Examples of Metaplectic Tensor Products

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Abstract

Let $F$ be a $p$-adic field and $U$ be a metaplectic cover of $G = GL(r, F)$. Suppose that $M = G_1 \times G_2$ is a Levi subgroup of $G$ with two blocks. This note discusses the tensor products of genuine admissible representations of $G_1$ and of $G_2$.

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1 Preliminaries

A totally disconnected group in this note means a topological group which is also a Hausdorff space with a countable basis consisting of open compact sets. Let $G$ be a totally disconnected group and $H$ be a normal subgroup of $G$ such that $G/H$ is finite abelian. If $\pi$ is an admissible irreducible representation of $G$, then the restriction $\pi|_H$ is a finite direct sum of admissible irreducible representations of $H$ (refer to [2, Lemma 2.1]). If $\sigma$ is a representation of $H$, and $g \in G$, denote by $\sigma^g$ the conjugate of $\sigma$ which is given by $\sigma^g(x) = \sigma(g^{-1}xg), \forall x \in H$. Denote by $\sigma^G$ the induced representation of $G$. Denote by $G^*$ the group of characters of $G$. We identify $(G/H)^*$ with the group of characters of $G$ which are trivial on $H$. For a character $\chi \in G^*$, denote by $\pi\chi$ the representation of $G$ given by $(\pi\chi)(g) = \chi(g)\pi(g)$. The following lemma follows from [5, Lemma 7.9] (also refer to [2, Lemma 2.3]).

Lemma 1.1 Let $\sigma$ be an irreducible representation of $H$. There is a subgroup $K$ of $G$ such that

1) $H < K < G$;
2) \( \sigma \) extends to an irreducible representation \( \rho \) of \( K \) on the same space in which \( \sigma \) operates;
3) \( \rho^G \) is irreducible.

Furthermore,

\[
\sigma^G = \bigoplus_{\omega \in (K/H)^*} (\rho\omega)^G,
\]

where each \((\rho\omega)^G\) is irreducible and equivalent to \( \rho^G \chi \) for any \( \chi \in (G/H)^* \) whose restriction to \( K \) equals \( \omega \).

**Corollary 1.2** Let \( \pi \) be an irreducible representation of \( G \). Then \( \sigma = \pi|_H \) is irreducible if and only if \( \pi \not\sim \pi \chi \) for any nontrivial character \( \chi \) of \( G \) trivial on \( H \).

**Corollary 1.3** For any two irreducible subrepresentations \( \pi_1 \) and \( \pi_2 \) of \( \sigma^G \), \( \pi_1|_H \cong \pi_2|_H \). Furthermore, if \( \sigma \) is subrepresentation of \( \pi_1|_H \), then so is \( \sigma^g \) for \( g \in G \).

**Corollary 1.4** If \( \pi \cong \pi \chi \) for any \( \chi \in (G/H)^* \), then \( \pi \cong \sigma^G \) for some irreducible representation \( \sigma \) of \( H \).

## 2 The Definition and Some Basic Properties

Let \( M \) be a totally disconnected group and \( G_1, G_2 \) be subgroups of \( M \) such that \( M = G_1 \times G_2 \). Let \( \mu \) be a finite subgroup of \( \mathbb{C}^\times \). We say a topological group \( \overline{M} \) is a covering group of \( M \) by \( \mu \) if we have a short exact sequence

\[
1 \longrightarrow \mu \longrightarrow \overline{M} \xrightarrow{p} M \longrightarrow 1,
\]

where \( p \) is a topological quotient map.

If \( H \) is a subgroup of \( M \), denote \( \overline{H} = p^{-1}(H) \). A representation \( \pi \) of \( \overline{H} \) is said to be genuine if \( \pi(\xi g) = \xi \pi(g) \) for any \( \xi \in \mu, g \in \overline{H} \). We assume that all the representations considered in this note are genuine and admissible.

Denote \([x, y] = xyx^{-1}y^{-1}\) for \( x, y \in \overline{M} \) and denote by \([\overline{G_1}, \overline{G_2}]\) the subgroup of \( M \) generated by \([g_1, g_2], g_1 \in \overline{G_1}, g_2 \in \overline{G_2} \). Observe in general \([\overline{G_1}, \overline{G_2}] \neq 1 \).

For \( g_2 \in G_2 \), define a character

\[
\chi_{g_2} : \overline{G_1} \to \mu, \quad g_1 \mapsto [g_2, g_1], \quad \forall g_1 \in G_1.
\]
Let 
\[
G_1^\circ = \{ g_1 \in G_1 : [g_1, g_2] = 1, \forall g_2 \in G_2 \}.
\]

It is easy to see that \( G_1^\circ \) is a normal subgroup \( G_1 \). Define the subgroup \( G_2^\circ \) similarly. The following condition on the covering is assumed in this section.

For \( i = 1, 2 \), \( G_i^\circ \) is of finite index in \( G_i \). \hfill (2)

Remark that the assumption is satisfied if \( M \) is a reductive group over \( p \)-adic field.

Let 
\[
M^\circ = G_1^\circ \times G_2^\circ.
\]

Let \( \Xi \) be the subgroup \( \{ (\xi, \xi^{-1}) : \xi \in \mu \} \). Then we have

\[
G_1 \times G_2 \cong (G_1^\circ \times G_2^\circ) / \Xi, \quad M^\circ \cong (G_1^\circ \times G_2^\circ) / \Xi.
\]

If \( (\pi_1, V_1) \) and \( (\pi_2, V_2) \) are irreducible representations of \( G_1^\circ \) and \( G_2^\circ \) respectively, define the tensor product representation \( \pi_1 \otimes \pi_2 \) of \( G_1^\circ \times G_2^\circ \) on the space \( V_1 \otimes V_2 \) by

\[
(\pi_1 \otimes \pi_2)(g_1 g_2)v_1 \otimes v_2 = \pi_1(g_1)v_1 \otimes \pi_2(g_2)v_2,
\]

\( \forall g_1 \in G_1^\circ, g_2 \in G_2^\circ, v_1 \in V_1, v_2 \in V_2 \).

This is well-defined since \( [G_1^\circ, G_2^\circ] = 1 \).

**Lemma 2.1** Notation as above, \( \pi_1 \otimes \pi_2 \) is admissible and irreducible. Any admissible irreducible representation of \( G_1^\circ \times G_2^\circ \) is of this form.

For \( x \in G_1^\circ, y \in G_2^\circ \) and \( g_2 \in G_2 \), we have

\[
(\pi_1 \otimes \pi_2)^y (xy) = (\pi_1 \otimes \pi_2)([g_2,x](g_2yg_2^{-1})) = (\pi_1 \chi_{g_2})(x) \otimes (\pi_2)^y (y). \quad (3)
\]

The next lemma follows from the above observation.

**Lemma 2.2** Let \( \pi \) be an admissible irreducible representation of \( G \). Then \( \pi |_{G_1^\circ} \) is a direct sum of at most countably many irreducible representations of \( G_1^\circ \). If \( \pi_1 \) is one of the irreducible component, then so is \( \pi_1 \chi \) for \( \chi \in (G_1^\circ/G_1^\circ)^* \). Any irreducible component of \( \pi |_{G_1^\circ} \) is equivalent to \( \pi_1 \chi \) for some \( \chi \in (G_1^\circ/G_1^\circ)^* \). 3
Let $M = G_1 \times G_2 \times \cdots \times G_r$ be a totally disconnected group and $\overline{M}$ is a covering of $M$ by a finite abelian group. Let

$$\overline{G}_i^\circ = \{g_i \in \overline{G}_i : [g_i, g] = 1, \forall g \in G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_r\}.$$ 

By induction, we can get a result for $M$ similar to Lemma 2.2.

**Definition 2.3** Let $\overline{M}$ be a covering of a totally disconnected group $M = G_1 \times \cdots \times G_r$ by a finite subgroup of $\mathbb{C}^\times$. Let $\pi_i$ be a genuine admissible irreducible representation of $G_i$ for $i \leq r$. An admissible irreducible representation of $\overline{M}$ is called a metaplectic tensor product of $\pi_1, \cdots, \pi_r$ if, for each $i$, its restriction to $\overline{G}_i$ contains the given representations $\pi_i$.

Denote by $\{\pi_1 \otimes \cdots \otimes \pi_r\}$ the set of all metaplectic tensor products of $\pi_1, \cdots, \pi_r$.

It follows from the definition that any irreducible representation of $\overline{M}$ is a tensor product.

If $\pi_i, i = 1, 2, 3$, are irreducible representations, we denote by $\{\{\pi_1 \otimes \pi_2\} \otimes \pi_3\}$ the union of the sets $\{\pi \otimes \pi_3\}$ where $\pi$ runs over $\{\pi_1 \otimes \pi_2\}$. We get the associative law immediately be the definition:

$$\{\{\pi_1 \otimes \pi_2\} \otimes \pi_3\} = \{\pi_1 \otimes \pi_2 \otimes \pi_3\} = \{\pi_1 \otimes \{\pi_2 \otimes \pi_3\}\}.$$

From now on, suppose we are given two representations $\pi_1$ of $\overline{G}_1$ and $\pi_2$ of $\overline{G}_2$. We want to see how much $\pi_1$ and $\pi_2$ can determine irreducible representations of $\overline{M}$. We start with a lemma.

**Lemma 2.4** The set of all metaplectic tensor products of $\pi_1$ and $\pi_2$ is the set of all inequivalent irreducible subrepresentations of $(\pi_1 \otimes \pi_2^\circ)^{\overline{M}}$ when $\pi_2^\circ$ runs over all irreducible subrepresentations of $\overline{G}_2$.

**Corollary 2.5** The number of metaplectic tensor products of $\pi_1$ and $\pi_2$ is not larger than $\min(|G_1/G_1^\circ|, |G_2/G_2^\circ|)$.

**Theorem 2.6** Let $\pi$ be a metaplectic tensor product of $\pi_1$ and $\pi_2$ Then the set of metaplectic tensor products of $\pi_1$ and $\pi_2$ is (up to equivalence) the set of $\pi \chi$ where $\chi$ runs over the set of characters of $\overline{M}/\overline{M}^\circ$. 

4
Proof

By Lemma 2.2, $\pi \chi$ is a tensor product of $\pi_1$ and $\pi_2$ for $\chi \in (\overline{M}/\overline{M}^\circ)^*$. To show any tensor product of $\pi_1$ and $\pi_2$ is of this form, we first observe the following fact: For any two irreducible subrepresentations $\pi_2^\circ$ and $\pi_2'^\circ$ of $\pi_2|_{G_2}$, $(\pi_1 \otimes \pi_2^\circ)^{\overline{M}} = (\pi_1 \otimes \pi_2'^\circ)^{\overline{M}} \chi$ for $\chi \in (\overline{M}/\overline{M}^\circ)^*$.

Indeed, assume $\pi_2'^\circ = (\pi_2^\circ)^{g_2}$ for some $g_2 \in G_2$. By (3), we get the following identities from which the claim would follow.

$$ (\pi_1 \otimes (\pi_2^\circ)^{g_2})^{\overline{M}} = \left( (\pi_1 \otimes (\pi_2^\circ)^{g_2})^{g_2^{-1}} \right)^{\overline{M}} \cong (\pi_1 \chi g_2^{-1} \otimes \pi_2^\circ)^{\overline{M}} \cong (\pi_1 \otimes \pi_2^\circ)^{\overline{M}} \chi $$

for some character $\chi$ of $\overline{M}$ which extends the character $(\chi g_2)^{-1} \otimes 1$ of $\overline{G_1} \times \overline{G_2}$.

By Lemma 1, Any two irreducible subrepresentations of $(\pi_1 \otimes \pi_2^\circ)^{\overline{M}}$ differ by a character on $\overline{M}/\overline{M}^\circ$. Now the theorem follows from Lemma 2.4. Q.E.D.

Corollary 2.7 The restrictions of any two metaplectic tensor products of $\pi_1$ and $\pi_2$ to the group $\overline{M}^\circ$ are equivalent. Hence the set $\{\pi_1 \otimes \pi_2\}$ of metaplectic tensor products of $\pi_1$ and $\pi_2$ is the set of inequivalent irreducible subrepresentations of $(\pi_1^\circ \otimes \pi_2^\circ)^{\overline{M}}$.

Remark that by the above corollary, the set of metaplectic tensor products of two irreducible representations $\pi_1$ and $\pi_2$ depend only on the restrictions of $\pi_1$ and $\pi_2$ to the subgroups $\overline{G_1}$ and $\overline{G_2}$ respectively.

We conclude this section by a lemma which will be used later.

Lemma 2.8 If $\pi_1|_{\overline{G_1}}$ is irreducible, then the number of metaplectic tensor products of $\pi_1$ and $\pi_2$ equals the number of inequivalent irreducible subrepresentations of $\pi_2|_{\overline{G_2}}$.

3 Applications to $\overline{GL}(r, F)$

Let $F$ be a $p$-adic field and $G = GL(r, F)$. Denote by $\mu_n$ the subgroup of $\mathbb{C}^\times$ of order $n$ and by $(\cdot, \cdot)$ the $n$-Hilbert symbol on $F$. Let $\overline{G}$ be the metaplectic cover of $G$ by $\mu_n$ as defined in [4, §0.1]. In particular, its cocycle on the diagonal subgroup $T$ is given by

$$ \sigma(a, b) = (\det(a), \det(b))^c \prod_{i<j} (a_i, b_j), $$

(5)
\[ a = \text{diag}(a_1, \ldots, a_r), b = \text{diag}(b_1, \ldots, b_r) \in T, \]

Suppose \( r = r_1 + r_2 \) and let \( \pi_i \) be an irreducible presentation of \( \overline{G_i} = \overline{GL(r_i)} \). In some cases, the set of metaplectic tensor products \( \{ \pi_1 \otimes \pi_2 \} \) can be distinguished by the central characters. For example, [3, Theorem 1] shows this is the case if the covering number \( n = 2 \). Also, this is the case if each \( \pi_i \) is the Langlands quotient of the induced representation \( \overline{G_i} \rho_i \), where \( T_i = T \cap G_i \) and \( \rho_i \) is an irreducible representation of \( T_i \) such that the exponent of the character \( \rho_i^n \) given by (the \( x \) below is at the \( i \)-th position)

\[ \rho_i^n(x) = \rho(\text{diag}(1, \ldots, 1, x, x^{-1}, 1, \ldots, 1)), \quad x \in F^\times \]

is nonnegative for any \( i < r \). The last example is when the base field \( F \) contains \( 2n \)-th roots of unity. Savin has shown that in this case, the exponent \( c \) in (5) can be chosen to be \(-1/2\) and hence \( \overline{M} \) is a quotient group of the direct product of \( \overline{G_1} \) and \( \overline{G_2} \). So there is only one metaplectic tensor product of two representations and it behaves just like the usual tensor product [6, Section 1]. It can be shown that the above construction can be realized only if the field \( F \) contains \( 2n \)-th roots of unity.

We now give an example showing that the central character can not always distinguish tensor products.

**Lemma 3.1** Let \( \rho \) be an irreducible representation of \( T \) such that \( \rho^w \cong \rho \) for any \( w \) in the normalizer of \( T \). Then \( \delta = i \overline{T} \rho \) is irreducible and \( \delta_{|\overline{\mathbb{C}^*}} \) is a direct sum of finite copies of an irreducible representation of \( \overline{\mathbb{C}^*} \).

Assume \( n, r, r_1, r_2 \) are natural numbers such that \( r = r_1 + r_2, \gcd(r_1, n) = \gcd(r_1 - 1, n) = 1, n|r_2 \). Let \( \overline{GL(r, F)} \) be the covering of \( GL(r, F) \) given by (5) with \( c = 0 \). Let \( M = G_1 \times G_2 \) with \( G_1 = GL(r_1) \) and \( G_2 = GL(r_2) \).

Take an irreducible representation \( \pi_1 \) of \( \overline{G_1} \) such that \( \pi_1|_{\overline{G_1}} \) is a direct sum of finite copies of an irreducible representation of \( \overline{G_1} \). The existence of such a representation is guaranteed by Lemma 3.1. Denote by \( \overline{Z_i} \) the center of \( \overline{G_i} \). Let \( \pi_2 \) be any irreducible representation of \( \overline{G_2} \). The center of \( \overline{M} \) is

\[ \overline{Z} = \{ \text{diag}(a^n I_{r_1}, b^n I_{r_2}) : a, b \in F^\times \} = \overline{Z_1} \overline{Z_2}. \]
So all metaplectic tensor products of \( \pi_1 \) and \( \pi_2 \) have the same central character. We show there are more than one metaplectic tensor products of \( \pi_1 \) and \( \pi_2 \).

Let \( \pi_2' \) be an irreducible subrepresentation of \( \pi_2|_{\overline{G}_2} \). First observe that
\[
(\pi_2')^g \not\cong \pi_2',
\]
for any \( g \in \overline{G_2} - \overline{G_2} \). Indeed, since \( n|r_2 \), the center of \( \overline{G_2} \) is
\[
Z(\overline{G_2}) = p^{-1}\{xI_{r_2} : x \in F^\times\}.
\]
Let \( \omega \) be the central character of \( \pi_2' \). Then the central character of \( (\pi_2')^g \) is \( \omega \chi_g \) where \( \chi_g \) is the character of the center of \( \overline{G_2} \) given by \( \chi_g(z) = [g, z] = (\det(g), x)^{r_2-1} \) for \( z \in Z(\overline{G^2}) \) with \( p(z) = xI_{r_2} \). Since \( \gcd(r_2 - 1, n) = 1 \), \( \chi_g \) is a nontrivial character. We then see \( (\pi_2')^g \) and \( \pi_2' \) have different central characters and hence are inequivalent.

We conclude that \( (\pi_1 \otimes \pi_2')^g \not\cong \pi_1 \otimes \pi_2' \) for any \( g \in \overline{G_2} - \overline{G_2} \) by (3). Hence \((\pi_1 \otimes \pi_2')^M\) is irreducible. By the choice of \( \pi_1 \), there is an \( h \in \overline{G_2} \) such that \( \pi_1 \chi_h \not\cong \pi_1 \) (Corollary 1.4). It then follows that \( \pi_1 \otimes (\pi_2')^h \not\cong (\pi_1 \otimes \pi_2')^g \) for any \( g \in \overline{G_2} - \overline{G_2} \) and we get
\[
(\pi_1 \otimes \pi_2')^M \not\cong (\pi_1 \otimes (\pi_2')^h)^M.
\]

Each side is a metaplectic tensor product of \( \pi_1 \) and \( \pi_2 \) by Lemma 2.4.

We close this note by a discussion on the metaplectic tensor product defined in [1].

Let \( B \) be a maximal subgroup of \( F^\times \) with the property that \( (b, b') = 1 \) for all \( b, b' \in F^\times \). Let \( r = r_1 + \cdots + r_l \) be a partition of \( r \) and \( M = G_1 \times \cdots \times G_l \), where \( G_i = GL(r_i) \). Denote
\[
M^B = \{ m = \text{diag}(g_1, \ldots, g_l) \in M : g_i \in G_i, \det(g_i) \in B \}.
\]

Let \( \overline{G} \) be the covering of \( G = GL(r, F) \) given by (5) and \( Z \) the center of \( \overline{G} \). Suppose \( \pi_i, i = 1, \ldots, l \), are admissible irreducible representations of \( \overline{Z}G \). Denote
\[
G_i^B = \{ g \in G_i : \det(g) \in B \}. \quad \text{Observe } [ZG_i^B, ZG_j^B] = 1, \forall i \neq j.
\]

The definition of the tensor product of \( \pi_i \)'s in [1, 26.2] is given as follows.

Pick up any irreducible subrepresentation \( \pi'_i \) of \( \pi_i|_{\overline{Z}G} \). Form the tensor product representation \( \pi' = \pi'_1 \otimes \cdots \otimes \pi'_l \) of \( M^B \). The tensor product of \( \pi_1, \ldots, \pi_l \) is defined to be the induced representation \( (\pi')^M \).

The well-definedness of the “tensor product” depends on the following claim: 

For any \( m \in M \setminus M^B Z \), \( (\pi')^m \) is not equivalent to \( \pi' \).
Actually the above claim is not true in general. Indeed, we are going to construct examples to show that the centralizer of $\overline{M^BZ}$ in $\overline{M}$ is not contained in $\overline{M^BZ}$ in general. Let

$$B^* = \{ x \in F^\times : (x, b) = 1, \forall b \in B \}.$$  

It is not hard to see that the centralizer of $\overline{M^BZ}$ in $\overline{M}$ is

$$\{ \text{ad} \text{iag} (\lambda_1I_r, \cdots, \lambda_lI_r) : \lambda_i \in B^*, a^{r-1} \in B^* \}.$$  \hspace{1cm} (7)

If the base field is the field $\mathbf{R}$ of real numbers, such an example is easy to find. In this case, $B = \mathbf{R}^+$, $B^* = \mathbf{R}^\times$. Let $r = 2s$ be any even number. Consider $M = GL(s) \times GL(s)$. By (7), any scalar matrix is in the centralizer of $\overline{M^BZ}$ in $\overline{M}$, while any scalar in $\overline{M^BZ}$ must be a positive multiple of the identity matrix.

For $p$-adic field, we construct as follows. Let $p$ be a prime and $n = 4p^2$. Let $F$ be a $p$-adic field containing $2n$-th roots of unity whose residue field has characteristic $p$, say, $F = \mathbf{Q}_p \left( e^{\pi \sqrt{-1}/4p^2} \right)$. Then $-1$ is an $n$-th power and $B^* = B$. Observe that $(x^{2p}, y^{2p}) = 1$ for any $x, y \in F^\times$. So we may choose $B$ containing $F^\times 2p$.

Let $r = 2p+1$ and consider $M = GL(p) \times GL(p+1)$. Take any $u \in F^\times - B$. By (7), $s(uI_r)$ is in the centralizer of $\overline{M^BZ}$ in $\overline{M}$. If $s(uI_r) \in \overline{M^BZ}$, then $uI_r = bI_r \cdot zI_r$ with $bI_r \in p(\overline{M})$ and $zI_r \in p(\overline{Z})$. By (6), both $b^p$ and $b^{p+1}$ are in $B$ and so is $b$. Hence $u = bz$ with $b \in B$ and $z^{2p} \in F^{\times 4p^2}$, i.e., $u = bz \in BF^{\times 2p} \mu_{2p} = B$. This contradicts with the choice of $u$ and hence the centralizer of $\overline{M^BZ}$ in $\overline{M}$ is not contained in $\overline{M^BZ}$.

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References


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