \quad y^2}} e_2$$

mientras que

$$e_1, e_2 \quad 0$$

donde $e_2 \quad \sqrt{x^2 \quad y^2}\, e_2$. **Obtenga las componentes de** e_1,e_2 **y** e_2 **en coordenadas polares**.

Sea $r \quad \sqrt{x^2 \quad y^2}$. Entonces:

$$e_1, e_2 \quad \left[\frac{x}{r}\frac{}{x} \quad \frac{y}{r}\frac{}{y}, \ \frac{y}{r}\frac{}{x} \quad \frac{x}{r}\frac{}{y} \right]$$

$$\left(\frac{x}{r}\frac{}{x} \quad \frac{y}{r}\frac{}{y} \right)\left(\frac{y}{r}\frac{}{x} \quad \frac{x}{r}\frac{}{y} \right) \quad \left(\frac{y}{r}\frac{}{x} \quad \frac{x}{r}\frac{}{y} \right)\left(\frac{x}{r}\frac{}{x} \quad \frac{y}{r}\frac{}{y} \right)$$

$$\left(\frac{x}{r}\frac{}{x}\left(\frac{y}{r} \right)\frac{}{x} \quad \frac{x}{r}\frac{}{x}\left(\frac{x}{r} \right)\frac{}{y} \quad \frac{y}{r}\frac{}{y}\left(\frac{y}{r} \right)\frac{}{x} \quad \frac{y}{r}\frac{}{y}\left(\frac{x}{r} \right)\frac{}{y} \right)$$

$$\left(\frac{y}{r}\frac{}{x}\left(\frac{x}{r} \right)\frac{}{x} \quad \frac{y}{r}\frac{}{x}\left(\frac{y}{r} \right)\frac{}{y} \quad \frac{x}{r}\frac{}{y}\left(\frac{x}{r} \right)\frac{}{x} \quad \frac{x}{r}\frac{}{y}\left(\frac{y}{r} \right)\frac{}{y} \right)$$

Las derivadas dan:

$$\frac{}{x}\left(\frac{y}{r} \right) \quad \frac{xy}{r^3}$$

$$\frac{}{x}\left(\frac{x}{r} \right) \quad \frac{y^2}{r^3}$$

$$\frac{}{y}\left(\frac{y}{r} \right) \quad \frac{x^2}{r^3}$$

$$\frac{}{y}\left(\frac{x}{r} \right) \quad \frac{xy}{r^3}$$

Entonces:

$$e_1, e_2$$

$$\frac{x}{r}\frac{xy}{r^3}\frac{}{x} \quad \frac{x}{r}\frac{y^2}{r^3}\frac{}{y} \quad \frac{y}{r}\frac{x^2}{r^3}\frac{}{x} \quad \frac{y}{r}\frac{xy}{r^3}\frac{}{y}$$

$$\frac{y}{r}\frac{y^2}{r^3}\frac{}{x} \quad \frac{y}{r}\frac{xy}{r^3}\frac{}{y} \quad \frac{x}{r}\frac{xy}{r^3}\frac{}{x} \quad \frac{x}{r}\frac{x^2}{r^3}\frac{}{y}$$

$$\left(\frac{x^2y}{r^4} \quad \frac{y^3}{r^4} \quad \frac{yx^2}{r^4} \quad \frac{x^2y}{r^4} \right)\frac{}{x}$$

$$\left(\frac{xy^2}{r^4} \quad \frac{xy^2}{r^4} \quad \frac{xy^2}{r^4} \quad \frac{x^3}{r^4} \right)\frac{}{y}$$

$$
\begin{aligned}
&\left(\frac{y^3 \quad x^2 y}{r^4}\right)\frac{}{x} \quad \left(\frac{xy^2 \quad x^3}{r^4}\right)\frac{}{y}\\
&\frac{y}{r^2}\left(\frac{y^2 \quad x^2}{r^2}\right)\frac{}{x} \quad \frac{x}{r^2}\left(\frac{y^2 \quad x^2}{r^2}\right)\frac{}{y} \quad \frac{y}{r^2}\frac{}{x} \quad \frac{x}{r^2}\frac{}{y}\\
&\frac{1}{r}\left(\frac{y}{r}\frac{}{x} \quad \frac{x}{r}\frac{}{y}\right) \quad \frac{1}{r}e_2 \quad \frac{1}{\sqrt{x^2 \quad y^2}}e_2
\end{aligned}
$$

Por lo tanto:

$$
\boxed{e_1, e_2 \quad \frac{1}{\sqrt{x^2 \quad y^2}}e_2}
$$

Si hacemos $e_2 \quad \sqrt{x^2 \quad y^2}\, e_2$, entonces:

$$
\begin{aligned}
e_1, e_2 \quad &\left[\frac{x}{r}\frac{}{x} \quad \frac{y}{r}\frac{}{y},\ y\frac{}{x} \quad x\frac{}{y}\right]\\
&\left(\frac{x}{r}\frac{}{x} \quad \frac{y}{r}\frac{}{y}\right)\left(y\frac{}{x} \quad x\frac{}{y}\right) \quad \left(y\frac{}{x} \quad x\frac{}{y}\right)\left(\frac{x}{r}\frac{}{x} \quad \frac{y}{r}\frac{}{y}\right)\\
&\left(\frac{x}{r}\frac{}{x} \quad \frac{y}{r}\frac{}{y}\right)\left(y\frac{}{x} \quad x\frac{}{y}\right)\\
&\frac{x}{r}\frac{}{x}\, y\, \frac{}{x} \quad \frac{x}{r}\frac{}{x}\, x\, \frac{}{y} \quad \frac{y}{r}\frac{}{y}\, y\, \frac{}{x} \quad \frac{y}{r}\frac{}{y}\, x\, \frac{}{y}\\
&y\frac{}{x}\left(\frac{x}{r}\right)\frac{}{x} \quad y\frac{}{x}\left(\frac{y}{r}\right)\frac{}{y} \quad x\frac{}{y}\left(\frac{x}{r}\right)\frac{}{x} \quad x\frac{}{y}\left(\frac{y}{r}\right)\frac{}{y}
\end{aligned}
$$

$$
\frac{x}{r}\frac{}{y} \quad \frac{y}{r}\frac{}{x} \quad y\frac{}{x}\left(\frac{x}{r}\right)\frac{}{x} \quad y\frac{}{x}\left(\frac{y}{r}\right)\frac{}{y} \quad x\frac{}{y}\left(\frac{x}{r}\right)\frac{}{x} \quad x\frac{}{y}\left(\frac{y}{r}\right)\frac{}{y}
$$

Tomando las derivadas, esto da:

$$
\begin{aligned}
&\frac{x}{r}\frac{}{y} \quad \frac{y}{r}\frac{}{x} \quad y\frac{}{x}\left(\frac{x}{r}\right)\frac{}{x} \quad y\frac{}{x}\left(\frac{y}{r}\right)\frac{}{y} \quad x\frac{}{y}\left(\frac{x}{r}\right)\frac{}{x} \quad x\frac{}{y}\left(\frac{y}{r}\right)\frac{}{y}\\
&\frac{x}{r}\frac{}{y} \quad \frac{y}{r}\frac{}{x} \quad y\frac{y^2}{r^3}\frac{}{x} \quad y\frac{xy}{r^3}\frac{}{y} \quad x\frac{xy}{r^3}\frac{}{x} \quad x\frac{x^2}{r^3}\frac{}{y}\\
&\left(\frac{x^2 y}{r^3} \quad \frac{y^3}{r^3} \quad \frac{y}{r}\right)\frac{}{x} \quad \left(\frac{x}{r}\frac{}{y} \quad \frac{xy^2}{r^3} \quad \frac{x^3}{r^3}\right)\frac{}{y}\\
&\left(\frac{x^2 y \quad y^3 \quad yr^2}{r^3}\right)\frac{}{x} \quad \left(\frac{xr^2 \quad xy^2 \quad x^3}{r^3}\right)\frac{}{y}\\
&\left(\frac{x^2 y \quad y^3 \quad yx^2 \quad yy^2}{r^3}\right)\frac{}{x} \quad \left(\frac{x^3 \quad xy^2 \quad xy^2 \quad x^3}{r^3}\right)\frac{}{y} \quad 0
\end{aligned}
$$

Por lo tanto:

$$
\boxed{e_1, e_2 \quad 0}
$$

En coordenadas polares, e_1 se ve como:

$$e_1 \quad \frac{r\cos}{r}\,\frac{}{x} \quad \frac{r\sin}{r}\,\frac{}{y}$$

donde $\frac{}{x} \quad \frac{r}{x}\,\frac{}{r} \quad \frac{}{x}\,\frac{}{}$ y $\frac{}{y} \quad \frac{r}{y}\,\frac{}{r} \quad \frac{}{y}\,\frac{}{}$

Calculamos las derivadas:

$$\begin{aligned}
&\frac{r}{x} \quad \frac{x}{\sqrt{x^2 \quad y^2}} \quad \frac{r\cos}{r} \quad \cos \\
&\frac{}{x} \quad \frac{y}{x^2 \quad y^2} \quad \frac{r\sin}{r^2} \quad \frac{\sin}{r} \\
&\frac{r}{y} \quad \frac{y}{\sqrt{x^2 \quad y^2}} \quad \frac{r\sin}{r} \quad \sin \\
&\frac{}{y} \quad \frac{x}{x^2 \quad y^2} \quad \frac{r\cos}{r^2} \quad \frac{\cos}{r}
\end{aligned}$$

Entonces e_1 es:

$$\begin{aligned}
e_1 \quad &\cos\left(\cos\ \frac{}{r} \quad \frac{\sin}{r}\,\frac{}{}\right) \quad \sin\left(\sin\ \frac{}{r} \quad \frac{\cos}{r}\,\frac{}{}\right) \\
&\left(\cos^2\ \frac{}{r} \quad \frac{\cos\ \sin}{r}\,\frac{}{}\right) \quad \left(\sin^2\ \frac{}{r} \quad \frac{\sin\ \cos}{r}\,\frac{}{}\right) \\
&\cos^2\ \frac{}{r} \quad \sin^2\ \frac{}{r} \quad \frac{\sin\ \cos}{r}\,\frac{}{} \quad \frac{\cos\ \sin}{r}\,\frac{}{} \\
&\frac{}{r}
\end{aligned}$$

$$\boxed{e_1 \quad \frac{}{r}}$$

De la misma forma, e_2 es:

$$\begin{aligned}
e_2 \quad &\frac{r\sin}{r}\,\frac{}{x} \quad \frac{r\cos}{r}\,\frac{}{y} \quad \sin\ \frac{}{x} \quad \cos\ \frac{}{y} \\
&\sin\left(\frac{r}{x}\,\frac{}{r} \quad \frac{}{x}\,\frac{}{}\right) \quad \cos\left(\frac{r}{y}\,\frac{}{r} \quad \frac{}{y}\,\frac{}{}\right) \\
&\sin\left(\cos\ \frac{}{r} \quad \frac{\sin}{r}\,\frac{}{}\right) \quad \cos\left(\sin\ \frac{}{r} \quad \frac{\cos}{r}\,\frac{}{}\right) \\
&\frac{\sin^2}{r}\,\frac{}{} \quad \sin\ \cos\ \frac{}{r} \quad \cos\ \sin\ \frac{}{r} \quad \frac{\cos^2}{r}\,\frac{}{} \\
&\frac{\sin^2}{r}\,\frac{}{} \quad \frac{\cos^2}{r}\,\frac{}{} \quad \frac{1}{r}\,\frac{}{}
\end{aligned}$$

Enonces e_2 es:

$$\boxed{e_2 \quad \frac{1}{r}\,\frac{}{}}$$

Por último, $e_2 \quad \sqrt{x^2 \quad y^2}\, e_2$, entonces:

$e_2 \quad r\frac{1}{r}\text{—} \quad \text{—}$

$$\boxed{e_2 \quad \text{—}}$$

14. **En** R^2 **la base ortonormal de** T_p **en coordenadas polares está dada como**

$$^1 \quad dr, \qquad ^2 \quad rd$$

¿Cuál es la base e_1, e_2 **de** T_p **dual a** $^1, ^2$ **? Recuerda quelas bases duales satisfacen** $^i e_j \quad {}^i_j$**, lo cual en particular para las bases naturales se ve como** $dx^i \; / \; x^j \quad {}^i_j$

Tenemos

$$^1 e_1 \quad 1$$
$$^2 e_2 \quad 1$$

Esto es:

$$dr \; e_1 \quad 1$$
$$rd \quad e_2 \quad 1$$

De aquí que:

$$e_1 \quad \frac{d}{dr} \quad \text{ya que } dr\left(\frac{d}{dr}\right) \quad \frac{dr}{dr} \quad 1$$

$$e_2 \quad \frac{1}{r}\frac{d}{d} \quad \text{ya que}$$

$$rd \left(\frac{1}{r}\frac{d}{d}\right) \quad r\frac{d}{d}d\left(\frac{1}{r}\right) \quad r\frac{1}{r}d\left(\frac{d}{d}\right) \quad 0 \quad \left(\frac{d}{d}\right) \quad 1$$

Por lo tanto:

$$\boxed{e_1 \quad \frac{d}{dr}}$$
$$\boxed{e_2 \quad \frac{1}{r}\frac{d}{d}}$$