# On the Numerical Solution of Black-Scholes Equation

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**Abstract:** *A novel adaptive radial basis function scheme based on the radial basis function methods is presented for the numerical solution of the Black-Scholes equation, which has been used extensively for the evaluation of European and American options. The accurate and efficient solution of this equation is very important and has remained as a long standing problem in financial engineering. We apply our novel approach to plain vanilla for the European case and compare the numerical solution of the Black-Scholes equation with the results of different numerical methods. It is shown that the new approach achieves a major improvement on all the previous numerical calculations for the solution of the Black-Scholes equation.*

*Keywords:* Meshless Methods, Black-Scholes Equation, European Options, Thin Plate Radial Basis Functions, Multiquadrics, Partial Differential Equation.

## 1 Introduction

Since 1973, in which Black and Scholes proposed an explicit formula for evaluating European call options without dividends, the formula of Black-Scholes is still extensively used with underlying. The numerical solution of this equation has been of paramount interest due to the governing partial differential equation, which is very difficult to generate stable and accurate solutions. The problem is due to the discontinuity of the function around the exercise price. That is to say: To implement a numerical valuation scheme, the underlying asset must be approximated. Although this can be done in a more or less straightforward way such that the option prices converge to the true value as we increase the refinement, it is by no means straightforward to find an approximation of the asset price such that the corresponding option prices have good convergence properties. As stated above, the slow convergence is a consequence of the discontinuity of the payoff function, and it has been known for some time that if we apply the usual procedure of discretizing the time and space axis, the option prices will converge, with increasing refinement [1, 2].

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Traditional numerical methods such as finite difference, finite element and binominal ones which are derived from local interpolation schemes and require a mesh to support the application, attempt to solve this equation using finite quotients. Then, they lead to intensive computation with hundreds of equations. In order to apply to financial market, where a large amount of data change dynamically, traditional approaches need to be improved to provide instantaneous accurate solution. However, these traditional methods have numerical problems of oscillations and damping.

On the other hand, Radial basis function known as a mesh-free method which aims to eliminate the structure of the mesh and approximate the solution using a set of quasi-random points rather than points from a grid discritization. This makes RBFs independent of the dimension of the problem. The proposed RBFs method provides an interpolation formula not only for the solution but also for its derivatives. Since the interpolation formulae arise from the RBFs approximation is globally defined in the computational domain, the computations of those important indicators like Delta values can be obtained as a bonus without a need to use extra interpolation technique.

In the next section, we introduce Black-Scholes model for option pricing then section 3 is devoted to a brief description of numerical methods for the solution of this equation. Section 4 shows our results and finally in section 5, we give our conclusion.

## 2 Black-Scholes Equation

The Black Scholes equation for the evaluation of an option price is stated as;

$$\frac{\partial}{\partial t}V(S,t)+\frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial S^2}V(S,t)+rS\frac{\partial}{\partial S}V(S,t)-rV(S,t)=0 \tag{1}$$

where r is the risk-free interest rate, $\sigma$ is the volatility, and $V(S,t)$ is the option price at time t and stock value $S$ where $S\in[0,\infty)$ and $t\in[0,T]$ respectively with T denoting the terminal expiry time of the option.

The Black-Scholes equation is a backward equation, meaning that the signs of the t derivatives and the second S derivative in the equation are the same when written on the same side of the equals sign. We therefore have to impose initial and final conditions which tell us how the solution must behave for all time at certain values of asset. The initial and boundary conditions are defined as:

$$V(S,T)=\begin{cases}\max(E-S,0) & \text{for Put}\\ \max(S-E,0) & \text{for Call}\end{cases} \tag{2}$$

where $E$ is the exercise price of the option.

## 3 Radial Basis Function Method

The radial basis functions we used in this paper are defined as following:

$$\begin{aligned}\text{TPS:}\quad \phi(\mathbf{x},\mathbf{x}_j) &= \phi(r_j)=r_j^4 log(r_j),\\ \text{MQ:}\quad \phi(\mathbf{x},\mathbf{x}_j) &= \phi(r_j)=\sqrt{c^2+r_j^2},\end{aligned}$$

$$\begin{aligned} \text{Cubic:} \quad \phi(\mathbf{x},\mathbf{x}_j) &= \phi(r_j) = r_j^3, \\ \text{Gaussian} \quad \phi(\mathbf{x},\mathbf{x}_j) &= \phi(r_j) = e^{-c^2 r_j^2} \end{aligned} \tag{3}$$

where $r_j = \|\mathbf{x} - \mathbf{x}_j\|$ is the Euclidean norm.

The approximation of the function $V(S,t)$ in Equation 1, using RBF, may be written as a linear combination of N radial functions:

$$V(S,t) \simeq \sum_{j=1}^{N} \lambda_j(t)\,\phi(S,S_j) \qquad \text{for } S \in \Omega \subset \mathbf{R}^d \tag{4}$$

where N is the number of data points, $\lambda$'s are the coefficients to be determined and $\phi$ is the radial basis function.

Equation 1 is discretized using the Crank-Nicholson ($\theta$-weighted) method. Note that it is backward integration in time.

$$\begin{aligned} V(S,t) - V(S,t+\delta t) + \delta t(1-\theta)\left[\frac{1}{2}\sigma^2 S^2 \nabla^2 V(S,t) + rS\nabla V(S,t) - rV(S,t)\right] \\ +\delta t\theta\left[\frac{1}{2}\sigma^2 S^2 \nabla^2 V(S,t+\delta t) + rS\nabla V(S,t+\delta t) - rV(S,t+\delta t)\right] = 0 \end{aligned} \tag{5}$$

where $0 \le \theta \le 1$. For implicit Crank-Nicholson scheme $\theta = 0.5$ and $\delta t$ is the time step size that is discritezed as $t_{final}/m$, where $m$ is the number of time steps. Using the notation, $V^n = V(S,t^n)$ where $t^n = t^{n-1} + \triangle t$, Equation 5 can be written as

$$\left[1 - \alpha\left(\frac{1}{2}\sigma^2 S^2 \nabla^2 + rS\nabla - r\right)\right] V^{n+1} = \left[1 + \beta\left(\frac{1}{2}\sigma^2 S^2 \nabla^2 - rS\nabla + r\right)\right] V^n \tag{6}$$

where $\alpha = \theta\triangle t$ and $\beta = (1-\theta)\triangle t$. We define new operators $H_+$ and $H_-$ by

$$H_+ = 1 - \alpha\left(\frac{1}{2}\sigma^2 S^2 \nabla^2 + rS\nabla - r\right), H_- = 1 + \beta\left(\frac{1}{2}\sigma^2 S^2 \nabla^2 - rS\nabla + r\right) \tag{7}$$

The operators $H_+$ and $H_-$ are applied to the approximation (4), yielding:

$$\sum_{j=1}^{N} \lambda_j^{n+1} H_+ \phi(S_{ij}) = \sum_{j=1}^{N} \lambda_j^n H_- \phi(S_{ij}) \tag{8}$$

Equation 8 generates a system of linear equation, which is solved using the Gaussian elimination with the partial pivoting method to obtain the unknowns, $\lambda_j^{n+1}$, from the known values of $\lambda_j^n$ at a previous time step and then they are transformed to the $V(S,t)$ by equation 4.

# 4 Results

For the purpose of numerical comparisons, we consider an European put option with $E$=10, $r$=0.05, $\sigma$=0.20, and $T$=0.5 (year). We assume the spatial domain as $[0,30]$ to approximate the semi-infinite domain $[0,\infty)$.

For a fixed time, $n_t$=100, we present the relative performance of the TPS, Cubic, Gaussian and MQ radial basis functions results for the asset and delta values in tables 1 and 2 for $n$=121. It can be perceived that TPS and MQ-RBFs are superior in comparison with the cubic and Gaussian ones. Table 3 and 4 show the variation of the error with the number of nodes. The calculations converge very fast to the true value as the number of nodes increase, but the Gaussian and Cubic radial basis functions still generate large errors in comparison with the TPS and MQ-RBF methods.

The relative error, which is given in terms of the analytical and RBF numerical solutions, shown in these tables are defined as

$$\varepsilon(t) = [1/(n-1)] \sum_{j=1}^{n} |V(S_j,t)_{RBF} - V(S_j,t)_{analytical}| \tag{9}$$

# 5 SUMMARY AND CONCLUSIONS

We have shown the numerical solution of the Black-Scholes equation using globally defined radial basis functions. The RBFs scheme is a truly meshless computational method which does not require the generation of a regular grid as in the finite difference or a mesh as in the finite element methods. This makes the RBFs particularly efficient in solving this kind of problems. As it can be seen from the results that the TPS and MQ-RBFs generate excellent results in comparison with the mesh dependent methods and meshfress Cubic and Gaussian radial basis functions methods.

# References

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Table 1: Using the parameters given in the text, the comparison of Radial Basis Functions for $n_t$=100, $n$=121

| Stock S | TPS | MQ | CUBIC | GAUSSIAN |
|---|---|---|---|---|
| 0.00 | 9.7531 | 9.7531 | 9.7531 | 9.7531 |
| 2.00 | 7.7531 | 7.7531 | 7.8423 | 7.7531 |
| 4.00 | 5.7531 | 5.7531 | 5.9158 | 5.7533 |
| 6.00 | 3.7532 | 3.7532 | 3.9760 | 3.7533 |
| 8.00 | 1.7983 | 1.7983 | 2.0252 | 1.7996 |
| 10.00 | 0.4409 | 0.4408 | 0.0655 | 0.4411 |
| 12.00 | 0.0479 | 0.0479 | 0.0500 | 0.0487 |
| 14.00 | 0.0027 | 0.0027 | 0.0302 | 0.0037 |
| 16.00 | 0.0001 | 0.0001 | 0.0086 | 0.0008 |
| 18.00 | 0.0000 | 0.0000 | -0.0001 | 0.0006 |
| ε | 0.00013971 | 0.00013637 | 0.06190414 | 0.00464602 |

Table 2: Same with Table 1 but for Delta value.

| Stock S | TPS | MQ | CUBIC | GAUSSIAN |
|---|---|---|---|---|
| 0.00 | -0.9995 | -1.0000 | 0.9559 | -0.7103 |
| 2.00 | -1.0000 | -1.0000 | 0.8351 | -0.9991 |
| 4.00 | -1.0000 | -1.0000 | 0.7224 | -1.0003 |
| 6.00 | -0.9996 | -0.9996 | 0.6178 | -0.9989 |
| 8.00 | -0.9088 | -0.9089 | 0.5215 | -0.9090 |
| 10.00 | -0.4022 | -0.4021 | 0.4333 | -0.4023 |
| 12.00 | -0.0618 | -0.0618 | 0.3532 | -0.0615 |
| 14.00 | -0.0042 | -0.0043 | 0.2814 | -0.0043 |
| 16.00 | -0.0002 | -0.0002 | 0.2177 | -0.0003 |
| 18.00 | 0.0000 | 0.0000 | 0.1622 | 0.0000 |
| ε | 0.00008954 | 0.00017647 | 0.63676377 | 0.00379306 |

Table 3: The variation of relative error for different value of the number of nodes for a fixed time, $n_t$=100

| Node (n) | TPS | MQ | CUBIC | GAUSSIAN |
|---|---|---|---|---|
| 40 | 0.00130294 | 0.00130526 | 0.06138752 | 7.71100769 |
| 41 | 0.00065001 | 0.00053212 | 0.06006126 | 7.70508390 |
| 50 | 0.00042345 | 0.00035310 | 0.05988777 | 7.10907512 |
| 51 | 0.00050754 | 0.00042699 | 0.06069781 | 6.85233992 |
| 70 | 0.00041497 | 0.00040278 | 0.06170775 | 0.00326177 |
| 71 | 0.00020256 | 0.00017067 | 0.06126799 | 0.00267421 |
| 100 | 0.00020239 | 0.00019625 | 0.06191282 | 0.00020160 |
| 101 | 0.00009659 | 0.00008336 | 0.06158383 | 0.00008586 |
| 120 | 0.00008250 | 0.00008250 | 0.06173100 | 0.00099817 |
| 121 | 0.00013971 | 0.00013637 | 0.06190414 | 0.00464602 |

Table 4: Same with Table 3 but for Delta

| Node (n) | TPS | MQ | CUBIC | GAUSSIAN |
|---|---|---|---|---|
| 40 | 0.00094501 | 0.00100075 | 0.60947830 | 0.31948619 |
| 41 | 0.00059034 | 0.00048427 | 0.61047430 | 0.32907185 |
| 50 | 0.00036618 | 0.00036037 | 0.61763871 | 0.59364399 |
| 51 | 0.00039951 | 0.00038904 | 0.61827842 | 0.63489574 |
| 70 | 0.00028038 | 0.00037626 | 0.62695586 | 0.13775177 |
| 71 | 0.00016520 | 0.00022685 | 0.62728377 | 0.13044955 |
| 100 | 0.00013199 | 0.00022412 | 0.63393781 | 0.00867712 |
| 101 | 0.00007368 | 0.00015430 | 0.63409905 | 0.00755410 |
| 120 | 0.00005907 | 0.00005907 | 0.63665164 | 0.00058427 |
| 121 | 0.00008954 | 0.00017647 | 0.63676377 | 0.00379306 |