Examples of Metaplectic Tensor Products

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Abstract

Let F be a p-adic field and $\overline{G}$ be a metaplectic cover of $G = GL(r, F)$. Suppose that $M = G_1 \times G_2$ is a Levi subgroup of G with two blocks. This note discusses the tensor products of genuine admissible representations of $\overline{G_1}$ and of $\overline{G_2}$.

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1 Preliminaries

A totally disconnected group in this note means a topological group which is also a Hausdorff space with a countable basis consisting of open compact sets. Let G be a totally disconnected group and H be a normal subgroup of G such that G/H is finite abelian. If π is an admissible irreducible representation of G, then the restriction $\pi|_H$ is a finite direct sum of admissible irreducible representations of H (refer to [2, Lemma 2.1]). If σ is a representation of H, and $g \in G$, denote by σ^g the conjugate of σ which is given by $\sigma^g(x) = \sigma(g^{-1}xg), \forall x \in H$. Denote by σ^G the induced representation of G. Denote by G^* the group of characters of G. We identify $(G/H)^*$ with the group of characters of G which are trivial on H. For a character $\chi \in G^*$, denote by $\pi\chi$ the representation of G given by $(\pi\chi)(g) = \chi(g)\pi(g)$. The following lemma follows from [5, Lemma 7.9] (also refer to [2, Lemma 2.3]).

Lemma 1.1 *Let σ be an irreducible representation of H. There is a subgroup K of G such that*
1) $H < K < G$;

2) σ extends to an irreducible representation ρ of K on the same space in which σ operates;
3) ρ^G is irreducible.

Furthermore,

$$\sigma^G = \bigoplus_{\omega \in (K/H)^*} (\rho\omega)^G, \tag{1}$$

where each $(\rho\omega)^G$ is irreducible and equivalent to $\rho^G\chi$ for any $\chi \in (G/H)^*$ whose restriction to K equals ω.

Corollary 1.2 *Let π be an irreducible representation of G. Then $\sigma = \pi|_H$ is irreducible if and only if $\pi \ncong \pi\chi$ for any nontrivial character χ of G trivial on H.*

Corollary 1.3 *For any two irreducible subrepresentations π_1 and π_2 of σ^G, $\pi_1|_H \cong \pi_2|_H$. Furthermore, if σ is subrepresentation of $\pi_1|_H$, then so is σ^g for $g \in G$.*

Corollary 1.4 *If $\pi \cong \pi\chi$ for any $\chi \in (G/H)^*$, then $\pi = \sigma^G$ for some irreducible representation σ of H.*

2 The Definition and Some Basic Properties

Let M be a totally disconnected group and G_1, G_2 be subgroups of M such that $M = G_1 \times G_2$. Let μ be a finite subgroup of $\mathbf{C}^\times$. We say a topological group $\overline{M}$ is a covering group of M by μ if we have a short exact sequence

$$1 \longrightarrow \mu \longrightarrow \overline{M} \xrightarrow{p} M \longrightarrow 1,$$

where p is a topological quotient map.

If H is a subgroup of M, denote $\overline{H} = p^{-1}(H)$. A representation π of $\overline{H}$ is said to be genuine if $\pi(\xi g) = \xi\pi(g)$ for any $\xi \in \mu, g \in \overline{H}$. We assume that *all the representations considered in this note are genuine and admissible.*

Denote $[x, y] = xyx^{-1}y^{-1}$ for $x, y \in \overline{M}$ and denote by $[\overline{G_1}, \overline{G_2}]$ the subgroup of M generated by $[g_1, g_2]$, $g_1 \in \overline{G_1}$, $g_2 \in \overline{G_2}$. Observe in general $[\overline{G_1}, \overline{G_2}] \neq 1$. For $g_2 \in G_2$, define a character

$$\chi_{g_2} : \overline{G_1} \to \mu, \quad g_1 \mapsto [g_2, g_1], \quad \forall g_1 \in G_1.$$

Let

$$\overline{G_1^\circ} = \{g_1 \in \overline{G_1} : [g_1, g_2] = 1, \forall g_2 \in \overline{G_2}\}.$$

It is easy to see that $\overline{G_1^\circ}$ is a normal subgroup $\overline{G_1}$. Define the subgroup $\overline{G_2^\circ}$ similarly. The following condition on the covering is assumed in this section.

$$\textit{For } i = 1, 2, \quad \overline{G_i^\circ} \textit{ is of finite index in } \overline{G_i} \tag{2}$$

Remark that the assumption is satisfied if M is a reductive group over p-adic field.

Let

$$M^\circ = G_1^\circ \times G_2^\circ.$$

Let Ξ be the subgroup $\{(\xi, \xi^{-1}) : \xi \in \mu\}$. Then we have

$$\overline{G_1 \times G_2^\circ} \cong \left(\overline{G_1} \times \overline{G_2^\circ}\right)/\Xi, \quad \overline{M^\circ} \cong \left(\overline{G_1^\circ} \times \overline{G_2^\circ}\right)/\Xi.$$

If (π_1, V_1) and (π_2°, V_2) are irreducible representations of $\overline{G_1}$ and $\overline{G_2^\circ}$ respectively, define the tensor product representation $\pi_1 \otimes \pi_2^\circ$ of $\overline{G_1 \times G_2^\circ}$ on the space $V_1 \otimes V_2$ by

$$(\pi_1 \otimes \pi_2^\circ)(g_1 g_2) v_1 \otimes v_2 = \pi_1(g_1) v_1 \otimes \pi_2^\circ(g_2) v_2,$$
$$\forall g_1 \in \overline{G_1}, g_2 \in \overline{G_2^\circ}, v_1 \in V_1, v_2 \in V_2.$$

This is well-defined since $[\overline{G_1}, \overline{G_2^\circ}] = 1$.

Lemma 2.1 *Notation as above, $\pi_1 \otimes \pi_2^\circ$ is admissible and irreducible. Any admissible irreducible representation of $\overline{G_1 \times G_2^\circ}$ is of this form.*

For $x \in \overline{G_1}, y \in \overline{G_2^\circ}$ and $g_2 \in \overline{G}_2$, we have

$$(\pi_1 \otimes \pi_2^\circ)^{g_2}(xy) = (\pi_1 \otimes \pi_2^\circ)\left([g_2, x] x \left(g_2 y g_2^{-1}\right)\right) = (\pi_1 \chi_{g_2})(x) \otimes (\pi_2^\circ)^{g_2}(y). \tag{3}$$

The next lemma follows from the above observation.

Lemma 2.2 *Let π be an admissible irreducible representation of $\overline{G}$. Then $\pi|_{\overline{G_1}}$ is a direct sum of at most countably many irreducible representations of $\overline{G_1}$. If π_1 is one of the irreducible component, then so is $\pi_1 \chi$ for $\chi \in \left(\overline{G_1}/\overline{G_1^\circ}\right)^*$. Any irreducible component of $\pi|_{\overline{G_1}}$ is equivalent to $\pi_1 \chi$ for some $\chi \in \left(\overline{G_1}/\overline{G_1^\circ}\right)^*$.*

Let $M = G_1 \times G_2 \times \cdots \times G_r$ be a totally disconnected group and $\overline{M}$ is a covering of M by a finite abelian group. Let

$$\overline{G_i^\circ} = \{g_i \in \overline{G_i} : [g_i, g] = 1, \forall g \in \overline{G_1 \times \cdots G_{i-1} \times G_{i+1} \times \cdots \times G_r}\}.$$

By induction, we can get a result for $\overline{M}$ similar to Lemma 2.2.

Definition 2.3 *Let $\overline{M}$ be a covering of a totally disconnected group $M = G_1 \times \cdots \times G_r$ by a finite subgroup of $\mathbf{C}^\times$. Let π_i be an genuine admissible irreducible representation of $\overline{G_i}$ for $i \leq r$. An admissible irreducible representation of $\overline{M}$ is called a metaplectic tensor product of $\pi_1, \cdots, \pi_r$ if, for each i, its restriction to $\overline{G_i}$ contains the given representations π_i.*

Denote by $\{\pi_1 \otimes \cdots \otimes \pi_r\}$ the set of all metaplectic tensor products of $\pi_1, \cdots, \pi_r$.

It follows from the definition that any irreducible representation of $\overline{M}$ is a tensor product.

If π_i, $i = 1, 2, 3$, are irreducible representations, we denote by $\{\{\pi_1 \otimes \pi_2\} \otimes \pi_3\}$ the union of the sets $\{\pi \otimes \pi_3\}$ where π runs over $\{\pi_1 \otimes \pi_2\}$. We get the associative law immediately be the definition.

$$\{\{\pi_1 \otimes \pi_2\} \otimes \pi_3\} = \{\pi_1 \otimes \pi_2 \otimes \pi_3\} = \{\pi_1 \otimes \{\pi_2 \otimes \pi_3\}\}.$$

From now on, suppose we are given two representations π_1 of $\overline{G_1}$ and π_2 of $\overline{G_2}$. We want to see how much π_1 and π_2 can determine irreducible representations of $\overline{M}$. We start with a lemma.

Lemma 2.4 *The set of all metaplectic tensor products of π_1 and π_2 is the set of all inequivalent irreducible subrepresentations of $(\pi_1 \otimes \pi_2^\circ)^{\overline{M}}$ when π_2° runs over all irreducible subrepresentations of $\pi_2|_{\overline{G_2^\circ}}$.*

Corollary 2.5 *The number of metaplectic tensor products of π_1 and π_2 is not larger than* $\min(|G_1/G_1^\circ|, |G_2/G_2^\circ|)$.

Theorem 2.6 *Let π be a metaplectic tensor product of π_1 and π_2 Then the set of metaplectic tensor products of π_1 and π_2 is (up to equivalence) the set of $\pi\chi$ where χ runs over the set of characters of $\overline{M}/\overline{M^\circ}$.*

Proof

By Lemma 2.2, $\pi\chi$ is a tensor product of π_1 and π_2 for $\chi \in (\overline{M}/\overline{M^\circ})^*$. To show any tensor product of π_1 and π_2 is of this form, we first observe the following fact: For any two irreducible subrepresentations π_2° and $\pi_2'^\circ$ of $\pi_2|_{\overline{G_2^\circ}}$, $(\pi_1 \otimes \pi_2'^\circ)^{\overline{M}} = (\pi_1 \otimes \pi_2^\circ)^{\overline{M}} \chi$ for $\chi \in (\overline{M}/\overline{M^\circ})^*$.

Indeed, assume $\pi_2'^\circ = (\pi_2^\circ)^{g_2}$ for some $g_2 \in \overline{G_2}$. By (3), we get the following identities from which the claim would follow.

$$(\pi_1 \otimes (\pi_2^\circ)^{g_2})^{\overline{M}} \cong \left((\pi_1 \otimes (\pi_2^\circ)^{g_2})^{g_2^{-1}}\right)^{\overline{M}} \cong \left(\pi_1 \chi_{g_2}^{-1} \otimes \pi_2^\circ\right)^{\overline{M}} \cong (\pi_1 \otimes \pi_2^\circ)^{\overline{M}} \chi \tag{4}$$

for some character χ of $\overline{M}$ which extends the character $(\chi_{g_2})^{-1} \otimes 1$ of $\overline{G_1 \times G_2^\circ}$.

By Lemma 1, Any two irreducible subrepresentations of $(\pi_1 \otimes \pi_2^\circ)^{\overline{M}}$ differ by a character on $\overline{M}/\overline{M^\circ}$. Now the theorem follows from Lemma 2.4. *Q.E.D.*

Corollary 2.7 *The restrictions of any two metaplectic tensor products of π_1 and π_2 to the group $\overline{M^\circ}$ are equivalent. Hence the set $\{\pi_1 \otimes \pi_2\}$ of metaplectic tensor products of π_1 and π_2 is the set of inequivalent irreducible subrepresentations of $(\pi_1^\circ \otimes \pi_2^\circ)^{\overline{M}}$.*

Remark that by the above corollary, the set of metaplectic tensor products of two irreducible representations π_1 and π_2 depend only on the restrictions of π_1 and π_2 to the subgroups $\overline{G_1^\circ}$ and $\overline{G_2^\circ}$ respectively.

We conclude this section by a lemma which will be used later.

Lemma 2.8 *If $\pi_1|_{\overline{G_1^\circ}}$ is irreducible, then the number of metaplectic tensor products of π_1 and π_2 equals the number of inequivalent irreducible subrepresentations of $\pi_2|_{\overline{G_2^\circ}}$.*

3 Applications to $\overline{GL(r,F)}$

Let F be a p-adic field and $G = GL(r,F)$. Denote by μ_n the subgroup of $\mathbf{C}^\times$ of order n and by $(\cdot,\cdot)$ the n-Hilbert symbol on F. Let $\overline{G}$ be the metaplectic cover of G by μ_n as defined in [4, §0.1]. In particular, its cocycle on the diagonal subgroup T is given by

$$\sigma(a,b) = (\det(a), \det(b))^c \prod_{i<j} (a_i, b_j), \tag{5}$$

$$a = \mathrm{diag}(a_1, \cdots, a_r), b = \mathrm{diag}(b_1, \cdots, b_r) \in T,$$

Suppose $r = r_1 + r_2$ and let π_i be an irreducible presentation of $\overline{G_i} = \overline{GL(r_i)}$. In some cases, the set of metaplectic tensor products $\{\pi_1 \otimes \pi_2\}$ can be distinguished by the central characters. For example, [3, Theorem 1] shows this is the case if the covering number $n = 2$. Also, this is the case if each π_i is the Langlands quotient of the induced representation $i_{\overline{T_i}}^{\overline{G_i}} \rho_i$, where $T_i = T \cap G_i$ and ρ_i is an irreducible representation of T_i such that the exponent of the character ρ_i^n given by (the x below is at the i-th position)

$$\rho_i^n(x) = \rho(\mathrm{diag}(1, \cdots, 1, x, x^{-1}, 1 \cdots, 1)), \quad x \in F^\times$$

is nonnegative for any $i < r$. The last example is when the base field F contains $2n$-th roots of unity. Savin has shown that in this case, the exponent c in (5) can be chosen to be $-1/2$ and hence $\overline{M}$ is a quotient group of the direct product of $\overline{G_1}$ and $\overline{G_2}$. So there is only one metaplectic tensor product of two representations and it behaves just like the usual tensor product [6, Section 1]. It can be shown that the above construction can be realized only if the field F contains $2n$-th roots of unity.

We now give an example showing that the central character can not always distinguish tensor products.

Lemma 3.1 *Let ρ be an irreducible representation of $\overline{T}$ such that $\rho^w \cong \rho$ for any w in the normalizer of $\overline{T}$. Then $\delta = i_{\overline{T}}^{\overline{G}} \rho$ is irreducible and $\delta|_{\overline{G^\circ}}$ is a direct sum of finite copies of an irreducible representation of $\overline{G^\circ}$.*

Assume n, r, r_1, r_2 are natural numbers such that $r = r_1 + r_2, \gcd(r_1, n) = \gcd(r_1 - 1, n) = 1, n | r_2$. Let $\overline{GL(r, F)}$ be the covering of $GL(r, F)$ given by (5) with $c = 0$. Let $M = G_1 \times G_2$ with $G_1 = GL(r_1)$ and $G_2 = GL(r_2)$.

Take an irreducible representation π_1 of $\overline{G_1}$ such that $\pi_1|_{\overline{G_1^\circ}}$ is a direct sum of finite copies of an irreducible representation of $\overline{G_1^\circ}$. The existence of such a representation is guaranteed by Lemma 3.1. Denote by $\overline{Z_i}$ the center of $\overline{G_i}$. Let π_2 be any irreducible representation of $\overline{G_2}$. The center of $\overline{M}$ is

$$\overline{Z} = \left\{ \mathrm{diag}(a^n I_{r_1}, b^n I_{r_2}) : a, b \in F^\times \right\} = \overline{Z_1 Z_2}.$$

So all metaplectic tensor products of π_1 and π_2 have the same central character. We show there are more than one metaplectic tensor products of π_1 and π_2.

Let π_2° be an irreducible subrepresentation of $\pi_2|_{\overline{G_2^\circ}}$. First observe that $(\pi_2^\circ)^g \ncong \pi_2^\circ$ for any $g \in \overline{G_2} - \overline{G_2^\circ}$. Indeed, since $n|r_2$, the center of $\overline{G_2^\circ}$ is $Z(\overline{G^\circ}) = p^{-1}\{xI_{r_2} : x \in F^\times\}$. Let ω be the central character of π_2°. Then the central character of $(\pi_2^\circ)^g$ is $\omega\chi_g$ where χ_g is the character of the center of $\overline{G_2^\circ}$ given by $\chi_g(z) = [g, z] = (\det(g), x)^{r_2-1}$ for $z \in Z(\overline{G^\circ})$ with $p(z) = xI_{r_2}$. Since $\gcd(r_2 - 1, n) = 1$, χ_g is a nontrivial character. We then see $(\pi_2^\circ)^g$ and π_2° have different central characters and hence are inequivalent.

We conclude that $(\pi_1 \otimes \pi_2^\circ)^{g_2} \ncong \pi_1 \otimes \pi_2^\circ$ for any $g_2 \in \overline{G_2} - \overline{G_2^\circ}$ by (3). Hence $(\pi_1 \otimes \pi_2^\circ)^{\overline{M}}$ is irreducible. By the choice of π_1, there is an $h \in \overline{G_2}$ such that $\pi_1\chi_h \ncong \pi_1$ (Corollary 1.4). It then follows that $\pi_1 \otimes (\pi_2^\circ)^h \ncong (\pi_1 \otimes \pi_2^\circ)^{g_2}$ for any $g_2 \in \overline{G_2} - \overline{G_2^\circ}$ and we get

$$(\pi_1 \otimes \pi_2^\circ)^{\overline{M}} \ncong \left(\pi_1 \otimes (\pi_2^\circ)^h\right)^{\overline{M}}.$$

Each side is a metaplectic tensor product of π_1 and π_2 by Lemma 2.4.

We close this note by a discussion on the metaplectic tensor product defined in [1].

Let B be a maximal subgroup of $F^\times$ with the property that $(b, b') = 1$ for all $b, b' \in F^\times$. Let $r = r_1 + \cdots + r_l$ be a partition of r and $M = G_1 \times \cdots \times G_l$, where $G_i = GL(r_i)$. Denote

$$M^B = \{m = \mathrm{diag}(g_1, \cdots, g_l) \in M : g_i \in G_i, \det(g_i) \in B\}. \qquad (6)$$

Let $\overline{G}$ be the covering of $G = GL(r, F)$ given by (5) and $\overline{Z}$ the center of $\overline{G}$. Suppose π_i, $i = 1, \cdots, l$, are admissible irreducible representations of $\overline{ZG_i}$. Denote $G_i^B = \{g \in G_i : \det(g) \in B\}$. Observe $[\overline{ZG_i^B}, \overline{ZG_j^B}] = 1, \forall i \neq j$. The definition of the tensor product of π_i's in [1, 26.2] is given as follows.

Pick up any irreducible subrepresentation π_i' of $\pi_i|_{\overline{ZG_i^B}}$. Form the tensor product representation $\pi' = \pi_1' \otimes \cdots \otimes \pi_l'$ of $\overline{M^B Z}$. The tensor product of $\pi_1, \cdots, \pi_l$ is defined to be the induced representation $(\pi')^{\overline{M}}$.

The well-definedness of the "tensor product" depends on the following claim: *For any $m \in \overline{M} - \overline{M^B Z}$, $(\pi')^m$ is not equivalent to π'.*

Actually the above claim is not true in general. Indeed, we are going to construct examples to show that *the centralizer of* $\overline{M^B Z}$ *in* $\overline{M}$ *is not contained in* $\overline{M^B Z}$ *in general.* Let

$$B^* = \{x \in F^\times : (x, b) = 1, \forall b \in B\}.$$

It is not hard to see that the centralizer of $\overline{M^B Z}$ in $\overline{M}$ is

$$\left\{a\mathrm{diag}\left(\lambda_1 I_{r_1}, \cdots, \lambda_l I_{r_l}\right) : \lambda_i \in B^*, a^{r-1} \in B^*\right\}. \tag{7}$$

If the base field is the field $\mathbf{R}$ of real numbers, such an example is easy to find. In this case, $B = \mathbf{R}^+$, $B^* = \mathbf{R}^\times$. Let $r = 2s$ be any even number. Consider $M = GL(s) \times GL(s)$. By (7), any scalar matrix is in the centralizer of $\overline{M^B Z}$ in $\overline{M}$, while any scalar in $\overline{M^B Z}$ must be a positive multiple of the identity matrix.

For p-adic field, we construct as follows. Let p be a prime and $n = 4p^2$. Let F be a p-adic field containing $2n$-th roots of unity whose residue field has characteristic p, say, $F = \mathbf{Q}_p\left(e^{\pi\sqrt{-1}/4p^2}\right)$. Then -1 is an n-th power and $B^* = B$. Observe that $(x^{2p}, y^{2p}) = 1$ for any $x, y \in F^\times$. So we may choose B containing $F^{\times 2p}$.

Let $r = 2p+1$ and consider $M = GL(p) \times GL(p+1)$. Take any $u \in F^\times - B$. By (7), $s(uI_r)$ is in the centralizer of $\overline{M^B Z}$ in $\overline{M}$. If $s(uI_r) \in \overline{M^B Z}$, then $uI_r = bI_r \cdot zI_r$ with $bI_r \in p(\overline{M^B})$ and $zI_r \in p(\overline{Z})$. By (6), both b^p and b^{p+1} are in B and so is b. Hence $u = bz$ with $b \in B$ and $z^{2p} \in F^{\times 4p^2}$, i.e., $u = bz \in BF^{\times 2p}\mu_{2p} = B$. This contradicts with the choice of u and hence the centralizer of $\overline{M^B Z}$ in $\overline{M}$ is not contained in $\overline{M^B Z}$.

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