

Some Unpleasant Monetarist Arithmetic

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In his presidential address to the American Economic Association (AEA), Milton Friedman (1968) warned not to expect too much from monetary policy. In particular, Friedman argued that monetary policy could not permanently influence the levels of real output, unemployment, or real rates of return on securities. However, Friedman did assert that a monetary authority could exert substantial control over the inflation rate, especially in the long run. The purpose of this paper* is to argue that, even in an economy that satisfies monetarist assumptions, if monetary policy is interpreted as open market operations, then Friedman's list of the things that monetary policy cannot permanently control may have to be expanded to include inflation.

In the context of this paper, an economy that satisfies monetarist assumptions (or, a *monetarist economy*) has two characteristics: the monetary base is closely connected to the price level, and the monetary authority can raise *seignorage*, by which we mean revenue from money creation. We will show that, under certain circumstances, the monetary authority's control over inflation in a monetarist economy is very limited even though the monetary base and the price level remain closely connected. In particular, we will demonstrate that this is true when monetary and fiscal policies are coordinated in a certain way and the public's demand for interest-bearing government debt has a certain form.¹

The public's demand for interest-bearing government debt constrains the government of a monetarist economy

in at least two ways. (For simplicity, we will refer to publicly held interest-bearing government debt as *government bonds*.) One way the public's demand for bonds constrains the government is by setting an upper limit on the real stock of government bonds relative to the size of the economy. Another way is by affecting the interest rate the government must pay on bonds. The extent to which these constraints bind the monetary authority and thus possibly limit its ability to control inflation permanently partly depends on the way fiscal and monetary policies are coordinated. To see this, consider two polar forms of coordination.

On the one hand, imagine that monetary policy dominates fiscal policy. Under this coordination scheme, the monetary authority independently sets monetary policy by, for example, announcing growth rates for base money for the current period and all future periods. By doing this, the monetary authority determines the amount of revenue it will supply the fiscal authority through seignorage. The fiscal authority then faces the constraints

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¹We will not exhaust the possible circumstances under which the monetary authority's control over inflation is very limited in monetarist economies. We will not even touch on the variety of nonmonetarist economies in which this is true. For examples of such nonmonetarist economies and a more general discussion of the ideas that underlie this paper, see Bryant and Wallace 1980. The messages of our paper are very similar to those of Miller 1981 and Lucas 1981a, b. Other related papers are McCallum 1978, 1981, and Scarth 1980.

imposed by the demand for bonds, since it must set its budgets so that any deficits can be financed by a combination of the seignorage chosen by the monetary authority and bond sales to the public. Under this coordination scheme, the monetary authority can permanently control inflation in a monetarist economy, because it is completely free to choose any path for base money.

On the other hand, imagine that fiscal policy dominates monetary policy. The fiscal authority independently sets its budgets, announcing all current and future deficits and surpluses and thus determining the amount of revenue that must be raised through bond sales and seignorage. Under this second coordination scheme, the monetary authority faces the constraints imposed by the demand for government bonds, for it must try to finance with seignorage any discrepancy between the revenue demanded by the fiscal authority and the amount of bonds that can be sold to the public. Although such a monetary authority might still be able to control inflation permanently, it is less powerful than a monetary authority under the first coordination scheme. If the fiscal authority's deficits cannot be financed solely by new bond sales, then the monetary authority is forced to create money and tolerate additional inflation.

Under the second coordination scheme, where the monetary authority faces the constraints imposed by the demand for government bonds, the form of this demand is important in determining whether or not the monetary authority can control inflation permanently. In particular, suppose that the demand for government bonds implies an interest rate on bonds greater than the economy's rate of growth. Then, if the fiscal authority runs deficits, the monetary authority is unable to control either the growth rate of the monetary base or inflation forever.

The monetary authority's inability to control inflation permanently under these circumstances follows from the arithmetic of the constraints it faces. Being limited simply to dividing government debt between bonds and base money and getting no help from budget surpluses, a monetary authority trying to fight current inflation can only do so by holding down the growth of base money and letting the real stock of bonds held by the public grow. If the principal and interest due on these additional bonds are raised by selling still more bonds, so as to continue to hold down the growth in base money, then, because the interest rate on bonds is greater than the economy's growth rate, the real stock of bonds will grow faster than the size of the economy. This cannot go on forever, since

the demand for bonds places an upper limit on the stock of bonds relative to the size of the economy. Once that limit is reached, the principal and interest due on the bonds already sold to fight inflation must be financed, at least in part, by seignorage, requiring the creation of additional base money. Sooner or later, in a monetarist economy, the result is additional inflation.

The first section of the paper establishes a version of this result in a model that is extremely monetarist. By imposing a simple quantity theory demand for base money, the model allows the government to raise seignorage and goes as far as anyone would go in assigning monetary policy influence over the price level. It is also monetarist in giving monetary policy influence over almost no real variables. Yet the model implies that, although fighting current inflation with tight monetary policy works temporarily, it eventually leads to higher inflation.

In the second section, we amend the model of the first section to include a more realistic demand for base money, one that depends on the expected rate of inflation. In a particular example of this second monetarist model, tighter money today leads to higher inflation not only eventually but starting today; tighter money today lacks even a temporary ability to fight inflation. While this example is extreme and may overstate the actual limits on tight money, it has the virtue of isolating a restrictive force on monetary policy that is omitted in the first section and that probably exists in the real world.

Tighter money now can mean higher inflation eventually

We describe a simple model that embodies unadulterated monetarism. The model has the following features:

- a. A common constant growth rate of n for real income and population.
- b. A constant real return on government securities that exceeds n .
- c. A quantity theory demand schedule for base or high-powered money, one that exhibits constant income velocity.²

A model with these features has the limitations on

²In Appendix A, we analyze a simple general equilibrium model that implies all our assumptions. The model of that appendix has the virtue that, since individual agents are identified, policies can be compared in terms of the welfare of the individuals in the model.

monetary policy stressed by Milton Friedman in his AEA presidential address: a natural, or equilibrium, growth rate of real income that monetary policy is powerless to affect and a real rate of interest on government bonds beyond the influence of monetary policy. We choose this model, one that embraces as unqualified a set of monetarist assumptions as we can imagine, to show that our argument about the limitations of monetary policy is not based on abandoning any of the key assumptions made by monetarists who stress the potency of monetary policy for controlling inflation. Instead, the argument hinges entirely on taking into account the future budgetary consequences of alternative current monetary policies when the real rate of return on government bonds exceeds n , the growth rate of the economy.

We describe fiscal policy by a time path or sequence $D(1), D(2), \dots, D(t), \dots$, where $D(t)$ is measured in real terms (time t goods) and is defined as real expenditures on everything except interest on government debt minus real tax collections. From now on we will refer to $D(t)$ as the *deficit*, but keep in mind that $D(t)$ equals the real deficit as ordinarily measured less real interest payments. For convenience, we label the current date $t = 1$. We describe monetary policy by a time path $H(1), H(2), \dots, H(t), \dots$, where $H(t)$ is the stock of base or high-powered money at time t . If, for simplicity, we assume that the entire government debt consists of one-period debt, then we can write the consolidated government budget constraint (consolidating the Treasury and the Federal Reserve System) as³

$$(1) \quad D(t) = \{[H(t) - H(t-1)]/p(t)\} + \{B(t) - B(t-1)[1 + R(t-1)]\}$$

for $t = 1, 2, \dots$. We are letting $p(t)$ be the price level at time t , while $R(t-1)$ is the real rate of interest on one-period government bonds between time $t-1$ and time t ; $B(t-1)[1 + R(t-1)]$ is the real par value of one-period privately held government bonds that were issued at time $t-1$ and fall due in period t , where $B(t-1)$ is measured in units of time $t-1$ goods and $[1 + R(t-1)]$ is measured in time t goods per unit of time $t-1$ goods. In equation (1), $B(t)$ is government borrowing from the private sector between periods t and $t+1$, measured in units of time t goods. Equation (1) states that the deficit must be financed by issuing some combination of currency and interest-bearing debt. Finally, we let $N(t)$ be the population at time t . We assume that $N(t)$ grows at the constant

rate n , or that

$$(2) \quad N(t+1) = (1+n)N(t)$$

for $t = 0, 1, 2, \dots$, with $N(0) > 0$ being given and n being a constant exceeding -1 .

Dividing both sides of (1) by $N(t)$ and rearranging gives the following per capita form of the government's budget constraint:

$$(3) \quad B(t)/N(t) = \{[1 + R(t-1)]/(1+n)\} \times [B(t-1)/N(t-1)] + [D(t)/N(t)] - \{[H(t) - H(t-1)]/[N(t)p(t)]\}.$$

We shall now use equation (3) and our monetarist model—assumptions a, b, and c—to illustrate a version of the following proposition: if fiscal policy in the form of the $D(t)$ sequence is taken as given, then tighter current monetary policy implies higher future inflation.

We specify alternative time paths for monetary policy in the following way. We take $H(1)$ as predetermined and let alternative monetary policies be alternative constant growth rates θ of $H(t)$ for $t = 2, 3, \dots, T$, where T is some date greater than or equal to 2. For $t > T$, we assume that the path of $H(t)$ is determined by the condition that the stock of interest-bearing real government debt per capita be held constant at whatever level it attains at $t = T$. The restriction on monetary policy from time T onward is consistent with there being a limit on the real debt per capita. Thus, with $H(1)$ taken as given, we assume that

³Although the government collects income taxes on the interest payments on government debt, the pre-tax yield is what belongs in equation (1), as long as private securities and government securities are taxed at a common rate and as long as any change in $B(t-1)$ is offset by an equal change in $K(t-1)$ in the opposite direction, where $K(t-1)$ is private investment measured in time $t-1$ goods. To see this, define $g(t)$ as government expenditures (not including interest payments) minus all taxes except taxes on private and government securities, and let τ be the tax rate on interest earnings. Then the government cash flow constraint can be written

$$g(t) - \tau RK(t-1) - \tau RB(t-1) = \{[H(t) - H(t-1)]/p(t)\} + \{B(t) - B(t-1)(1+R)\}.$$

Our Appendix A model implies complete crowding out, which can be expressed as $B(t-1) + K(t-1) = \bar{B}$, a constant. Substituting $\bar{B}$ into the last equation gives

$$g(t) - \tau R\bar{B} = \{[H(t) - H(t-1)]/p(t)\} + \{B(t) - B(t-1)(1+R)\}$$

which is equivalent to (1) above, with $D(t) = g(t) - \tau R\bar{B}$.

$$(4) \quad H(t) = (1+\theta) H(t-1)$$

for $t = 2, 3, \dots, T$ and examine the consequences of various choices of θ and T .⁴ We will say that one monetary policy is tighter than another if it is characterized by a smaller θ .

Notice that we have written equation (1) in terms of real debt and real rates of return. If we want to analyze a setting in which government bonds are not indexed, which is the situation in the United States today, then we must insure that anticipated inflation is the same as actual inflation. We impose that condition, in part, by supposing that both the path of fiscal policy, the $D(t)$ sequence, and the path of monetary policy, θ and T , are announced at $t = 1$ and known by private agents. Once we assume that, it does not matter whether nominal or indexed debt is issued from $t = 1$ onward.⁵

Now note that assumptions a and c imply that the price level at any time t is proportional to the time t stock of base money per capita, $H(t)/N(t)$, namely, that

$$(5) \quad p(t) = (1/h) [H(t)/N(t)]$$

for some positive constant h .

From equation (5) it follows that, for $t = 2, \dots, T$, one plus the inflation rate is given by $p(t)/p(t-1) = (1+\theta)/(1+n)$. Thus, when we specify monetary policy, a θ and a T , we are simultaneously choosing the inflation rate for periods $t = 2, 3, \dots, T$. We are interested in determining how the inflation rate for the periods after T depends on the inflation rate chosen for the periods before T .

We do this in two simple steps. We first determine how the inflation rate after T depends on the stock of interest-bearing real government debt per capita attained at T and to be held constant thereafter, denoting that per capita stock by $b_\theta(T)$. We then show how $b_\theta(T)$ depends on θ .

To find the dependence of the inflation rate for $t > T$ on $b_\theta(T)$, we use equation (3) for any date $t > T$, substituting into it $B(t)/N(t) = B(t-1)/N(t-1) = b_\theta(T)$ and $H(t) = hN(t)p(t)$ as implied by (5). The result can be written as

$$(6) \quad 1 - [1/(1+n)] [p(t-1)/p(t)] \\ = ([D(t)/N(t)] + \{[R(t-1) - n]/(1+n)\} b_\theta(T))/h.$$

Note that equation (6) makes sense only if the right-hand side is less than unity, a condition which itself places an upper bound on $b_\theta(T)$ if $[R(t-1) - n]$ is positive, as we are assuming. If that condition holds and $[R(t-1) - n]$ is a positive constant, as stated by assumption b, then the right-hand side of (6) is higher the higher $b_\theta(T)$ is. This in turn implies that the inflation rate is higher the higher $b_\theta(T)$ is, a conclusion that holds for all $t > T$.

To complete the argument that a tighter monetary policy now implies higher inflation later, we must show that the smaller θ is, the higher $b_\theta(T)$ is. To find $b_\theta(T)$ and its dependence on θ , we first find $B(1)/N(1) \equiv b(1)$ and then show how to find the entire path $b(1)$, $b_\theta(2)$, $b_\theta(3)$, $\dots$, $b_\theta(T)$.

We solve for $b(1)$ from the $t = 1$ version of equation (3), namely,

$$(7) \quad b(1) = \{\bar{B}(0)/[N(1)p(1)]\} + [D(1)/N(1)] \\ - \{[H(1) - H(0)]/[N(1)p(1)]\}.$$

Here, in place of $B(0)[1 + R(0)]$, we have inserted $\bar{B}(0) \div p(1)$, $\bar{B}(0)$ being the nominal par value of the debt issued at $t = 0$. By making this substitution, we avoid assuming anything about the relationship between actual and expected inflation from time $t = 0$ to time $t = 1$. In conjunction with equation (5), equation (7) lets us solve for $b(1)$ in terms of $D(1)$, $N(1)$, $H(1)$, $H(0)$, and $\bar{B}(0)$. Note that $b(1)$ does not depend on θ .

We now proceed to find $b_\theta(2)$, $b_\theta(3)$, $\dots$, $b_\theta(T)$. Using equations (4) and (5) and the definition $b(t) = B(t)/N(t)$,

⁴The reader may have noted that the argument presented above does not depend on the magnitude of the $D(t)$ sequence. For the same economy, another way to specify policy is to have (4) hold until some given bound on per capita real debt is reached and have monetary policy be determined thereafter by the condition that the per capita real debt be held constant at that bound. Under assumptions a, b, and c, the following proposition is true for rules of this kind: If an $H(t)$ growth rate θ and a $D(t)$ sequence are such that the debt bound is reached at time T_θ and if $\bar{\theta} < \theta$, then, under the $H(t)$ growth rate $\bar{\theta}$, the given debt bound is reached at $T_{\bar{\theta}} < T_\theta$ and the inflation rate during the period from $T_{\bar{\theta}}$ to T_θ is higher under the $\bar{\theta}$ policy than under the θ policy.

⁵This assumes a *rational expectations equilibrium*, which is equivalent to perfect foresight here because the model has no randomness. Thus, our statements involve comparing alternative paths for monetary and fiscal variables which are known in advance. The authorities are assumed to stick with the plans that they announce and not to default, in real terms, on the interest-bearing debt issued from time 1 onward, so that it is as if all interest-bearing debt were indexed. Such an assumption is appropriate for analyzing the alternative time sequences or strategies for monetary policy variables, despite the fact that governments have historically defaulted on substantial fractions of their interest-bearing debt by inflating it away. Such a default option is not available as a policy to which a government can plan to resort persistently.

we can write equation (3) as

$$(8) \quad b(t) = \{[1 + R(t-1)]/(1+n)\} b(t-1) + [D(t)/N(t)] - [h\theta/(1+\theta)]$$

for $t = 2, 3, \dots, T$. By repeated substitution, it follows for any $t > 2$ and $t \leq T$ that

$$(9) \quad b_t(t) = \phi(t, 1)b(1) + \left(\sum_{s=2}^t \phi(t, s)[D(s)/N(s)]\right) - \left([h\theta/(1+\theta)] \sum_{s=2}^t \phi(t, s)\right)$$

where $\phi(t, t) = 1$ and, for $t > s$,

$$\phi(t, s) = \left(\prod_{j=s}^{t-1} [1 + R(j)]\right) / (1+n)^{t-s}.$$

It follows from (9) that $b_t(T)$ is larger the smaller θ is.⁶

This completes our demonstration of a version of the proposition that less inflation now achieved through monetary policy on its own implies more inflation in the future. It is crucial for such a result that the real rate of return on government securities exceed n from T onward [see equation (6)] and that the path of fiscal policy given by $D(1), D(2), \dots, D(t), \dots$ not depend on θ .

Tighter money now can mean higher inflation now

In the last section, we described circumstances in which tighter monetary policy lowers inflation in the present, but at the cost of increasing inflation in the future. Our having assumed a money demand schedule of the simplest quantity theory form [equation (5)] not only much simplified the analysis but also had the substantive aspect of ignoring any dependence of the demand for base money on the expected rate of inflation. This dependence is widely believed to be important; Bresciani-Turroni (1937) and Cagan (1956) found substantial evidence that it exists by studying countries that had undergone rapid inflation. This dependence complicates the dynamics of the influence of monetary policy on the price level. If the demand for money depends on the expected rate of inflation, then it turns out (see Sargent and Wallace 1973) that the current price level depends on the current level and all anticipated future levels of the money supply. This sets up a force whereby high rates of money creation anticipated in the future tend to raise the current rate of inflation. As we shall show, this force can limit the power

of tighter monetary policy to deliver even a temporarily lower inflation rate.

We maintain all of the features of the last section except one: we replace equation (5) by⁷

$$(10) \quad H(t)/[N(t)p(t)] = (\gamma_1/2) - [(\gamma_2/2)p(t+1)/p(t)]$$

for $t \geq 1$, with $\gamma_1 > \gamma_2 > 0$. Equation (10) is a version of the demand schedule for money that Cagan (1956) used in studying hyperinflations. The equation is shown in our Appendix B to imply the following equation for the price level at t :

$$p(t) = (2/\gamma_1) \sum_{j=0}^{\infty} (\gamma_2/\gamma_1)^j [H(t+j)/N(t+j)].$$

This equation expresses the current price level in terms of the current value and all future values of the per capita supply of base money. So the current price level and inflation rate depend not only on how tight money is today, but also on how tight it is for all tomorrows. If the situation is, as in the last section, that tighter money now causes looser money later, then this equation for $p(t)$ suggests the possibility that tighter money today might fail to bring about a lower inflation rate and price level even today. We shall now provide an example in which this possibility is in fact realized.

As in the last section, policy consists of a deficit sequence $D(t)$, a date T after which monetary policy is determined by the condition that the real interest-bearing government debt per capita be held constant, and θ , the growth rate of the monetary base for periods before T . In the model of this section, the path of the price level before T depends on all of these aspects of policy and not just on θ , as was true in the model of the last section.

Appendix B describes a way of solving for the paths of the endogenous variables. Here we simply present an example in which a tighter monetary policy in the form of a lower θ implies a uniformly higher price level and inflation rate.

The economy of this example is characterized by $\gamma_1 = 3.0$, $\gamma_2 = 2.5$, $R = .05$, and $n = .02$. The common features of policy are a per capita deficit sequence $d(t)$ with $d(t) =$

⁶Equation (9) can be used to determine the $D(t)$ sequences and the values of θ that satisfy the "if" clause of the proposition given in footnote 4.

⁷Note that equation (5) is a special case of equation (10) with $h = \gamma_1/2$ and $\gamma_2 = 0$. See Appendix A for an underlying model that implies (10) and all of our other assumptions.

A Spectacular Example of the Potential Effects of Tight and Loose Monetary Policy

Tight Money: $\theta = .106$ Loose Money: $\theta = .120$

Date (t)	Inflation Rate [p(t+1)/p(t)]		Per Capita Bond Holdings [B(t)/N(t)]		Per Capita Real Money Balances {H(t)/[N(t)p(t)]}	
	Tight	Loose	Tight	Loose	Tight	Loose
1	1.0842	1.0825	0.0811	0.0815	0.1202	0.1469
2	1.0841	1.0808	0.1196	0.1180	0.1448	0.1490
3	1.0841	1.0789	0.1592	0.1552	0.1449	0.1514
4	1.0841	1.0768	0.2000	0.1933	0.1449	0.1540
5	1.0841	1.0743	0.2420	0.2321	0.1449	0.1571
6	1.0840	1.0716	0.2853	0.2718	0.1450	0.1606
7	1.0840	1.0684	0.3297	0.3121	0.1450	0.1641
8	1.0840	1.0647	0.3755	0.3532	0.1450	0.1691
9	1.0839	1.0605	0.4227	0.3949	0.1451	0.1744
≥ 10	1.0839	1.0556	0.4712	0.4372	0.1451	0.1805

Parameters

$$\begin{array}{lll} \gamma_1 = 3.0 & R = .05 & d(t) = \begin{cases} .05 & \text{for } t = 1, 2, \dots, 10. \\ 0 & \text{for } t > 10. \end{cases} \\ \gamma_2 = 2.5 & n = .02 & [H(0) + \bar{B}(0)]/H(1) = 200/164.65 \end{array}$$

.05 for $t = 1, 2, \dots, 10$ and $d(t) = 0$ for $t > 10$; $T = 10$; and $[H(0) + \bar{B}(0)]/H(1) = 200/164.65$. Two different base money growth rates are studied: $\theta = .106$ and $\theta = .120$. The accompanying table compares the inflation rates, per capita bond holdings, and per capita real money balances for the economy under the two policies. It turns out that the price level at $t = 1$ is 1.04 percent higher under the smaller θ , that is, the tighter policy.

This example is spectacular in that the easier, or looser, monetary policy is uniformly better than the tighter policy. (In terms of the model of Appendix A, the equilibrium for the looser monetary policy is Pareto superior to that for the tighter monetary policy.) In this example, the tighter current monetary policy fails to even temporarily reduce inflation below the level it would be under the looser policy.⁸

Concluding Remarks

We have made two crucial assumptions to obtain our results.

One is that the real rate of interest exceeds the growth

rate of the economy. We have made that assumption because it seems to be maintained by many of those who argue for a low rate of growth of money no matter how big the current deficit is. If we were to replace that assumption, we would instead assume that the public's demand for government bonds is an increasing function of the real rate of return on bonds, with an initial range over which that demand is positive at rates of return that are negative or less than the growth rate of the economy. We would still assume that the quantity of bonds demanded per capita has an upper bound. A demand function for government bonds like this would imply that monetary policy helps determine the real rate of interest on government bonds and that, for some monetary policies entailing low enough bond supplies, seignorage can be earned on bonds as well as on base money. However, an analysis that included such a demand schedule for bonds would share with ours the implication that a sufficiently

⁸See Appendix C for a discussion of how to find parameter values which imply this seemingly paradoxical price level behavior.

tight current monetary policy can imply growth in government interest-bearing indebtedness so rapid that inflation in the future is higher than it would have been with an easier current monetary policy.

The other crucial assumption that we have made is that the path of fiscal policy $D(t)$ is given and does not depend on current or future monetary policies. This assumption is not about the preferences, opportunities, or behavior of private agents, as is our first crucial assumption, but is, rather, about the behavior of the monetary and fiscal authorities and the game that they are playing. Since the monetary authority affects the extent to which seignorage is exploited as a revenue source, monetary and fiscal policies simply have to be coordinated. The question is, Which authority moves first, the monetary authority or the fiscal authority? In other words, Who imposes discipline on whom? The assumption made in this paper is that the fiscal authority moves first, its move consisting of an entire $D(t)$ sequence. Given that $D(t)$ sequence, monetary policy must be determined in a way consistent with it, if that is possible. [As we have seen, it may not be possible if the $D(t)$ sequence is too big for too long.] Given this assumption about the game played by the authorities, and given our first crucial assumption, the monetary authority can make money tighter now only by making it looser later.

One can interpret proposals for monetary restraint differently than we have in this paper, in particular, as calls to let the monetary authority move first and thereby impose discipline on the fiscal authority. In this interpretation, the monetary authority moves first by announcing a fixed θ rule like (4) not just for $t = 2, 3, \dots, T$, but for all $t \geq 1$. By doing this in a binding way, the monetary authority forces the fiscal authority to choose a $D(t)$ sequence consistent with the announced monetary policy. This form of permanent monetary restraint is a mechanism that effectively imposes fiscal discipline. Alternative monetary mechanisms that do impose fiscal discipline have been suggested, for example, fixed exchange rates or a commodity money standard such as the gold standard. Nothing in our analysis denies the possibility that monetary policy can permanently affect the inflation rate under a monetary regime that effectively disciplines the fiscal authority.

Appendix A

An Overlapping Generations Model That Generates Our Assumptions

This appendix describes a simple formal model that implies the assumptions used in the preceding paper. The model is a version of Samuelson's (1958) model of overlapping generations.

We describe the evolution of the economy from time $t = 1$ onward. The economy is populated by agents who each live two periods. In each period, only one type of good exists. At each time $t \geq 1$, there are born $N_1(t)$ identical poor people who are endowed after taxes with α_1 units of the good when young and α_2 units when old. At each date $t \geq 1$ there are also born $N_2(t)$ identical rich people who are endowed after taxes with β units of the good when young and zero units when old. We assume that $N_1(t) = (1+n)N_1(t-1)$ and $N_2(t) = (1+n)N_2(t-1)$ for $t \geq 1$, with $N_1(0)$ and $N_2(0)$ given and positive and $n > -1$. The total population is $N(t) = N_1(t) + N_2(t)$.

There is available in this economy a physical technology for converting the time t good into the time $t+1$ good. In particular, if $k(t) \geq \underline{k}$ goods are stored at time $t \geq 1$, then $(1+R)k(t)$ goods become available at time $t+1$. This is a constant returns-to-scale technology with a constant real rate of return on investment of $R > 0$. We assume that there is a minimum scale of $\underline{k}$ at which this investment can be undertaken and that this minimum scale and the endowments satisfy $\beta/2 > \underline{k} > \alpha_1$. We also assume that a legal restriction on intermediation prevents two or more of the poor from sharing investments, thereby preventing the poor from holding the real investment.

The government issues both currency, which doesn't bear interest, and bonds, which do. The currency is held by the poor because government bonds are issued in such large minimum denominations that the poor cannot afford them. (Again, a legal restriction on intermediation is relied on to prevent two or more people from sharing a government bond.) There is no uncertainty in the model, so that the rich will hold government bonds only if the real interest rate on bonds at least equals that on private investment, which must be at least as large as the yield on currency.

As in our paper, the government finances a real deficit $D(t)$ by some combination of currency creation and bond creation. The government's budget constraint is

$$(A1) \quad D(t) = \{[H(t) - H(t-1)]/p(t)\} \\ + [B(t) - B(t-1)](1+R)$$

for $t \geq 1$, where $H(t)$ is the stock of base or high-powered money

(currency) measured in dollars, $p(t)$ is the price level in dollars per time t goods, and $B(t)$ is government borrowing (from the private sector) in time t goods. The government's real deficit $D(t)$ is, then, measured in time t goods.

In addition, at time $t = 1$ there are $N_1(0)$ and $N_2(0)$ old poor and rich people, respectively, who hold $H(0)$ units of currency and maturing bonds of par nominal value $\bar{B}(0)$. The old alive at time $t = 1$ simply offer all of their currency inelastically in exchange for goods to those young at that time.

The young of each generation $t \geq 1$ are assumed to maximize the utility function $c_t^h(t)c_t^h(t+1)$ where $c_t^h(s)$ is consumption of the s -period good by an agent of type h born at time t . Letting $w_t^h(s)$ be the endowment of the s -period good of an agent of type h born at t , and assuming that each agent faces a single rate of return R^h , a young agent h at generation t chooses a lifetime consumption bundle to maximize utility subject to the present-value constraint,

$$c_t^h(t) + [c_t^h(t+1)/(1+R^h)] \\ = w_t^h(t) + [w_t^h(t+1)/(1+R^h)].$$

The solution to this problem is the saving function:

$$(A2) \quad w_t^h(t) - c_t^h(t) \\ = \{w_t^h(t) - [w_t^h(t+1)/(1+R^h)]\}/2.$$

Since all saving of poor people is in the form of currency, if h is poor, $1 + R^h = p(t)/p(t+1)$. Moreover, in the range where $p(t)/p(t+1) < 1+R$, only the poor hold currency. Thus, in this range, the money market equilibrium condition is that $H(t)/p(t)$ equals the total real saving of the poor, which by (A2) is $N_1(t) \{ \alpha_1 - [\alpha_2 p(t+1)/p(t)] \}/2$. Dividing by $N(t)$, we can write this condition as

$$(A3) \quad H(t)/[N(t)p(t)] = \{ \alpha_1 - [\alpha_2 p(t+1)/p(t)] \} \\ \times N_1(t)/2N(t).$$

This is equation (10) if we let $\gamma_1/2 = \alpha_1 N_1(t)/2N(t)$ and $\gamma_2/2 = \alpha_2 N_1(t)/2N(t)$. [Recall that $N_1(t)/N(t)$ is constant.] We get equation (5) if $\alpha_2 = 0$.

According to (A2), each rich person saves a constant amount $\beta/2$ per period. As long as government bonds bear the

real rate of return R , each rich person is indifferent between holding government bonds or holding private capital. However, in the aggregate, the rich only wish to save $N_2(t)\beta/2$ per period. The number $\beta/2$ determines an upper bound on per capita holdings of interest-bearing government debt, the sort of bound alluded to in the paper. We let $K(t)$ denote the total amount of real investment (storage), measured in goods, undertaken by the young members of generation t , all of them rich. We then have

$$(A4) \quad K(t) + B(t) = N_2(t)\beta/2 = \bar{B}(t)$$

where $B(t)$ is the amount of loans to the government. Equation (A4) expresses the result that additional government borrowing merely crowds out private investment on a one-for-one basis.

The national income identity can be written like this:

$$\begin{aligned} (A5) \quad & N_1(t)c_1^1(t) + N_1(t-1)c_{t-1}^1(t) + N_2(t)c_t^2(t) \\ & + N_2(t-1)c_{t-1}^2(t) + K(t) + G(t) \\ & = N_1(t)\alpha_1 + N_1(t-1)\alpha_2 + N_2(t)\beta \\ & + T(t) + (1+R)K(t-1). \end{aligned}$$

Here $G(t)$ denotes government purchases and $T(t)$ denotes total direct taxes. The government deficit as defined in our paper is related to $G(t)$ and $T(t)$ by $D(t) = G(t) - T(t)$.

Thus, as long as solutions satisfy $p(t)/p(t+1) < 1+R$ and the total real bond supply is less than $\bar{B}(t)$, the model just described implies all the assumptions made in the paper. This particular model also implies how different agents fare under different policies. The present-value budget constraint set out above indicates that each poor person is better off the lower the inflation rate, that each rich person is unaffected by the inflation rate, and that those who at $t = 1$ are in the second period of their lives and are holding currency or maturing bonds are better off the lower the initial price level, $p(1)$. These observations are what lie behind our claim in the paper that, for the example in the second section, the tight money policy is Pareto inferior to the loose money policy.*

*By pursuing the example in the second section of the paper and other examples comparing the welfare of agents across stationary states, the model can be used to support Milton Friedman's 1948 prescription that the entire government deficit be financed by creating base money.

Appendix B

A Model in Which Tighter Money Now Can Cause Higher Inflation Now

In this appendix, we analyze the model in the second section of the paper which generalizes the model of the first section by assuming that the demand schedule for base money depends on the expected rate of inflation. The particular demand schedule that we use resembles Cagan's (1956) famous demand schedule and can be deduced formally from the model in Appendix A by assuming that the poor of each generation are endowed with $\gamma_1 N(0)/N_1(0) > 0$ units of the consumption good when they are young and $\gamma_2 N(0)/N_1(0) > 0$ units when they are old. (The model in the first section of the paper emerges when we set $\gamma_2 = 0$.) Except for this generalization, all other features of the model remain as they were in the first section of the paper.

As before, we assume a demand schedule for base money of the form

$$(B1) \quad H(t)/[N(t)p(t)] = (\gamma_1/2) - [(\gamma_2/2)p(t+1)/p(t)]$$

for $t \geq 1$, where $\gamma_1 > \gamma_2 > 0$. [This is equation (10) in the second section of the paper.] Except for replacing equation (5) with this equation, we retain the features of the model in the paper's first section, including the budget restraint (1) and the law of motion of total population (2). We describe experiments similar to the one in that section: we hold the per capita real government debt $b(t)$ constant for $t > T$ and examine the choice of alternative rates of growth of base money θ for $t = 2, \dots, T$. The step of replacing (5) with (B1) substantially complicates the dynamics of the system, as we shall see.

We begin by examining the behavior of the system for $t > T$. For $t > T + 1$ we specify as before that monetary policy is determined so that $b(t) = b(t-1) = b(T)$. Using the budget constraint (1) together with this condition implies

$$(B2) \quad [H(t) - H(t-1)]/[N(t)p(t)] \\ = \{[R(t-1) - n]/(1+n)\} b(T) + [D(t)/N(t)]$$

for $t \geq T + 1$. We now assume that

$$D(t)/N(t) = d$$

for $t \geq T$, where d is a constant. This is a computationally convenient assumption, although the general flavor of our results does not depend on making it.

We now define per capita real balances as $m(t) \equiv$

$H(t)/[N(t)p(t)]$ and the one-period gross inflation rate as $\pi(t) = p(t)/p(t-1)$. In terms of these variables, equations (B1) and (B2) become

$$(B3) \quad m(t) = (\gamma_1/2) - (\gamma_2/2)\pi(t+1)$$

for $t \geq 1$ and

$$(B4) \quad m(t) - \{m(t-1)/[\pi(t)(1+n)]\} = \xi$$

for $t \geq T + 1$, where

$$\xi = [(R-n)/(1+n)]b(T) + d.$$

The variable ξ has the interpretation of the per capita deficit that must be financed by seignorage from time $T+1$ onward. Eliminating $m(t)$ and $m(t-1)$ from these equations by substituting (B3) into (B4) leads to the following nonlinear difference equation in $\pi(t)$ for $t \geq T + 1$:

$$(B5) \quad \pi(t+1) = \lambda - (\gamma_1/\gamma_2)[1/(1+n)][1/\pi(t)]$$

where

$$\lambda \equiv (\gamma_1/\gamma_2) + [1/(1+n)] - (2\xi/\gamma_2).$$

Equation (B5) is graphed in the accompanying figure. It is readily verified that if

$$(B6) \quad \lambda^2 - \{4\gamma_1/[\gamma_2(1+n)]\} > 0$$

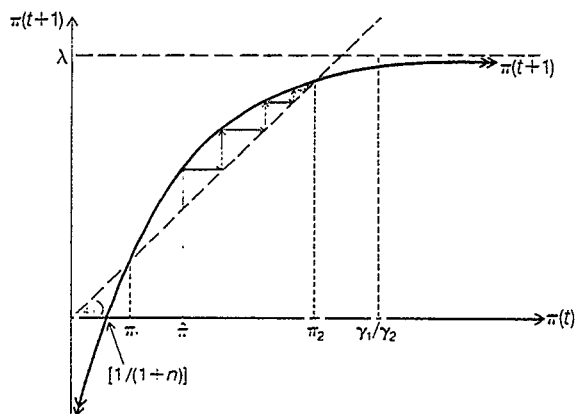
then (B5) has two stationary points, their values being given by

$$(B7) \quad \pi_1 = (1/2)[\lambda - (\lambda^2 - \{4\gamma_1/[\gamma_2(1+n)]\})^{1/2}] \\ \pi_2 = (1/2)[\lambda + (\lambda^2 - \{4\gamma_1/[\gamma_2(1+n)]\})^{1/2}].$$

We let $\bar{\xi}$ be the value of ξ for which the left-hand side of (B6) equals zero. Evidently, $\bar{\xi}$ is a function of γ_1 , γ_2 , and n and represents the maximum stationary per capita deficit that can be financed by seignorage. From (B7), it follows that, if $\xi = 0$, then $\pi_1 = 1/(1+n)$, $\pi_2 = \gamma_1/\gamma_2$. From the graph of (B5), it immediately follows that, for $\bar{\xi} > \xi > 0$, $\pi_1 > 1/(1+n)$, $\pi_2 < \gamma_1/\gamma_2$, and

Equation (B5)

$$\pi(t+1) = \lambda - (\gamma_1/\gamma_2)[1/(1+n)][1/\pi(t)]$$



raising ξ causes π_1 to rise and π_2 to fall.

Inequality (B6) is a necessary and sufficient condition for it to be possible to finance the per capita deficit ξ by seignorage. Assuming (B6) is satisfied, there exists a multiplicity of inflation-real balance paths that finance the deficit. Any setting of $\pi(T)$ satisfying $\pi_1 < \pi(T) < \gamma_1/\gamma_2$, with $\pi(t)$ for $t > T + 1$ being given by (B5), results in the deficit being financed. (Later we shall describe the money supply paths needed to accomplish these paths for inflation.) The graph of (B5) shows that, for any $\pi_1 < \pi(T) < \gamma_1/\gamma_2$, $\pi(t) - \pi_2$ as $t \rightarrow \infty$. Thus, there are three classes of inflation paths which finance the deficit:

1. The stationary path with $\pi(t) = \pi_1$ for $t > T$.
2. The stationary path with $\pi(t) = \pi_2$ for $t > T$.
3. Nonstationary paths with $\gamma_1/\gamma_2 > \pi(T) > \pi_1$ and $\lim_{t \rightarrow \infty} \pi(t) = \pi_*$.

We assume that the government selects the money supply path so that $\pi(T) = \pi_1$, that is, so that the deficit is financed by the uniformly lowest inflation rate path and, therefore, in view of (B3), the lowest price level path. This assumption is reasonable, since this selection leaves the government with the same resources as any other selection, while leaving holders of money better off.

Having determined the inflation rate $p(t)/p(t-1) = \pi_1$ for $t > T+1$ from (B6), we can determine the time T real balances and price level by setting $t = T$ in (B3):

$$H(T)/[N(T)p(T)] = (\gamma_1/2) - (\gamma_2/2)\pi,$$

OT

$$(B8) \quad p(T) = (2/\gamma_1)\{1/[1 - (\gamma_2/\gamma_1)\pi_1]\}[H(T)/N(T)].$$

Since $H(T)$ and $N(T)$ are given at T , this equation determines $p(T)$ as a function of π_1 . Also, since $\pi(t)$ is constant for $t > T + 1$, we have from (B3) and the definition of $m(t) \equiv H(t)/[N(t)p(t)]$ that

$$H(t)/N(t) = \pi_1[H(t-1)/N(t-1)]$$

for $t \geq T + 1$, so that per capita nominal balances grow at the constant gross rate π_t , which is the rate of inflation for $t > T + 1$.

It is instructive to describe briefly the following alternative way to solve the system for $t > T + 1$ by obtaining a pair of linear difference equations. Define $h(t) \equiv H(t)/N(t)$, and write the budget constraint (B2) as

$$h(t) = [1/(1+n)]h(t-1) + \xi p(t)$$

for $t \geq T + 1$ and the demand function for base money (B1) as

$$p(t) = (\gamma_2/\gamma_1)p(t+1) + (2/\gamma_1)h(t)$$

for $t \geq 1$. Using the lag operator L , we write these two equations as

$$(B9) \quad \{1 - [1/(1+n)]L\}h(t) = \xi p(t)$$

for $t \geq T + 1$ and

$$[1 - (\gamma_2/\gamma_1)L^{-1}]p(t) = (2/\gamma_1)h(t)$$

for $t \geq 1$. Solving the second equation in terms of $h(t)$ gives

$$(B10) \quad p(t) = (2/\gamma_1)[1 - (\gamma_2/\gamma_1)L^{-1}]^{-1}h(t) + c(\gamma_1/\gamma_2)'$$

or

$$(B11) \quad p(t) = (2/\gamma_1) \sum_{i=0}^{\infty} (\gamma_2/\gamma_1)^i h(t+i) + c(\gamma_1/\gamma_2)^t$$

for $t \geq 1$, where c is any nonnegative constant. Substituting (B10) into (B9) and operating on both sides of the result with $[1 - (\gamma_2/\gamma_1)\mathcal{L}^{-1}]$ gives the following homogenous difference equation in $h(t)$:

$$(B12) \quad L^{-1}[-1 + \lambda L - [1/(1+n)](\gamma_1/\gamma_2)L^2]h(t) = 0.$$

The characteristic polynomial in L can be factored in the usual way so that

$$(B13) \quad L^{-1}[(1 - \pi_1 L)(1 - \pi_2 L)]h(t) = 0$$

where π_1 and π_2 are the same roots given in (B7).

Since for $\xi > 0$ we have $\pi_1 < \pi_2 < \gamma_1/\gamma_2$, it follows that the geometric sum in current and future $h(t)$ that appears in (B11) converges for any $h(t)$ paths that satisfy (B13), or equivalently,

$$(B14) \quad h(t) = (\pi_1 + \pi_2)h(t-1) - \pi_1\pi_2h(t-2)$$

for $t \geq T+1$, with $h(T)$ given and $h(T+1)$ free. To insure that the deficit is financed each period, we have to add two side conditions to those listed under (B14): we must set $c = 0$ in (B10) and set $h(T+1)$ so that (B10) implies that $\pi(T+1) < \gamma_1/\gamma_2$. All of the price level paths with $c > 0$ have $\lim_{t \rightarrow \infty} \pi(t) = \gamma_1/\gamma_2$, which in view of equations (B3) and (B4) implies that $\lim_{t \rightarrow \infty} m(t) = 0$ and that a positive deficit cannot be financed. Any path with $\pi(T) > \gamma_1/\gamma_2$ implies nonpositive real balances at T . Since we are assuming that the government selects $h(t) = \pi_1 h(t-1)$, for $t > T+1$, and $h(T)$ is given, equation (B11) with $t = T$ becomes equivalent to equation (B8). We note that the admissible paths given by (B14) with $h(T+1) \neq \pi_1 h(T)$ have $\lim_{t \rightarrow \infty} [h(t)/h(t-1)] = \pi_2$ and so constitute the per capita nominal money supply paths that correspond to the inflation paths with $\pi(T) > \pi_1$ in the graph of (B5).

In summary, we have that for $t \geq T$ the price level and the stock of base money per capita evolve according to

$$(B15) \quad p(t) = (2/\gamma_1) \{1/[1 - (\gamma_2/\gamma_1)\pi_1]\} h(t)$$

$$(B16) \quad h(t+1) = \pi_1 h(t)$$

subject to $h(T)$ given, where π_1 is given by (B7).

We now describe the behavior of the price level, the supply of base money, and the stock of real government debt per capita for $t < T$. As in the first section of the paper, we assume a constant growth rate of base money [see equation (4) in the paper, which we repeat here as (B17)]:

$$(B17) \quad H(t) = (1+\theta)H(t-1)$$

for $t = 2, 3, \dots, T$. Equation (B10) with $c = 0$ implies that for all $t \geq 1$

$$(B18) \quad p(t) = (2/\gamma_1) \sum_{j=0}^{\infty} (\gamma_2/\gamma_1)^j h(t+j).$$

Further, we know from (B17) and (B16) that for $t = 1, 2, \dots, T-1$

$$(B19) \quad h(t+1) = \mu h(t)$$

where

$$(B20) \quad \mu = (1+\theta)/(1+n)$$

and for $t = T, T+1, \dots$

$$(B16) \quad h(t+1) = \pi_1 h(t).$$

Let us define the parameter ϕ by

$$(B21) \quad \phi = \gamma_2/\gamma_1$$

and write (B18) for $t \leq T$ as

$$(B22) \quad p(t) = (2/\gamma_1) \sum_{j=0}^{T-t} \phi^j h(t+j) + (2/\gamma_1) \sum_{j=T-t+1}^{\infty} \phi^j h(t+j).$$

Substituting (B19) and (B16) into (B22) and using some algebra implies

$$(B23) \quad p(t) = (2/\gamma_1) \{ [1 - \phi\pi_1 + (\pi_1 - \mu)\phi^{T-t+1}\mu^{T-t}] \div [(1 - \phi\pi_1)(1 - \phi\mu)] \} h(t)$$

for $t \leq T$.

Next, we define $s(t)$ as per capita seignorage:

$$s(t) \equiv [H(t) - H(t-1)]/[N(t)p(t)].$$

For $t \leq T$, we have that

$$s(t) = [h(t-1)/p(t)][\theta/(1+n)]$$

or

$$s(t) = [\theta/(1+\theta)][h(t)/p(t)].$$

Using (B23) in the above equation gives

$$(B24) \quad s(t) = [\theta/(1+\theta)](\gamma_1/2) \{ (1 - \phi\mu)(1 - \phi\pi_1) \div [1 - \phi\pi_1 + (\pi_1 - \mu)\phi^{T-t+1}\mu^{T-t}] \}$$

for $t \geq 2$. Using (1) from the paper, the definition of $s(t)$, and the definition $d(t) = D(t)/N(t)$, we have the law of motion for per capita real interest-bearing government debt:

$$(B25) \quad b(t) = [(1+R)/(1+n)]b(t-1) + d(t) - s(t)$$

for $T \geq t \geq 2$. Finally, we repeat equation (7) as equation (B26), which is the special version of (B25) for $t = 1$:

$$(B26) \quad b(1) = \{\bar{B}(0)/[N(1)p(1)]\} + d(1) - \{[H(1) - H(0)]/[N(1)p(1)]\}$$

where $\bar{B}(0)$ is the nominal par value of the one-period interest-bearing debt that was issued at time $t = 0$.

In Table B1 we have collected the equations describing the

Table B1
Equations Describing the Behavior of the System Before and After T ,
the Date Interest-Bearing Government Debt Per Capita is Stabilized

Path of	Before T ($1 \leq t \leq T$)	After T ($t \geq T$)
Base Money Per Capita	(B19) $h(t+1) = \mu h(t)$ (B20) $\mu = (1+\theta)/(1+n)$	(B16) $h(t+1) = \pi_1 h(t)$
Price Level	(B23) $p(t) = (2/\gamma_1) \{ [1 - \phi\pi_1 + (\pi_1 - \mu)\phi^{T-t+1}\mu^{T-t}] \div [(1-\phi\pi_1)(1-\phi\mu)] \} h(t)$	(B15) $p(t) = (2/\gamma_1) \{ 1/[1-(\gamma_2/\gamma_1)\pi_1] \} h(t)$
Real Interest-Bearing Government Debt Per Capita	(B25) $b(t) = [(1+R)/(1+n)]b(t-1) + d(t) - s(t)$ for $2 \leq t \leq T$ (B26) $b(1) = \{\bar{B}(0)/[N(1)p(1)]\} + d(1)$ - $\{[H(1)-H(0)]/[N(1)p(1)]\}$	$b(t) = b(T)$
Seignorage Per Capita	(B24) $s(t) = [\theta/(1+\theta)](\gamma_1/2) \{ (1-\phi\mu)(1-\phi\pi_1) \div [1 - \phi\pi_1 + (\pi_1 - \mu)\phi^{T-t+1}\mu^{T-t}] \}$ for $2 \leq t \leq T$	$s(t) = (\gamma_1/2) \{ 1 - [1/(\pi_1(1+n))] \} \times [1 - (\gamma_2/\gamma_1)\pi_1]$
Real Government Deficit Net of Interest Payments Per Capita	$d(t) = D(t)/N(t)$	$d(t) = d$

Parameters and Definitions

$$\begin{aligned}
 h(t) &\equiv H(t)/N(t) & \phi &= \gamma_2/\gamma_1 & \pi_1 &= (1/2)[\lambda - (\lambda^2 - 4\gamma_1/\gamma_2(1+n))]^{1/2} \\
 N(t) &= (1+n)N(t-1) & \xi &= [(R-n)/(1+n)]b(T) + d & \lambda &= (\gamma_1/\gamma_2) + [1/(1+n)] - (2\xi/\gamma_2) \\
 & & \mu &= (1+\theta)/(1+n)
 \end{aligned}$$

equilibrium before and after T . Starting at $t = 1$, the system works as follows. We take as exogenous a time path of the per capita deficit net of interest payments, $\{d(t); t \geq 1\}$, with $d(t) = d$ for $t \geq T$. We further take as exogenous $\bar{B}(0)$ and $H(0)$, which give the nominal par value of government debt inherited from the past. The date T is also taken as exogenous. The monetary authority chooses settings for $H(1)$ and θ . Then equations (B19), (B20), and (B23)–(B26) simultaneously determine $p(t)$ and $b(t)$ for $t = 1, \dots, T$, while equation (B15) determines $p(t)$

for $t > T$.

The equations of the model are linear in the endogenous variables, given a value for π_1 . However, from (B7) and the fact that $\xi = [(R-n)/(1+n)]b(T) + d$, we see that π_1 is itself a function of $b(T)$, which in turn depends on the value of π_1 through its effect on the behavior of $p(t)$ and $s(t)$ for $1 \leq t < T$, via equation (B23). Thus, determining the equilibrium of the system involves solving a nonlinear system of equations.

While the system can be solved in a variety of ways, we have

Table B2
Another Spectacular Example of the Potential Effects
of Tight and Loose Monetary Policy

Tight Money: $\theta = .106$ Loose Money: $\theta = .120$

Date (t)	Inflation Rate [p(t+1)/p(t)]		PerCapita Bond Holdings [B(t)/N(t)]		Per Capita Real Money Balances {H(t)/[N(t)p(t)]}	
	Tight	Loose	Tight	Loose	Tight	Loose
1	1.0842	1.0824	0.0811	0.0811	0.1448	0.1470
2	1.0841	1.0807	0.1196	0.1175	0.1448	0.1491
3	1.0841	1.0788	0.1592	0.1547	0.1449	0.1515
4	1.0841	1.0766	0.2000	0.1927	0.1449	0.1542
5	1.0841	1.0742	0.2420	0.2316	0.1449	0.1573
6	1.0840	1.0714	0.2852	0.2711	0.1450	0.1608
7	1.0840	1.0682	0.3297	0.3115	0.1450	0.1648
8	1.0840	1.0645	0.3755	0.3525	0.1450	0.1694
9	1.0839	1.0602	0.4227	0.3941	0.1451	0.1748
≥ 10	1.0839	1.0552	0.4712	0.4363	0.1451	0.1810

Parameters

$\gamma_1 = 3.0$	$R = .05$	$d(t) = \begin{cases} .05 & \text{for } t = 1, 2, \dots, 10. \\ 0 & \text{for } t > 10. \end{cases}$	$H(0) = 100$	$N(0) = 1,000$
$\gamma_2 = 2.5$	$n = .02$		$\bar{B}(0) = 100$	$b(1) = .08109$

found it convenient to use the following procedure based on backwards recursions. We begin by taking θ , but not $H(1)$, as given. We choose a value for $b(T)$ and solve (B7) for π_1 . Then we recursively solve (B24) and (B25) backwards for values of $\{b(t), s(t+1); t = T-1, T-2, \dots, 1\}$. Also, from (B23) we can determine per capita real balances $h(t)/p(t)$ for $t = 1, \dots, T$. Finally, given the values of $b(1)$ and $h(1)/p(1)$ thus determined, we solve equation (B26) for the value of $H(1)$ [or, equivalently, of $p(1)$]. This procedure produces a choice of $H(1)$ and θ and associated sequences for $b(t)$, $p(t)$, $h(t)$, and $s(t)$ that solve the system.

By employing iterations on this procedure, the model can be solved taking $b(1)$ as given. The method is simply to search over solutions of the type described in the previous paragraph, varying $b(T)$ until the specified initial value of $b(1)$ is found. In this way, a set of equilibria with different θ 's can be calculated, each one of which starts from the same value of $b(1)$. In a similar fashion, equilibria can be generated with different θ 's, each one of which starts from the same value of $H(1)$. [Of course, $b(1)$ will then differ across the different θ 's.] This last procedure was the one used to generate the examples in the paper, each of which started with $H(1) = 164.65$.

We now describe the results of using this solution procedure to compute the equilibria of an economy with the parameters $\{\gamma_1, \gamma_2, N(0), d(t), \bar{B}(0), H(0), T, b(1)\}$ under different monetary policies, that is, different values of θ . Since the values of θ are different, the values of the economy's endogenous variables $\{p(t); t \geq 1\}$ and $\{b(t); t \geq 2\}$ will, in general, be different.

Table B2 compares two very different monetary policies in a particular economy. Under both policies, the economy has $\gamma_1 = 3.0, \gamma_2 = 2.5, N(0) = 1,000, n = .02, d(t) = .05$ for $1 \leq t \leq T$, $d(t) = d = 0$ for $t > T, \bar{B}(0) = 100, H(0) = 100, T = 10, b(1) = .08109$, and $R = .05$. The tight money policy is $\theta = .106$, while the loose money policy is $\theta = .120$. As can be seen from the table, for all $t \geq 1$, the tight money policy produces a uniformly higher inflation rate than the loose money policy. Note that, as expected, the loose money policy is associated with a slower rate of bond creation from $t = 1$ to $t = 10$ and that therefore that policy ends up permitting slower growth in base money from T on than does the tight money policy. Thus, tighter money now implies looser money later, as in the economy described in the first section of the paper.

In the present example, however, the effect of expected future rates of money creation on the current rate of inflation is

Table B3
An Intermediate Example of the Potential Effects
of Tight and Loose Monetary Policy

Tight Money: $\theta = .01$ Loose Money: $\theta = .03$

Date (t)	Inflation Rate $[p(t+1)/p(t)]$		PerCapita Bond Holdings $[B(t)/N(t)]$		Per Capita Real Money Balances $\{H(t)/[N(t)p(t)]\}$	
	Tight	Loose	Tight	Loose	Tight	Loose
1	1.0043	1.0192	0.1500	0.1500	0.2468	0.2356
2	1.0089	1.0221	0.2020	0.1976	0.2433	0.2335
3	1.0150	1.0258	0.2556	0.2467	0.2388	0.2307
4	1.0227	1.0306	0.3108	0.2973	0.2330	0.2249
5	1.0326	1.0367	0.3677	0.3496	0.2256	0.2225
6	1.0449	1.0444	0.4264	0.4036	0.2163	0.2167
7	1.0601	1.0539	0.4869	0.4594	0.2030	0.2096
8	1.0781	1.0656	0.5493	0.5170	0.1915	0.2008
9	1.0989	1.0796	0.6137	0.5767	0.1759	0.1903
≥ 10	1.1221	1.0960	0.6802	0.6385	0.1585	0.1780

Parameters

$$\begin{array}{llll} \gamma_1 = 2.0 & R = .05 & d(t) = \begin{cases} .05 & \text{for } t = 1, 2, \dots, 10. \\ 0 & \text{for } t > 10. \end{cases} & \begin{array}{l} H(0) = 100 \\ \bar{B}(0) = 100 \end{array} \\ \gamma_2 = 1.5 & n = .02 & & \begin{array}{l} N(0) = 1,000 \\ b(1) = 1.4999 \end{array} \end{array}$$

sufficiently strong that tighter money initially produces higher inflation in both the present and the future. This happens because, via equation (B18), the higher eventual rate of money creation associated with the lower path more than offsets the downward effects on the initial inflation rates that are directly associated with the lower initial rate of money creation. Like the closely related example in the paper, this comparison provides a spectacular example in which tighter money now fails to buy even a temporarily lower inflation rate than does looser money now.

Table B3 compares different θ 's in an economy that provides an intermediate example, one between the paper's first section economy and the later spectacular examples. This economy maintains the parameters $\gamma_1 = 2.0$, $\gamma_2 = 1.5$, $N(0) = 1,000$, $n = .02$, $d(t) = .05$ for $1 \leq t \leq T$, $\bar{d}(t) = d = 0$ for $t > T$, $\bar{B}(0) = 100$, $H(0) = 100$, $T = 10$, $b(1) = 1.4999$, and $R = .05$. Here the tight money policy is $\theta = .01$, while the loose money policy is $\theta = .03$. Under tight money, the economy experiences a lower inflation rate for $1 \leq t \leq 5$, but a higher rate for $t \geq 5$. [Here the gross inflation rate at t is defined as the right-hand rate $p(t+1)/p(t)$.] In this case, the effect of the higher eventual rate of money creation that is associated with the initially tighter policy

causes inflation to be higher even before T , when money actually becomes looser. But this effect is not strong enough to eliminate completely the temporary benefits of tight money on the current inflation. Still, notice that, compared to the paper's first section example, the effect of the initial tight money on the initial inflation rate is considerably weakened. With all other parameters the same, but $\gamma_2 = 0$ (the first section case), we would have had $p(t+1)/p(t) = (1+\theta)/(1+n) = .9902$ for $1 \leq t \leq T$.

Appendix C

Sufficient Conditions for Tighter Money Now to Cause Higher Inflation Now

This appendix* establishes sufficient conditions for the case where a tighter monetary policy (lower θ) leads to a uniformly higher price level and inflation rate for all $t \geq 1$. The method is by construction: a pair of inequalities will be reduced to a single relation by the correct choice of certain parameter values. We satisfy the inequalities by making the implicit discount rate $[1 - (\gamma_2/\gamma_1)]$ sufficiently low, while maintaining convergence of the relevant infinite sum.

Let θ_h and θ_l denote a higher and a lower monetary growth policy, respectively; that is, $\theta_h > \theta_l$. Then we want both¹

$$(C1) \quad p_l(\theta_l) > p_h(\theta_h)$$

and

$$(C2) \quad p_{t+1}(\theta_l)/p_t(\theta_l) > p_{t+1}(\theta_h)/p_t(\theta_h)$$

for all t . By (B15) and (B16) in Table B1, for $t \geq T$, $p_{t+1}(\theta)/p_t(\theta) = \pi_t(\theta)$. For policy experiments that fix b_1 , it is clear that (over the relevant range) a lower θ leads to a higher b_T and hence to a higher ξ . This is exactly the statement that a tighter monetary policy now implies a higher deficit to be financed by seignorage from time $T+1$ on. From the graph of (B5) in Appendix B, it is clear that an increase in ξ increases the value of the root π_t . Therefore, $\pi_t(\theta_l) > \pi_t(\theta_h)$. Hence, condition (C2) is satisfied for $t \geq T$. Condition (C1) follows, at most, T' periods after T (where T' is finite), given (C2) for $t \geq T$.

Hence, we restrict attention to $t < T$. It is clear that, if (C2) holds for $t < T$, then $p_t(\theta_l) > p_t(\theta_h)$ implies (C1) for $t < T$ and therefore for all t . From (B26),

$$p_t(\theta) = [(\bar{B}_0 + H_0)/N_1]/(b_1 - d_1 + \{H_1(\theta)/[N_1 p_t(\theta)]\}).$$

But by (B23),

$$\begin{aligned} H_1(\theta)/[N_1 p_t(\theta)] &= h_1(\theta)/p_t(\theta) \\ &= (\gamma_1/2)[1 - \phi\pi_t(\theta)][1 - \phi\mu(\theta)]/[1 - \phi\pi_t(\theta) \\ &\quad + [\pi_t(\theta) - \mu(\theta)]\phi^T \mu(\theta)^{T-1}]. \end{aligned}$$

Calling this $m_t(\theta)$, $p_t(\theta) = k_1/[k_2 + m_t(\theta)]$, where $k_1 \equiv (\bar{B}_0 + H_0)/N_1$ and $k_2 \equiv b_1 - d_1$. Clearly, $k_1 > 0$. Then $k_2 + m_t(\theta) > 0$ for positive $p_t(\theta)$. Then $p_t(\theta_l) > p_t(\theta_h)$ if and only if $m_t(\theta_l) > m_t(\theta_h)$.

Define the function

$$\begin{aligned} \Gamma(\phi, \theta, t) &= 1 - \phi\pi_t(\theta) \\ &\quad + [\pi_t(\theta) - \mu(\theta)]\phi^{T-t+1}\mu(\theta)^{T-t}. \end{aligned}$$

Then, using (B19), (B20), and (B23) to write out explicitly $p_{t+1}(\theta)/p_t(\theta)$ and the above characterization of the price level condition, (C1) and (C2) for $t < T$ are equivalent to

$$(C3) \quad [1 - \phi\pi_t(\theta_h)][1 - \phi\mu(\theta_h)]/\Gamma(\phi, \theta_h, 1) > [1 - \phi\pi_t(\theta_l)][1 - \phi\mu(\theta_l)]/\Gamma(\phi, \theta_l, 1)$$

and

$$(C4) \quad (1 + \theta_l)[\Gamma(\phi, \theta_l, t+1)/\Gamma(\phi, \theta_l, t)] > (1 + \theta_h)[\Gamma(\phi, \theta_h, t+1)/\Gamma(\phi, \theta_h, t)]$$

for $t > T$. We need to choose $\phi = (\gamma_2/\gamma_1)$, θ_l, θ_h that satisfy (C3) and (C4) and support positive values for nominal balances, prices, and bond holdings and real values for π_1 and π_2 .

Recall that, given b_1, b_T can be found if π_1 is known. But π_1 is a function of b_T . The only case where π_1 is determined independently of b_T is $\pi_1 = \pi_2 = [\phi(1+n)]^{-1/2}$, as is easily seen by comparing (B12) and (B13). This occurs at the maximum value of ξ that yields real roots for the characteristic polynomial in (B13). Using this, we pick a θ_l to simplify (C3) and (C4). Conditions on parameter values that satisfy these two inequalities will then become transparent.

Let θ_l solve $\mu(\theta_l) = \pi_1(\theta_l) = [\phi(1+n)]^{-1/2}$. Since $\mu(\theta_l) = (1 + \theta_l)/(1 + n)$, this gives $\theta_l = (1 + n)^{1/2}\phi^{-1/2}$. Choosing $\pi_1 = \pi_2 = [\phi(1+n)]^{-1/2}$ implies a value for ξ (and hence for b_T) by comparing (B12) and (B13). Then fixing θ_l determines b_1 by recursively solving (B24) and (B25) backwards. This value of b_1 is kept constant across policy experiments (different θ settings).

Choosing $\mu(\theta_l) = \pi_1(\theta_l)$ simplifies (C3) and (C4) to

$$(C5) \quad [1 - \phi\pi_t(\theta_h)][1 - \phi\mu(\theta_h)]/\Gamma(\phi, \theta_h, 1) > 1 - \phi\mu(\theta_l)$$

and

$$(C6) \quad 1 + \theta_l > (1 + \theta_h)[\Gamma(\phi, \theta_h, t+1)/\Gamma(\phi, \theta_h, t)]$$

*This appendix was written by Danny Quah, a graduate student at Harvard University.

¹For simplicity with the θ_h and θ_l notation, time is indicated by a subscript in this appendix, rather than parenthetically, as in the paper and other appendices.

for $t \leq T-1$. It will be shown below that we want to set $\phi \cong [\pi_1(\theta_h)]^{-1}$. Then $\Gamma(\phi, \theta_h, t+1)/\Gamma(\phi, \theta_h, t) \cong [\phi\mu(\theta_h)]^{-1}$, so that the right-hand side of (C6) is approximately t -independent. Therefore, consider (C6) for $t=1$, and rewrite (C5):

$$(C7) \quad [1 - \phi\mu(\theta_h)]^{-1} > \Gamma(\phi, \theta_h, 1)/[1 - \phi\pi_1(\theta_h)][1 - \phi\mu(\theta_h)]$$

$$(C8) \quad 1 + \theta_l > (1 + \theta_h)[\Gamma(\phi, \theta_h, 2)/\Gamma(\phi, \theta_h, 1)].$$

Maintain $[1 - \phi\mu(\theta_h)]$ and $[1 - \phi\pi_1(\theta_h)]$ positive; multiply the left- and right-hand sides of (C7) by the corresponding sides of (C8) to get, after some manipulation,²

$$(C9) \quad [(1 + \theta_l)/(1 + \theta_h)][[1 - \phi\mu(\theta_h)]/[1 - \phi\mu(\theta_l)]] > \Gamma(\phi, \theta_h, 2)/[1 - \phi\pi_1(\theta_h)].$$

The left-hand side of (C9) is the product of two terms, each of which is easily seen to be slightly less than unity for small $(\theta_h - \theta_l) > 0$. Therefore, the left-hand side of (C9) is $(1 - \varepsilon)$ for small $\varepsilon > 0$.

Write the right-hand side of (C9) as

$$1 + \{[\pi_1(\theta_h) - \mu(\theta_h)]\phi^{T-t+1}\mu(\theta_h)^{T-t}/[1 - \phi\pi_1(\theta_h)]\} = 1 + \delta.$$

By the choice of θ_h , $\pi_1(\theta_h) = \mu(\theta_l)$. Therefore, $\theta_h > \theta_l$ implies $\pi_1(\theta_h) < \mu(\theta_h)$. Hence, $\delta < 0$ and can be made arbitrarily large in magnitude when ϕ approaches arbitrarily close to $\pi_1(\theta_h)^{-1}$ from below. This will satisfy condition (C9).

The condition for real and positive π_1 given b_T (for $d=0$) is

$$b_T < (\gamma_2/2)[(1+n)/(R-n)]((1/\phi) + [1/(1+n)] - \{2/[\phi(1+n)]^{1/2}\}).$$

Values for b_T that are too low will imply negative b_1 . To guarantee strictly positive b_1 , set b_T as high as desired by increasing both γ_2 and γ_1 , keeping $\phi = \gamma_2/\gamma_1$ at the chosen value.

In recapitulation, the method involves carrying out the steps above in reverse order. Choose γ_2/γ_1 sufficiently close to 1; set γ_2 so that maximum b_T appears high enough. Calculate θ_l and work backwards from b_T to b_1 . Then, using this value of b_1 , set θ_h so that $(\theta_h - \theta_l)$ is small and positive.

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²This procedure almost always obtains the desired example. I say "almost" because, strictly speaking, (C7) and (C8) imply (C9), but the converse is not true.

The views expressed herein are those of the author and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.