

A weaker constraint qualification and a second order necessary optimality condition for mathematical programming*

Yiran He[†] and Jie Sun[‡]

Abstract. We discuss a new constraint qualification and establish a second order necessary optimality condition of the Bonnans-Shapiro type for mathematical programming problems. This qualification condition is weaker than Robinson's constraint qualification. An example is presented to show the usefulness of the result.

Key words: optimality conditions, nonlinear programming, tangent cones

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[†]Department of Decision Sciences, National University of Singapore, Republic of Singapore and Department of Mathematics, Sichuan Normal University, Chengdu, Sichuan, P.R. China. Email: yiranhe@hotmail.com

[‡]Department of Decision Sciences and Singapore-MIT Alliance, National University of Singapore, Republic of Singapore. Fax (65)6779-2621. Email: jsun@nus.edu.sg

1 Introduction

We derive a new second order necessary optimality condition for mathematical programming problems. As is well known, the necessary optimality condition is valid only if a certain constraint qualification (CQ) is satisfied, or as is often called in the literature, only if the problem is “nondegenerate”. Examples of CQ include the Mangasarian-Fromovitz constraint qualification, the Robinson constraint qualification, and the linear independence constraint qualification, etc. However, many practical problems are “degenerate” problems in some sense. It is therefore natural to look for new second order optimality conditions under weaker constraint qualifications.

We start from considering problems where the feasible region is an abstract closed set, then by using second order tangency formulae, we deduce a more explicit result for the case where the feasible region is defined by nonlinear equalities and inequalities. This approach can help to understand the roles of various constraint qualifications in the establishment of second order necessary optimality conditions.

2 A second order necessary condition for optimality

Let $f : \mathfrak{R}^n \rightarrow \mathfrak{R}$ be twice continuously differentiable and let K be a closed (not necessarily convex) subset of \mathfrak{R}^n . Consider

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to} \quad & x \in K. \end{aligned} \tag{1}$$

It is known that if $x_0 \in K$ is a locally optimal solution to (1), then the first order necessary optimality conditions can be written as $Df(x_0)h \geq 0$ for all $h \in T_K(x_0)$, where $T_K(x_0)$ denotes the (Bouligand-) contingent cone of K at x_0 . We derive a second order optimality condition for problem (1). All basic concepts and notations can be found in Bonnans and Shapiro [4]. For a linear operator A , $\ker(A)$ denotes the kernel of A . D and D^2 denote respectively the first and second order derivatives of a function. For a nonempty set $K \subset \mathfrak{R}^n$, $\sigma_K(x) := \sup_{y \in K} \langle x, y \rangle$ denotes its support function; for $x_0 \in K$ and $h \in \mathfrak{R}^n$, $T_K^2(x_0, h)$ denotes the outer second order tangent set at x_0 in the direction h :

$$T_K^2(x_0, h) := \{w \in \mathfrak{R}^n : \text{dist}(x_0 + t_n h + (t_n^2/2)w, K) = o(t_n^2) \text{ for some } t_n \downarrow 0\}.$$

We denote $S(x_0)$ the set $T_K(x_0) \cap \{h : Df(x_0)h \leq 0\}$. For a nonempty set S in X ,

$$S^+ := \{\xi \in X^* : \langle \xi, x \rangle \geq 0, \text{ for all } x \in S\}$$

and $S^- := -S^+$.

Proposition 1. If $x_0 \in K$ is a locally optimal solution to (1), then for every $h \in S(x_0)$,

$$Df(x_0)w + D^2f(x_0)(h, h) \geq 0, \quad \text{for every } w \in T_K^2(x_0, h).$$

Proof. Let $h \in S(x_0)$ and $w \in T_K^2(x_0, h)$. By definition of outer second order tangent set, there exists $t_n \downarrow 0$ such that $\text{dist}(x_0 + t_n h + (1/2)t_n^2 w, K) = o(t_n^2)$. Therefore there exists $r(t_n) = o(t_n^2)$ such that $x_n := x_0 + t_n h + (1/2)t_n^2 w + r(t_n) \in K$. Since f is twice continuously differentiable, the second order Taylor expansion yields that

$$\begin{aligned} f(x_n) &= f(x_0) + t_n Df(x_0)h + (1/2)t_n^2 [Df(x_0)w + D^2f(x_0)(h, h)] + o(t_n^2) \\ &\leq f(x_0) + (1/2)t_n^2 [Df(x_0)w + D^2f(x_0)(h, h)] + o(t_n^2), \end{aligned}$$

where the inequality follows from that $h \in S(x_0)$ and hence $Df(x_0)h \leq 0$. Since x_0 is a locally optimal solution and $x_n \in K$, we have $f(x_0) \leq f(x_n)$ for sufficiently large n and hence $Df(x_0)w + D^2f(x_0)(h, h) \geq 0$. \square

Corollary 2. Let x_0 be a locally optimal solution to (1). Then for every $h \in S(x_0)$ and any convex set $T(h) \subset T_K^2(x_0, h)$

$$D^2f(x_0)(h, h) - \sigma_{T(h)}(-Df(x_0)) \geq 0;$$

and hence $\sigma_{T(h)}(-Df(x_0)) < \infty$.

Proof.

$$\begin{aligned} 0 &\leq D^2f(x_0)(h, h) + \min\{Df(x_0)w : w \in T_K^2(x_0, h)\} \\ &= D^2f(x_0)(h, h) - \max\{-Df(x_0)w : w \in T_K^2(x_0, h)\} \\ &= D^2f(x_0)(h, h) - \sigma_{T_K^2(x_0, h)}(-Df(x_0)) \\ &\leq D^2f(x_0)(h, h) - \sigma_{T(h)}(-Df(x_0)). \end{aligned}$$

This completes the proof. \square

Remark. Corollary 2 is a special case of [4, Theorem 3.45] with $G(x) = x$. We give a direct and simplified proof here. Note that we have not used any assumptions on constraint qualification so far.

Now we consider minimization problems with explicit constraints:

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to} \quad & G(x) \in C, \end{aligned} \tag{2}$$

where $f : \mathfrak{R}^n \rightarrow \mathfrak{R}$ and $G : \mathfrak{R}^n \rightarrow \mathfrak{R}^m$ are twice continuously differentiable and C is a closed convex set in \mathfrak{R}^m . Let $K = G^{-1}(C)$. Then K is a closed set and problem (2) is a special case of problem (1).

Proposition 3. Let $K = G^{-1}(C)$. For every $x_0 \in K$ and every $h \in Y$ with $DG(x_0)h \in T_C(G(x_0))$,

$$T_K^2(x_0, h) \subset \{w : D^2G(x_0)(h, h) + DG(x_0)w \in T_C^2(G(x_0), DG(x_0)h)\}. \tag{3}$$

Proof. Let $w \in T_K^2(x_0, h)$. Then there exist sequences $\{t_n\} \rightarrow 0+$ and $\{y_n\} \subset K$ such that the sequence $x_n := x_0 + t_n h + (1/2)t_n^2 w$ satisfying $y_n = x_n + o(t_n^2)$. This implies that $G(y_n) = G(x_n) + o(t_n^2)$ as the mapping G is continuously differentiable and hence locally Lipschitz at x_0 . By the second order Taylor expansion,

$$G(x_n) = G(x_0) + t_n DG(x_0)h + (1/2)t_n^2 [DG(x_0)w + D^2G(x_0)(h, h)] + o(t_n^2).$$

It follows that $DG(x_0)w + D^2G(x_0)(h, h) \in T_C^2(G(x_0), DG(x_0)h)$. \square

By virtue of Proposition 1, if x_0 is a locally optimal solution to the problem (2), then for $K = G^{-1}(C)$ and for every $h \in S(x_0)$,

$$Df(x_0)w + D^2f(x_0)(h, h) \geq 0, \quad \text{for all } w \in T_K^2(x_0, h). \tag{4}$$

Though (3) provides an upper estimate (in the sense of inclusion relation) of $T_K^2(x_0, h)$, this does not help us to get further significant reformulation of the condition (4) in terms of the second order derivative of the mapping G and the second order tangent cone of the set C . In order to obtain a significant second order optimality condition, it is enough to require that (4) become an equality:

$$T_K^2(x_0, h) = \{w : D^2G(x_0)(h, h) + DG(x_0)w \in T_C^2(G(x_0), DG(x_0)h)\}, \quad (5)$$

which we call the *second order Abadie constraint qualification* at x_0 in the direction h . Throughout this paper, when mentioning the second order Abadie CQ, we always assume that the sets in both sides of (5) are nonempty. We use the terminology “second order Abadie CQ” because it is somewhat related to the first order Abadie constraint qualification at x_0 which says

$$T_K(x_0) = \{h : DG(x_0)h \in T_C(G(x_0));$$

see [1, 2] for a special version stated for nonlinear programming. Actually, it can be seen that the first order Abadie constraint qualification at x_0 is just the second order Abadie constraint qualification at x_0 in the direction $h = 0$. Therefore, if the second order Abadie constraint qualification at x_0 holds in the direction $h = 0$, then the set $S(x_0)$ becomes

$$\{v : DG(x_0)v \in T_C(G(x_0)) \text{ and } Df(x_0)v \leq 0\} =: C(x_0),$$

which is the critical cone of the problem (2).

Another important reason that makes us assume the equality (5) as a constraint qualification is the following fact: When *Robinson’s constraint qualification* holds at x_0 , that is,

$$0 \in \text{int}(G(x_0) + DG(x_0)\mathbb{R}^m - C), \quad (6)$$

the equality (5) holds naturally; see [4].

In the following, we let

$$\Lambda(x_0) := \{\lambda \in \mathbb{R}^m : Df(x_0) + DG(x_0)^*\lambda = 0, \lambda \in N_C(G(x_0))\}$$

denote the set of Lagrange multipliers of the problem (2).

Theorem 4. Let x_0 be a locally optimal solution to (2) and $h \in C(x_0)$. Assume that $\Lambda(x_0)$ is nonempty and the second order Abadie CQ holds at x_0 in the direction h . Then for any convex set $\mathcal{T} \subset T_C^2(G(x_0), DG(x_0)h)$,

$$\limsup_{y \rightarrow -Df(x_0)} \sup_{\lambda \in \Lambda(x_0, y)} \{D^2f(x_0)(h, h) + \langle \lambda, D^2G(x_0)(h, h) \rangle - \sigma_{\mathcal{T}}(\lambda)\} \geq 0,$$

where $\Lambda(x_0, y) := \{\lambda \in N_C(G(x_0)) : DG(x_0)^*\lambda = -y\}$.

Proof. Let $\mathcal{T} \subset T_C^2(G(x_0), DG(x_0)h)$ be a convex set and let

$$M := \mathcal{T} + T_C(G(x_0)) - D^2G(x_0)(h, h). \quad (7)$$

By [4, (3.63)], $\mathcal{T} + T_C(G(x_0))$ is contained in $T_C^2(G(x_0), DG(x_0)h)$. It follows that $DG(x_0)^{-1}(M)$ is a convex set and is contained in $T_K^2(x_0, h)$, where the latter is from the second order Abadie CQ at x_0 in the direction h . Since $h \in C(x_0)$, in view of Corollary 2, there holds

$$D^2f(x_0)(h, h) - \sigma_{DG(x_0)^{-1}(M)}(-Df(x_0)) \geq 0. \quad (8)$$

By virtue of [7, Corollary 16.3.1], $\sigma_{DG(x_0)^{-1}(M)}(\cdot)$ is the closure of the function

$$g(x) := (DG(x_0)^* \sigma_M)(x); \quad (9)$$

the latter is by definition [7, P.38.] equal to $\inf \{\sigma_M(\lambda) : DG(x_0)^* \lambda = x\}$. It follows from (7) that

$$\begin{aligned} g(x) &= \inf \{\sigma_M(\lambda) : DG(x_0)^* \lambda = x\} \\ &= \inf \{\sigma_{\mathcal{T}}(\lambda) + \sigma_{T_C(G(x_0))}(\lambda) - \langle \lambda, D^2G(x_0)(h, h) \rangle : DG(x_0)^* \lambda = x\} \\ &= \inf \{\sigma_{\mathcal{T}}(\lambda) - \langle \lambda, D^2G(x_0)(h, h) \rangle : DG(x_0)^* \lambda = x, \lambda \in N_C(G(x_0))\} \\ &= \inf \{\sigma_{\mathcal{T}}(\lambda) - \langle \lambda, D^2G(x_0)(h, h) \rangle : \lambda \in \Lambda(x_0, -x)\}. \end{aligned}$$

Note that the set $\Lambda(x_0, Df(x_0)) = \Lambda(x_0)$ is assumed to be nonempty. It follows that $-Df(x_0)$ is in the domain of g , and hence the domain of g is nonempty. Recalling that [7, p.52] the closure of a proper convex function g is defined to be the function $x \mapsto \liminf_{y \rightarrow x} g(y)$, we have

$$\begin{aligned} \sigma_{DG(x_0)^{-1}(M)}(-Df(x_0)) &= \sigma_{DG(x_0)^{-1}(M)}(-Df(x_0)) = \liminf_{y \rightarrow -Df(x_0)} g(y) \\ &= \liminf_{y \rightarrow -Df(x_0)} \inf_{\lambda \in \Lambda(x_0, y)} \{\sigma_{\mathcal{T}}(\lambda) - \langle \lambda, D^2G(x_0)(h, h) \rangle\}. \end{aligned}$$

This together with (8) yields the conclusion. \square

Throughout the rest of this paper, we assume that the set C in the problem (2) is polyhedral. Thus the indicator function of C

$$\delta_C(x) := \begin{cases} 0 & x \in C, \\ \infty & x \notin C \end{cases}$$

is a piecewise linear-quadratic function [8]. Since C is polyhedral, it follows from [4, (3.63)] that

$$T_C^2(G(x_0), DG(x_0)h) = T_{T_C(G(x_0))}(DG(x_0)h), \quad (10)$$

which we write as $\mathcal{O}(h)$ in the following.

Theorem 5. Let x_0 be a locally optimal solution to (2) and let $h \in C(x_0)$. If the second order Abadie CQ holds at x_0 in the direction h , then $\Lambda(x_0)$ is nonempty and

$$\sup_{\lambda \in \Lambda(x_0)} \{D^2f(x_0)(h, h) + \langle \lambda, D^2G(x_0)(h, h) \rangle\} \geq 0.$$

Proof. Let $h \in C(x_0)$. The formula (10) implies that

$$\mathcal{O}(h)^- \subset N_C(G(x_0)) \quad \text{and} \quad \sigma_{\mathcal{O}(h)}(\lambda) = \begin{cases} 0 & \lambda \in \mathcal{O}(h)^- \\ \infty & \text{otherwise.} \end{cases} \quad (11)$$

Let $T := DG(x_0)^{-1}(M)$, where $M := \mathcal{O}(h) - D^2G(x_0)(h, h)$. Then T is a nonempty convex subset of the second order tangent set $T_K^2(x_0, h)$. Applying Corollary 2, we obtain that

$$\sigma_T(-Df(x_0)) \leq D^2f(x_0)(h, h). \quad (12)$$

Since C is a polyhedral set, we have the sets $\mathcal{O}(h)$, M and hence T are also polyhedral. Since the indicator function $\delta_T(x)$ is equal to $(\delta_M \circ DG(x_0))(x)$ and since δ_M is a proper convex piecewise linear-quadratic as M is a polyhedral, it follows from [8, Corollary 11.33] that

$$\sigma_T(-Df(x_0)) = (\delta_T)^*(-Df(x_0)) = \inf \{ \sigma_M(\lambda) : DG(x_0)^* \lambda = -Df(x_0) \}.$$

This together with (12) implies that

$$\begin{aligned} D^2 f(x_0)(h, h) &\geq \sigma_T(-Df(x_0)) & (13) \\ &= \inf \{ \sigma_M(\lambda) : DG(x_0)^* \lambda = -Df(x_0) \} \\ &= \inf \{ \sigma_{\mathcal{O}(h)}(\lambda) - \langle \lambda, D^2 G(x_0)(h, h) \rangle : DG(x_0)^* \lambda = -Df(x_0) \} \\ &= \inf \{ -\langle \lambda, D^2 G(x_0)(h, h) \rangle : DG(x_0)^* \lambda = -Df(x_0), \lambda \in \mathcal{O}(h)^- \} \\ &\geq \inf \{ -\langle \lambda, D^2 G(x_0)(h, h) \rangle : DG(x_0)^* \lambda = -Df(x_0), \lambda \in N_C(G(x_0)) \} \\ &= \inf \{ -\langle \lambda, D^2 G(x_0)(h, h) \rangle : \lambda \in \Lambda(x_0) \} \\ &= -\sup \{ \langle \lambda, D^2 G(x_0)(h, h) \rangle : \lambda \in \Lambda(x_0) \}, & (14) \end{aligned}$$

where the second equality and the second inequality follow from (11). The assertion of $\Lambda(x_0) \neq \emptyset$ is an immediate consequence of (14) as by convention supremum over empty set is $-\infty$. By virtue of (14), we obtain that

$$\begin{aligned} 0 &\leq D^2 f(x_0)(h, h) - \sigma_T(-Df(x_0)) \\ &\leq \sup_{\lambda \in \Lambda(x_0)} \{ D^2 f(x_0)(h, h) + \langle \lambda, D^2 G(x_0)(h, h) \rangle \}. \end{aligned}$$

The proof is complete. \square

An example

We provide an example to which Theorem 5 is applicable but the classical results in the literature is not.

Example. For $x \in \mathbb{R}^2$, define $f(x) = x_1^2 - x_2$

$$g_i(x) = \begin{cases} -x_1 & i = 1, \\ -x_2 & i = 2, \\ x_2 - x_1^2 & i = 3. \end{cases} \quad (15)$$

Let $K := \{x : g_i(x) \leq 0, i = 1, 2, 3\}$ and let $x_0 = (0, 0)$. Then x_0 minimizes f over K . It can be seen that the active indices $I(x_0) = \{1, 2, 3\}$,

$$T_K(x_0) = \{h : h_1 \geq 0, h_2 = 0\} = \{h : Dg_i(x_0)h \leq 0 \text{ for all } i \in I(x_0)\}, \quad (16)$$

and $\mathcal{I}(x_0, h) = \{2, 3\}$ for $h \in T_K(x_0) \setminus \{0\}$. Since $Df(x_0) = (0, -1)$, we have

$$C(x_0) = \{h : h_1 > 0, h_2 = 0\}.$$

We will show that for every $h \in C(x_0)$,

$$\begin{aligned} T_K^2(x_0, h) &= \{ \omega : Dg_i(x_0)\omega + D^2 g_i(x_0)(h, h) \leq 0 \text{ for all } i \in \mathcal{I}(x_0, h) \} \\ &\equiv \{ \omega : 0 \leq \omega_2 \leq 2h_1^2 \}, & (17) \end{aligned}$$

and hence the second order Abadie CQ holds at x_0 in every direction h in $C(x_0)$. By Proposition 3, it suffices to prove that the right-hand side of the first equality is contained in $T_K^2(x_0, h)$. Assume that $\omega \in \mathbb{R}^2$ be such that $0 \leq \omega_2 \leq 2h_1^2$ and $h = (h_1, 0)$ with $h_1 > 0$. We consider the following limit

$$\lim_{t \rightarrow 0^+} \frac{\text{dist}(th + \frac{1}{2}t^2\omega, K)}{t^2}.$$

It suffices to consider this limit for the case when $th + \frac{1}{2}t^2\omega \notin K$. In this case, for sufficiently small $t > 0$, $th + \frac{1}{2}t^2\omega$ belongs to \mathbb{R}_{++}^2 and is above the curve $g_3(x) = 0$. Let $(u(t), v(t))$ be the projection of $th + \frac{1}{2}t^2\omega$ onto K , which always exists as K is closed. That is to say, $(u(t), v(t)) \in K$ solves the following minimization problem:

$$\begin{aligned} \min \quad & \frac{1}{2}\|x - (th + \frac{1}{2}t^2\omega)\|^2 \\ \text{subject to} \quad & x \in K \equiv \{x : g_i(x) \leq 0, i = 1, 2, 3\}. \end{aligned} \quad (18)$$

It can be seen that for sufficiently small $t > 0$, $u(t) > 0$ and $(u(t), v(t))$ is on the curve $g_3(x) = 0$: $v(t) = u(t)^2$. In other words, $i = 3$ is the unique active index at $(u(t), v(t))$. Since the gradient of $g_3(\cdot)$ at the point $(u(t), v(t))$ is equal to $(-2u(t), 1)^T$, which is nonsingular, applying Lagrange multiplier theorem (cf. [3, Proposition 3.3.1]) to the problem (18) yields that there exists unique $\lambda(t) > 0$ such that

$$th + \frac{1}{2}t^2\omega - (u(t), u(t)^2) = \lambda(t)Dg_3(u(t), u(t)^2) = \lambda(t)(-2u(t), 1);$$

that is,

$$\begin{cases} (1 - 2\lambda(t))u(t) = th_1 + \frac{1}{2}t^2\omega_1 \\ u(t)^2 + \lambda(t) = \frac{1}{2}t^2\omega_2, \end{cases} \quad (19)$$

which implies that $u(t)$ is the real root of the cubic equation $u^3 + p(t)u + q(t) = 0$ for sufficiently small $t > 0$, where $p(t) = \frac{1}{2}(1 - t^2\omega_2)$ and $q(t) = -\frac{1}{2}(th_1 + \frac{1}{2}t^2\omega_1)$. It can be seen that this cubic equation has a unique real root

$$u(t) = \frac{3z(t)^2 - p(t)}{3z(t)}, \quad \text{where } z(t) = \sqrt[3]{-\frac{q(t)}{2} + \sqrt{\frac{q(t)^2}{4} + \frac{p(t)^3}{27}}}.$$

Note that when $t \rightarrow 0^+$, $p(t) \rightarrow \frac{1}{2}$, $q(t) \rightarrow 0$, $p'(t) \rightarrow 0$, $q'(t) \rightarrow -\frac{1}{2}h_1$ and $z(t) \rightarrow \frac{1}{\sqrt[3]{6}}$, and hence

$$z'(t) = \frac{1}{3}z(t)^{-2} \left\{ -\frac{1}{2}q'(t) + \frac{\frac{1}{2}q(t)q'(t) + \frac{1}{9}p(t)^2p'(t)}{2\sqrt{\frac{q(t)^2}{4} + \frac{p(t)^3}{27}}} \right\} \rightarrow \frac{1}{2}h_1.$$

Applying L'Hôpital rule, we have

$$\lim_{t \rightarrow 0^+} \frac{3z(t)^2 - p(t)}{t} = \lim_{t \rightarrow 0^+} 6z(t)z'(t) - p'(t) = \frac{\sqrt{6}}{2}h_1,$$

and hence it follows that

$$\lim_{t \rightarrow 0^+} \frac{u(t)}{t} = \lim_{t \rightarrow 0^+} \frac{3z(t)^2 - p(t)}{3z(t)t} = h_1.$$

Thus for sufficiently small $t > 0$,

$$\begin{aligned} 0 &\leq \frac{\text{dist}(th + \frac{1}{2}t^2\omega, K)}{t^2} \leq \frac{\|th + \frac{1}{2}t^2\omega - (u(t), u(t)^2)\|}{t^2} \\ &= \frac{\lambda(t)\sqrt{4u(t)^2 + 1}}{t^2} = \frac{u(t)^2}{t^2} - \frac{1}{2}\omega_2 \\ &\rightarrow h_1^2 - \frac{1}{2}\omega_2 \leq 0, \end{aligned}$$

where the two equalities follow from (19), and the last inequality follows from our assumption that $\omega_2 \leq 2h_1^2$. Therefore for any sequence $t_n \rightarrow 0+$,

$$\lim_{n \rightarrow \infty} \frac{\text{dist}(t_n h + \frac{1}{2}t_n^2\omega, K)}{t_n^2} = 0,$$

which implies that $\omega \in T_K^2(x_0, h)$. Thus (17) is verified.

In view of (16) and (17), the second order Abadie CQ holds at x_0 in every direction $h \in C(x_0)$ and hence the assumption in Theorem 5 is satisfied. By Theorem 5, we conclude that the second order necessary optimality condition holds at the optimal solution $(0, 0)$.

On the other hand, the inequalities system $g_i(x) \leq 0$ ($i = 1, 2, 3$) do not satisfy Robinson's constraint qualification at x_0 . Otherwise, one would have

$$0 \in \text{int}(G(x_0) + DG(x_0)\mathfrak{R}^2 - \mathfrak{R}_-^3). \quad (20)$$

Noting that

$$\begin{aligned} G(x_0) &= (g_1(x_0), g_2(x_0), g_3(x_0)) = (0, 0, 0) \quad \text{and} \\ DG(x_0) &= (g'_1(x_0), g'_2(x_0), g'_3(x_0)) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}; \end{aligned}$$

by (20), we obtain the existence of $h = (h_1, h_2) \in \mathfrak{R}^2$ such that $DG(x_0)h$ belongs to $\text{int}(\mathfrak{R}_-^3)$: $g'_i(x_0)h < 0$ for each $i = 1, 2, 3$. This is impossible as $g'_2(x_0)h = -h_2$ and $g'_3(x_0)h = h_2$.

Before ending this paper, we give a weaker sufficient condition than Robinson's CQ for the Abadie CQ to hold. This sufficient condition is essentially a local error bound for the system $G(x) \in C$. It is said that a *local error bound* holds for the system $K = \{x : G(x) \in C\}$ at $x_0 \in K$ if there are positive scalars γ and η such that

$$\text{dist}(x, K) \leq \gamma \text{dist}(G(x), C), \quad \text{for all } x \in B(x_0; \eta), \quad (21)$$

where dist is the distance function and $B(x_0; \eta)$ is the open ball centered at x_0 with radius η . A systematic discussion on the theory of error bound can be found in the survey paper [6] and Chapter 6 of the book [5].

It is known that Robinson's CQ at x_0 is equivalent to the so-called metrical regularity which says that there exist $\gamma > 0$ and a neighborhood V of $(x_0, 0)$ such that

$$\text{dist}(x, G^{-1}(C - y)) \leq \gamma \text{dist}(G(x) + y, C), \quad \text{for all } (x, y) \in V; \quad (22)$$

see [4, Proposition 2.89].

Comparing the metric regularity (22) with local error bound (21), it is clear that (21) is weaker than (22) and hence (21) is weaker than Robinson's CQ at x_0 (Fixing $y = 0$ in (22), we obtain the local error bound (21)).

Proposition 6. Let K be the feasible set of the problem (2). If the local error bounds (21) holds at $x_0 \in S$ then the second order Abadie CQ holds at x_0 in every direction h .

Proof. In view of (3), it suffices to prove that

$$\{w : D^2G(x_0)(h, h) + DG(x_0)w \in T_C^2(G(x_0), DG(x_0)h)\} \subset T_K^2(x_0, h).$$

Let w belong to the set in the left-hand side of the above expression. Then there exists a sequence $t_n \rightarrow 0+$ such that

$$\text{dist}(G(x_0) + t_n DG(x_0)h + (1/2)[D^2G(x_0)(h, h) + DG(x_0)w], C) = o(t_n^2).$$

Set $x_n := x_0 + t_n h + (1/2)t_n^2 w$ and

$$y_n := G(x_0) + t_n DG(x_0)h + (1/2)[D^2G(x_0)(h, h) + DG(x_0)w].$$

Then $\text{dist}(y_n, C) = o(t_n^2)$. Since G is twice continuously differentiable, by the second order Taylor expansion, $G(x_n) = y_n + o(t_n^2)$. Since local error bound (21) holds, it follows that for sufficiently large n ,

$$\text{dist}(x_n, K) \leq \gamma \text{dist}(G(x_n), C) \leq \gamma \|G(x_n) - y_n\| + \gamma \text{dist}(y_n, C) = o(t_n^2).$$

This verifies that $w \in T_K^2(x_0, h)$ and hence the conclusion. \square

References

- [1] J. ABADIE, *On the Kuhn-Tucker theorem*, in Nonlinear Programming (NATO Summer School, Menton, 1964), North-Holland, Amsterdam, 1967, pp. 19–36.
- [2] M. S. BAZARAA AND C. M. SHETTY, *Nonlinear programming*, John Wiley & Sons, New York-Chichester-Brisbane, 1979. Theory and algorithms.
- [3] D. P. BERTSEKAS, *Nonlinear Programming*, Athena Scientific, Boston, 1999.
- [4] J. F. BONNANS AND A. SHAPIRO, *Perturbation Analysis of Optimization Problems*, Springer Series in Operations Research, Springer-Verlag, New York, 2000.
- [5] F. FACCHINEI AND J.-S. PANG, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer-Verlag, New York, 2003.
- [6] J.-S. PANG, *Error bounds in mathematical programming*, Math. Programming, 79 (1997), pp. 299–332.
- [7] R. T. ROCKAFELLAR, *Convex Analysis*, Princeton University Press, Princeton, N.J., 1970. Princeton Mathematical Series, No. 28.
- [8] R. T. ROCKAFELLAR AND R. J.-B. WETS, *Variational Analysis*, Springer-Verlag, Berlin, 1998.