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A Generalization of Roth Inequality on Distribution

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Abstract

The present paper develops an inequality of Roth to the general case.

Let $\Omega\subset R^m, f\in C^m(\dot{\Omega})$, and $P_1,\cdots,P_N$ be N points contained in Ω. Let $S(x^1,\cdots,x^m)$ denote the number of points contained in $(-\infty,x^1)\times\cdots\times(-\infty\ x^m)$. $\Delta(t):=\{x\in\Omega\mid \left|\frac{\partial^m f(x)}{\partial x^1\cdots\partial x^m}\right|\geqslant t\}$, $\rho(x,\partial\Delta(t))$ denote the distance between x and the edge of $\Delta(t)$. Then $\int_\Omega(S(x)-f(x))^2\mathrm{d}v\geqslant c(m)(\log N)^{m-1}N^{-2}\int_0^\infty t\int_{\Delta(t)}\rho(x,\partial\Delta(t))^m\,\mathrm{d}v\mathrm{d}t$, where $c(m)>0$ is a constant and depends on m only.

Key words: disribution, irregularities, inequality

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1 Introduction

Roth[1] proved the following theorem:

Let N be a large integer, and $P_1,P_2,\cdots,P_N$ be points, not necessarily distinct, in the square $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$, for any point (u,v) in this square, let $S(u,v)$ denote the number of points in the rectangle $0\leqslant x<u, 0\leqslant y<v$. Then

$$\int_0^1\int_0^1(S(x,y)-Nxy)^2\mathrm{d}x\mathrm{d}y>c\log N \tag{1}$$

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where c is a positive absolute constant.

Suppose domain $\Omega\subset R^m, f\in C^m(\Omega)$. $x=(x^1,\cdots,x^m)\in R^m$. Let $dv=dx^1dx^2\cdots dx^m$, $\rho(x,\partial\Omega)$ be the distance from x to the boundary of Ω. Suppose $P_1,\cdots,P_N\in R^m$, and $S(x)=S(x^1,\cdots,x^m)$ is the number of points in $[0,x^1)\times\cdots\times[0,x^m)$. In this paper we will generalize (1) to the following theorem:

Theorem Let $\Omega\subset R^m, f\in C^m(\Omega), P_1,P_2,\cdots,P_N$ be N points in Ω. $\Delta(t)=\{x\in\Omega\,|\,|\partial^m f(x)/\partial x^1\cdots\partial x^m|\geq t\}$ for $t\geq 0$. Then

$$\int_\Omega [S(x)-f(x)]^2 dv\geq c(m)\ (\log N)^{m-1}N^{-2}\int_0^\infty (t\int_{\Delta(t)}\rho(x,\partial\Delta(t))^m dv)dt \tag{2}$$

where $c(m)$ is a positive constant related to m only.

2 The Case in $[0,1]^m$

Firstly, we introduce some notations.

Let $P_i=X_i=(X_i^1,X_i^2,\cdots,X_i^m)$ be points in $[0,1]^m$, not necessarily distinct, $i=1,2,\cdots,N$ where N is a large integer. Any $x\in[0,1)$ has the unique representation with the base 2 as the form

$$x=x_1/2+x_2/2^2+x_3/2^3+\cdots, x_i=0\ or\ 1$$

Let $x_i(x)$ denote the value of x_i in the representation of x. Let $\psi_r(x)=(-1)^{x_r+1}$ for $r=1,2,\cdots$

Let $n>2$ be an integer to be chosen later and $R_m=\{r=(r_1,\cdots,r_m)\in N^m\,|\,r_1+r_2+\cdots+r_m=n\}$. We define a function $F_r(x)$ related to $\{P_i, i\leq N\}$ as follows.

Firstly, if there is at least one i for which

$$\left.\begin{array}{l} x_1(X_i^1)=x_i(x^1),\cdots,x_{r_1-1}(X_i^1)=x_{r_1-1}(x^1)\\ \cdots\qquad\qquad\qquad\qquad\cdots\\ x_1(X_i^m)=x_i(x^m),\cdots,x_{r_m-1}(X_i^m)=x_{r_m-1}(x^m)\end{array}\right\} \tag{3}$$

take $F_r(x)=0$.

Secondly, if there is no such i, we take $F_r(x)=\psi_1(x^1)\psi_2(x^2)\cdots\psi_{r_m}(x^m)$.

We now show that

$$\int F_r(x)dx^k=0, \text{for } k=1,2,\cdots,m \tag{4}$$

provided the limits of integration are integral multiples of 2^{-r_k+1}.

This is because that the distinction into two cases in the definition of $F_r(x)$ for fixed $(x^1,x^2,\cdots,x^{k-1},x^{k+1},\cdots,x^s)$ depends only on $x_1(x^k),\cdots,x_{r_k-1}(x^k)$, and the integral over x^k gives equal and opposite contributions with $x_{r_k}(x^k)=0$ and $x_{r_k}(x^k)=1$.

Denote $F(x)=\sum_{r\in R_m}F_r(x)$.

In the following of the section, we prove six lemmas.

Lermma 1 If $f(x)\in C^m(R^m)$ and $\partial^m f/\partial x^1\partial x^2\cdots\partial x^m\geq 0$, then $\int_{[0,1]^m} f(x)F_r(x)dv\geq 0$.

Proof By induction on dimension m.

When $m=1$, for any fixed k, if there is an $X_i\in[k/2^{r-1},(k+1)/2^{r-1})$, then $F_r(x)=0$. Otherwise, $F_r(x) = -1$ or $+1$ corresponding to $x \in [k/2^{r-1}, (2k+1)/2^r)$ or $[(2k+1)/2^r, (k+1)/2^{r-1})$. Hence, since $\partial f/\partial x \geqslant 0$, $\int_{k/2^{r-1}}^{(k+1)/2^{r-1}} f(x) F_r(x) \mathrm{d}x = \left(\int_{(2k+1)/2^r}^{(k+1)/2^{r-1}} - \int_{k/2^{r-1}}^{(2k+1)/2^r} \right) f(x) \mathrm{d}x \geqslant 0$

Therefore, the summation over $k = \{0, 1, \cdots, 2^{r-1}-1\}$ yields $\int_0^1 f(x) F_r(x) \mathrm{d}x \geqslant 0$.

Now suppose that the proposition is true with m replaced by $m-1$.

Denote $\mathrm{d}v' = \mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^{m-1}$. $x' = (x^1, \cdots, x^{m-1})$. It suffices to prove that, for each $k \in \{0, 1, \cdots, 2^{r_m-1}\}$, $\int_{k2^{1-r_m}}^{(k+1)2^{1-r_m}} \mathrm{d}x^m \int_{[0,1]^{m-1}} f(x) F_r(x) \mathrm{d}v' \geqslant 0$.

There is no loss generality in assuming that among $P_i (i = 1, 2, \cdots, N)$, the points satisfying $x_i^m \in [k2^{1-r_m}, (k+1)2^{1-r_m}]$ are $P_1, P_2, \cdots, P_M (M \leqslant N)$. Let $P_i' = (X_i^1, X_i^2, \cdots, X_i^{m-1})$, $(i = 1, 2, \cdots, M)$ and let F_r' be the corresponding function related to $\{P_i', i = 1, 2, \cdots, M\}$. It is obvious that

$$F_r(x) = \psi_{r_m}(x^m) F_{r'}(x')$$

Hence $\int_{k2^{1-r_m}}^{(k+1)2^{1-r_m}} \mathrm{d}x^m \int_{[0,1]^{m-1}} f(x) F_r(x) \mathrm{d}v'$

$$= \int_{k2^{1-r_m}}^{(2k+1)2^{-r_m}} \mathrm{d}x^m \int_{[0,1]^{m-1}} [f(x', x^m + 2^{-r_m}) - f(x', x^m)] F_r'(x') \mathrm{d}v'$$

Since $\partial^m f/\partial x^1 \cdots \partial x^m \geqslant 0$, $\partial^{m-1} f(x^1, x^m)/\partial x^1 \cdots \partial x^{m-1}$ is a nondecreasing function of x^m, then

$$\frac{\partial^{m-1}}{\partial x^1 \cdots \partial x^{m-1}} [f(x', x^m + 2^{-r_m}) - f(x')] \geqslant 0$$

By the inductive hypothesis on $m-1$, we have

$$\int_{[0,1]^{m-1}} [f(x', x^m + 2^{-r_m}) - f(x', x^m)] F_r'(x') \mathrm{d}v' \geqslant 0$$

Therefore, lemma 1 follows.

Lemma 2 $\int_{[0,1]^m} x^1 x^2 \cdots x^m F_r(x) \mathrm{d}v \geqslant 2^{-2n}(2^{n-m} - N)$, where $r \in R_m$.

Proof $\int_{[0,1]^m} x^1 x^2 \cdots x^m F_r(x) \mathrm{d}v$

$$= \sum_{\substack{1 \leqslant k_i \leqslant 2^{r_i - 1} \\ i = 1, 2, \cdots, m}} \int_{(k_1 - 1)2^{1-r_1}}^{k_1 2^{1-r_1}} \cdots \int_{(k_m - 1)2^{1-r_m}}^{k_m 2^{1-r_m}} x^1 \cdots x^m F_r(x) \mathrm{d}v$$

$$= \sum_{\substack{x_i(\xi^j) = 0, 1 \\ 1 \leqslant j \leqslant m, 1 \leqslant i \leqslant r_j - 1}} \int_{-2^{-r_1}}^{2^{-r_1}} \cdots \int_{-2^{-r_m}}^{2^{-r_m}} (\xi^1 + x^1) \cdots (\xi^m + x^m). F_r(\xi^1 + x^1, \cdots, \xi^m + x^m) \mathrm{d}v$$

$$\left(\text{where} \quad \xi^j = (2k_j - 1)2^{-r_j} = \frac{x_1(\xi^j)}{2} + \cdots + \frac{x_{r_m - 1}(\xi^j)}{2^{r_m - 1}} + \frac{1}{2^{r_m}} \right)$$

$$= \sum{}^* \int_{-2^{-r_1}}^{2^{-r_1}} \cdots \int_{-2^{-r_m}}^{2^{-r_m}} (\xi^1 + x^1) \cdots (\xi^m + x^m) \mathrm{sgn} x^1 \cdots \mathrm{sgn} x^m \mathrm{d}v$$

where the summation is over all those sets of values of $x_i(\xi^j) = 0, 1$ $(1 \leqslant j \leqslant m, 1 \leqslant i \leqslant r_j - 1)$ for which (3) does not hold for any $k \in \{1, 2, \cdots, N\}$.

Now $\int_{-a}^{a} (\xi + x) \mathrm{sgn} x \mathrm{d}x = \int_{-a}^{a} x \mathrm{sgn} x \mathrm{d}x = a^2$. Hence, the above expression is $\sum{}^* 2^{-2n}$.

The total number of possibilities for $x_i(\xi^j)$ $(1 \leqslant j \leqslant m, 1 \leqslant i \leqslant r_j - 1)$ is 2^{n-m}, and there can be at most N of those for which (3) holds for some $k \in \{1, 2, \cdots, N\}$. Hence the total number of terms in

$\sum^{*}$ is at least $2^{n-m}-N$, whence the result $\int_{[0,1]^m} x^1\cdots x^m F_r(x)\,\mathrm{d}v \geqslant (2^{n-m}-N)2^{-2n}$. (We shall eventually choose n so that $2^{n-m}>N$.)

Lemma 3 Let $c(n)=|R_m|$ be the number of solutions of $r_1+r_2+\cdots+r_m=n$ $(r_i\geqslant 1, i=1,\cdots,n)$ and $A=\min\limits_{[0,1]^m}|\partial^m f/\partial x^1\cdots\partial x^m|$, then $|\int_{[0,1]^m} f(x)F(x)\mathrm{d}v| \geqslant 2^{-2n}(2^{n-m}-N)c(n)A$.

Proof: It if easy to understand that we may suppose $\partial^m f/\partial x^1\cdots\partial x^m\geqslant 0$ in $[0,1]^m$. The proofs in other cases are similiar or trivial.

Let $R(x)=f(x)-Ax^1\cdots x^m$, then $\partial^m R/\partial x^1\cdots\partial x^m=\partial^m f/\partial x^1\cdots\partial x^m - A\geqslant 0$. By lemma 1, $\int_{[0,1]^m} R(x)F_r(x)\mathrm{d}v\geqslant 0$. By lemma 2, $\int_{[0,1]^m} Ax^1\cdots x^m F_r(x)\mathrm{d}v\geqslant 2^{-2n}(2^{n-m}-N)A$. Therefore

$$\begin{aligned}&\int_{[0,1]^m} f(x)F(x)\mathrm{d}v\\ &=\sum_{r\in R_m}\left[\int_{[0,1]^m} Ax^1\cdots x^m F_r(x)\mathrm{d}v+\int_{[0,1]^m} R(x)F_r(x)\mathrm{d}v\right]\\ &\geqslant\sum_{r\in R_m} 2^{-2n}(2^{n-m}-N)A=2^{-2n}(2^{n-m}-N)c(n)A\end{aligned}$$

Lemma 4 $\int_{[0,1]^m} F^2(x)\mathrm{d}v\leqslant c(n)$.

Proof $$\int_{[0,1]^m} F(x^2)\mathrm{d}v=\sum_{r\in R_m}\int_{[0,1]^m} F_r^2(x)\mathrm{d}v+\sum_{\substack{s\neq t\\ s,t\in R_m}}\int_{[0,1]^m} F_s(x)F_t(x)\mathrm{d}v$$

Since $|F_r(x)|\leqslant 1$, the first expression on the right is at most $c(n)$. We shall prove the sceond expression is 0 by proving if $s\neq t$, then

$$\int_{[0,1]^m} F_s(x)\,F_t(x)\mathrm{d}v=0 \tag{5}$$

Without loss of generality, suppose $s_m<t_m$. Hence $F_s(x)$ is constant when $x^m\in[(k-1)2^{1-t_m}, k2^{1-t_m}]$. Then by (4).

$$\int_{(k-1)2^{1-t_m}}^{k2^{1-t_m}} F_s(x)F_r(x)\mathrm{d}x^m=0,\ \int_0^1 F_s(t)\,F_t(x)\mathrm{d}x^m=0$$

Immediately (5) holds, and lemma 4 follows.

Lemma 5 For $i=1,2,\cdots N$, $\int_{x_i^1}^1\int_{x_i^2}^1\int_{x_i^m}^1 F(x)\mathrm{d}v=0$.

Proof It suffices to prove that $\int_{x_i^1}^1\cdots\int_{x_i^m}^1 F_r(x)\mathrm{d}v=0$.

Let X^k be the least multiple of 2^{1-r_k} which is no less than $X_i^k (k=1,2,\cdots,m)$.

$$\int_{x_i^1}^1\cdots\int_{x_i^m}^1 F_r(x)\mathrm{d}v=\left(\int_{x_i^1}^{X^1}+\int_{X^1}^1\right)\cdots\left(\int_{x_i^m}^{X^m}+\int_{X^m}^1\right)F_r(x)\mathrm{d}v$$

Thus, we split the integral into 2^m terms.

By the definition of $F_r(x)$, $\int_{x_i^1}^{X^1}\cdots\int_{x_i^m}^{X^m} F_r(x)\mathrm{d}v=0$, since $F_r(x)=0$ and the other 2^m-1 terms are also 0 by (4). Then lemma 5 follows.

Lemma 6 For $f(x)\in C^m(R^m)$,

$$\int_{[0,1]^m}[S(x)-f(x)]^2\mathrm{d}v \geqslant c_1 N^{-2}\log^{m-1}N \min_{[0,1]^m}\left(\frac{\partial^m f}{\partial x^1\cdots\partial x^m}\right)^2$$

(in this paper, $c_1, c_2\cdots$ are used to denote the constants depending only on m)

Proof By lemma 5,

$$\int_{[0,1]^m} S(x)\cdot F(x)\mathrm{d}v = \int_{[0,1]^m} F(x)\Big(\sum_{\substack{1\leqslant i\leqslant n\\ X_i^j\leqslant x^j\\ 1\leqslant j\leqslant m}} 1\Big)\mathrm{d}v$$

$$= \sum_{i=1}^{N}\int_{X_i^1}^{1}\cdots\int_{X_i^m}^{1} F_{(x)}\mathrm{d}v = 0$$

By lemma 3, $\left|\int_{[0,1]^m}[f(x)-S(x)]F(x)\mathrm{d}v\right| \geqslant 2^{-2n}(2^{n-m}-N)c(n)A$

By Schwartz Inequality and lemma 4, we have

$$\int_{[0,1]^m}[f(x)-S(x)]^2\mathrm{d}v \geqslant c(n)[2^{-2n}(2^{n-m}-N)A]^2 \tag{6}$$

Choose n so that $2^{m+2}N\geqslant 2^n > 2^{m+1}N$. For $c(n)\sim n^{m-1}/(m-1)!$, when N is large enough, we have $c(n)\geqslant(\log N)^{m-1}/m!$. Thus lemma 6 follows after a short calculation of (6).

3 General Case

Lemma 7 Let $a+[0,r]^m\subset\Omega$. Then

$$\int_{a+[0,r]^m}[S(x)-f(x)]^2\mathrm{d}v \geqslant c_2(\log N)^{m-1}N^{-2}r^{2m}\min_{a+[0,r]^m}\left(\frac{\partial^m f}{\partial x^1\cdots\partial x^m}\right)^2$$

Proof Let $a=(a^1,a^2,\cdots,a^m)$,

$$P_i^* = (Y_i^1,\cdots,Y_i^m) = \frac{1}{r}(X_i^1-a^1,\cdots,X_i^m-a^m),\quad 1\leqslant i\leqslant N$$

$$P_i^{**} = (\max(0,Y_i^1),\cdots,\max(0,Y_i^m)), 1\leqslant i\leqslant N$$

$$P_{N+1}^{**}=P_{N+2}^{**}=\cdots=P_M^{**}=\mathbf{0}, M=N+[e^m]+1.$$

$S^*(x^1,\cdots,x^m)$ and $S^{**}(x^1,\cdots,x^m)$ denote the number of points $\{P_i^*, i=1,\cdots,N\}$ and $\{P_i^{**}, i=1,\cdots,M\}$ in $(-\infty,x^1)\times\cdots\times(-\infty,x^m)$ respectively. Then $S^*(x)=S(x\cdot r+a)$ and for $x\in[0,1]^m$, $S^{**}(x)=S^*(x)+M-N$.

By lemma 6, we have

$$\int_{a+[0,r]^m}[S(x)-f(x)]^2\mathrm{d}v$$

$$= r^m\int_{[0,1]^m}[S^*(y)-f(ry+a)]^2\mathrm{d}y'\cdots\mathrm{d}y^m$$

$$= r^m\int_{[0,1]^m}[S^{**}(y)-f(ry+a)+M-N]^2\mathrm{d}y^1\cdots\mathrm{d}y^m$$

$$\geqslant r^m c_1(\log N_1)^{m-1}N_1^{-2}[\min_{[0,1]^m}\frac{\partial^m f}{\partial y^1\cdots\partial y^m}f(ry+a)]^2$$

$$= c_1(\log N_1)^{m-1}N_1^{-2}r^{2m}\left(\min_{a+[0,r]^m}\frac{\partial^m f}{\partial x^1\cdots\partial x^m}\right)^2$$

where N_1 is the number of points P_i^{**} $(1\leqslant i\leqslant M)$ in $[0,1]^m$.

Then $N_1 \leqslant M = N + [e^m] + 1$. It is easy to see that $N_1 \geqslant e^m$, $(\log N_1)^{m-1} N_1^{-2} \geqslant (\log M)^{m-1} M^{-2} \geqslant \frac{1}{2} \cdot (\log N)^{m-1} N^{-2}$ (N is large). We obtain lemma 7 from (7).

Lemma 8 For any $A: 0 \leqslant A \leqslant \sup_{\Omega} \left| \frac{\partial^m f}{\partial x^1 \cdots \partial x^m} \right|$,

$$\int_{\Omega} [S(x) - f(x)]^2 dv \geqslant c_3 (\log N)^{m-1} N^{-2} A^2 \int_{\Delta(A)} \rho(x, \partial\Delta(A))^m dv$$

Proof Consider the close set $\Delta_\varepsilon(A)$ in $\Delta(A)$, where $\Delta_\varepsilon(A) = \{x \in \Delta(A) \mid \rho(x, \partial\Delta(A)) \geqslant \varepsilon\}$. All the cubes mentioned in the following are m-dimensional and with the edges parallel to coordinate axes respectively.

For any $x \in \Delta_\varepsilon(A)$, construct an m-dimensional open cube $U(x)$ with center x satisfying

$$U(x) \subset \Delta(A), U(x) \cap \partial\Delta_\varepsilon(A) \neq \varnothing.$$

Then $\Delta_\varepsilon(A) \subset \bigcup_{x \in \Delta_\varepsilon(A)} U(x)$, thus there exist finite cubes $U_1, U_2, \cdots, U_n$ covering $\Delta_\varepsilon(A)$.

Let U_1 be the biggest in them. $U_i (i \geqslant 2)$ be the biggest in those without intersection with $U_j (j < i)$. Suppose we finally get $l \leqslant n$ nonintersecting cubes $U_i (i = 1, \cdots, l)$. . Let r_i be the length of the edge of U_i, V_i be the cube with the same center as U_i and the length $3r_i$ of edges ($i \leqslant l$). Then it is easy to see for any $U_j (j = 1, \cdots, n)$, there is some $i \leqslant l, U_j \subset V_i$. So

$$\Delta_\varepsilon(A) \subset \bigcup_{i=1}^{l} V_i, V_i \cap \partial\Delta_\varepsilon(A) \neq \varnothing.$$

Thus for any $x \in V_i, \rho(x, \partial\Delta_\varepsilon(A)) \leqslant 3\sqrt{m} r_i$. Then

$$\int_{\Delta_\varepsilon(A) \cap V_i} \rho(x, \partial\Delta_\varepsilon(A))^m dv \leqslant (3\sqrt{m} r_i)^m \int_{\Delta_\varepsilon(A) \cap V_i} dv \leqslant 9^m m^{\frac{m}{2}} r_i^{2m}$$

Since $\Delta_\varepsilon(A) = \bigcup_{i=1}^{l} (\Delta_\varepsilon(A) \cap V_i)$, we have

$$\int_{\Delta_\varepsilon(A)} \rho^m dv \leqslant \sum_{i=1}^{l} \int_{\Delta_\varepsilon(A) \cap V_i} \rho(x, \partial\Delta_\varepsilon(A))^m dv \leqslant 9^m m^{\frac{m}{2}} \sum_{i=1}^{l} r_i^{2m} \tag{8}$$

By lemma 7, since $U_i \subset \Delta(A)$,

$$\int_{U_i} (S(x) - f(x))^2 dv \geqslant c_2 (\log N)^{m-1} N^{-2} r_i^{2m} \min_{U_i} \left(\frac{\partial^m f}{\partial x^1 \cdots \partial x^m} \right)^2 \geqslant \frac{c_2}{4} (\log N)^{m-1} N^{-2} r_i^{2m} A^2$$

Since $\bigcup_{i=1}^{l} U_i \subset \Omega$ and $U_i \cap U_j = \phi$ for $i \neq j$, $i, j \leqslant l$, by (8)

$$\int_{\Omega} (S - f)^2 dv = \sum_{i=1}^{l} \int_{U_i} (S - f)^2 dv \geqslant \frac{c_2}{4} (\log N)^{m-1} N^{-2} A^2 \sum_{i=1}^{l} r_i^{2m} \geqslant \frac{c^2}{4} (\log)^{m-1} N^{-2} A^2 9^{-m} m^{-\frac{m}{2}} \int_{\Delta_\varepsilon(A)} \rho^m(x, \partial\Delta_\varepsilon(A)) dv$$

$$= c_2(\log N)^{m-1}N^{-2}A^2\int_{\Delta_\varepsilon(A)}\rho(x,\partial\Delta_\varepsilon(A))^m\mathrm{d}v$$

Lemma 8 follows by $\varepsilon\to 0$.

Proof of the throrem Suppose $M=\sup\limits_{\Omega}|\partial^m f/\partial x\cdots\partial x^m|$ is finite, and let Ω be a bounded domain. Let $\varepsilon=M/n,(n=1,2,\cdots)$.

$$\int_\Omega(S-f)^2\mathrm{d}v=\sum_{k=0}^{n-1}\int_{\Delta(k\varepsilon)-\Delta((k+1)\varepsilon)}(S-f)^2\mathrm{d}v$$

$$\geqslant c_3(\log N)^{m-1}N^{-2}\sum_{k=0}^{n-1}(k\varepsilon)^2\int_{\Delta(k\varepsilon)-\Delta((k+1)\varepsilon)}\rho^m(x,\partial\Delta(k\varepsilon))\mathrm{d}v$$

$$=c_3(\log N)^{m-1}N^{-2}\Big[\sum_{k=0}^{n-1}(k\varepsilon)^2\int_{\Delta(k\varepsilon)}-\sum_{k=0}^{n-1}(k\varepsilon)^2\int_{\Delta((k+1)\varepsilon)}\Big]\rho^m\mathrm{d}v$$

$$=c_3(\log N)^{m-1}N^{-2}\sum_{k=1}^{n-1}(2k-1)\varepsilon^2\int_{\Delta(k\varepsilon)}\rho^m\mathrm{d}v$$

$$\geqslant c_3(\log N)^{m-1}N^{-2}\sum_{k=1}^{n-1}\Big(k\varepsilon\int_{\Delta(k\varepsilon)}\rho^m\mathrm{d}v\Big)\varepsilon \tag{9}$$

Let $n\to\infty$ and $\varepsilon\to 0$, we obtain the theorem by the definition of integration.

For the general case. Let

$$\Omega_n=\{x\in\Omega\,|\,\|x\|\leqslant n,\rho(x,\partial\Omega)\geqslant\frac{1}{n}\}$$

Thus Ω_n is a bounded close set, so f is bounded in Ω_n. By the above proof,

$$\int_{\Omega_n}(S-f)^2\mathrm{d}v\geqslant c_3(\log N)^m N^{-2}\int_0^\infty\Big(t\int_{\Delta(t)\cap\Omega_n}\rho(x,\partial\Delta(t))\mathrm{d}v\Big)\mathrm{d}t$$

The two sides of the above inequality are nondecreasing of n. The theorem follows by $n\to\infty$.

In the theorem, let $\Omega=[0,1]^m, f(x)=N_{x^1\cdots x^m}$, By (2)

$$\int_{[0,1]^m}[S(x^1,\cdots,x^m)-N_{x^1\cdots x^m}]^2\mathrm{d}x^1\cdots\mathrm{d}x^m\geqslant c(m)(\log N)^{m-1}$$

The above inequality is the direct generaization of (1).

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Reference

[1] Roth K F, On Irregularities of Distribution, *Mathematika*, 1 (1954), 73-79.

关于 Roth 分布不等式的一个推广

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摘 要

本文推广了 Roth 的关于分布不均匀性的一个不等式到很一般的情况.

设 Ω 为 R^m 中一区域,$f\in C^m(\Omega)$. $P_1\cdots P_N$ 为 Ω 内 N 个点. 记 $S(x^1,\cdots,x^m)$ 为在 $(-\infty,x^1)\times\cdots\times(-\infty,x^m)$ 内的点数. 记 $\Delta(t)=\{x\in\Omega\,|\,|\partial^m f(x/\partial x^1\cdots\partial x^m|\geqslant t\}$. $\rho(x,\partial\Delta(t))$ 为 x 到 $\Delta(t)$ 的边界距离,则

$$\int_\Omega[S(x)-f(x)]^2\mathrm{d}v\geqslant c(m)(\log N)^{m-1}N^{-2}\int_0^\infty(t\int_{\Delta(t)}\rho(x,\partial\Delta(t))^m\mathrm{d}v)\mathrm{d}t.$$

关键词: 分布,奇异性,不均匀性

中图法分类号: O156·4