The Residual Spectrum of $\overline{GL(N)}$: The Borel Case

THE PROBLEM

F: number field such that $-1 \in F^{\times n}$ and that $\mu_n = \{x \in F : x^n = 1\} \subset F$.
$\mathbf{A}$: ring of adeles over F. $G = GL(N)$.
$\overline{G} = \overline{G(\mathbf{A})}$: the metaplectic covering of $G(\mathbf{A})$ by μ_n given by the cocycle

$$\sigma\left(h, h'\right) = \prod_{i<j}\left(h_i, h_j'\right)_{\mathbf{A}}\left(\det h, \det h'\right)_{\mathbf{A}}^{q}.$$

for diagonal matrices $h, h' \in H(\mathbf{A})$.
$G(F)^*$: the lift of $G(F)$ in $\overline{G(\mathbf{A})}$.
$Z =$ the center of $\overline{G(\mathbf{A})}$.
χ: character of Z s.t. χ: $\mu_n \hookrightarrow \mathbf{C}^\times$.

$$\begin{aligned} L^2\left(\overline{G}, \chi\right) &= \{f : \overline{G} \to \mathbf{C} : f(zg) = \chi(z) f(g), \\ &\qquad \forall z \in Z, |f| \in L^2\left(ZG(F)^* \backslash \overline{G}\right)\}. \end{aligned}$$

If $M = GL(N_1) \times \cdots \times GL(N_k)$, denote $X(M) = \{\underline{s} \in \mathbf{C}^k : \sum_{i\leq k} N_i s_i = 0\}$.

Denote $\rho[\underline{s}](m) = \rho(m)\prod_{i\leq k}|\det g_i|^{s_i}$, for $\underline{s} \in X(M)$, $m = \text{diag}(g_1, \cdots, g_k)$. Define

$$E(\phi, \underline{s})(g) = \sum_{\nu \in P(F)^* \backslash G(F)^*} \phi(\nu g, \underline{s})$$

for $\phi : \underline{s} \to \text{Ind}_{\overline{M}}^{\overline{G}} \rho[\underline{s}]$ of Paley-Wiener type and $\overline{K}$-finite.

Then $L^2\left(\overline{G}, \chi\right) = \oplus_{\Theta=\{(M,\rho)\}} L^2\left(\overline{G}, \chi\right)_{\Theta}$;

$$L^2\left(\overline{G}, \chi\right)_{\Theta} = \int^{\oplus}_{\substack{\underline{s}\in X(M) \\ \text{Re}(\underline{s})=\underline{\lambda}}} E\left(\text{Ind}_{\overline{M}}^{\overline{G}} \rho, \underline{s}\right) d\underline{s}.$$

where ρ is the isotropic subspace of an irreducible representation in $L_0^2\left(\overline{M}, \chi_M\right)$.

The inner product is

$$\left\langle \phi', \phi \right\rangle = (2\pi)^{1-k} \int_{\substack{\underline{s}\in X(M) \\ \text{Re}\underline{s}=\underline{\lambda}}} \sum_{\sigma \in W(\rho,\rho')} \tag{1}$$

$$\left\langle M\left(\sigma^{-1}, \sigma(\rho), -\sigma(\overline{\underline{s}})\right) \phi'(-\sigma\overline{\underline{s}}), \phi(\underline{s}) \right\rangle d\underline{s}.$$

This thesis decomposes $L^2\left(\overline{G}, \chi\right)_{\Theta}$ when $\Theta = \{(H, \rho)\}$.

THE RESULT

$\underline{p} = (p_1, \cdots, p_r)$: $\sum_{i \leq r} p_i = N$

$\longleftrightarrow$ block diagonal $M = M_{\underline{p}}$.

$\pi_i \, (i \leq r)$: unitary character of $\mathbf{A}^{\times n}/F^{\times n}$.

$\pi_{\underline{p}} = \mathrm{Ind}_{Z\overline{H^n}H(F)^*}^{\overline{H}} \left(\cdots \otimes \underbrace{\pi_i \otimes \cdots \otimes \pi_i}_{p_i} \otimes \cdots \right)$.

$\lambda\left(\underline{p}\right) = \left(\frac{p_1-1}{2n}, \frac{p_1-3}{2n}, \cdots, \frac{1-p_1}{2n}, \frac{p_2-1}{2n}, \cdots, \frac{1-p_r}{2n}\right)$.

$J_{\underline{p}}$: the unique irreducible quotient of

$$\mathrm{Ind}_{\overline{H}}^{\overline{M}} \pi_{\underline{p}} \left[\lambda\left(\underline{p}\right)\right].$$

$Z\left(\underline{p}\right) = \left\{\underline{z} \in i\mathbf{R}^r : \sum_{i \leq r} p_i z_i = 0\right\}$.

Then

$$L^2\left(\overline{G}, \chi\right)_{\Theta} = \oplus_{\left\{\left(H, \pi_{\underline{p}}\right)\right\} \in \Theta} L_{\left\{\pi_{\underline{p}}\right\}};$$

$$L_{\left\{\pi_{\underline{p}}\right\}} = \int_{Z\left(\underline{p}\right)/\mathfrak{S}_r}^{\oplus} \mathrm{Ind}_{\overline{M}}^{\overline{G}} \left(J_{\underline{p}}\left[\underline{z}\right]\right) d\underline{z}.$$

Especially, (π is a character of $\mathbf{A}^{\times n}$)

$L^2\left(\overline{G}, \chi\right)_{\Theta, disc} = \sum_{\{\otimes \pi\}}$ Langlands quotient of

$\mathrm{Ind}_{Z\overline{H^n}H(F)^*}^{\overline{G}} \left(\pi||^{\frac{N-1}{2n}} \otimes \pi||^{\frac{N-3}{2n}} \otimes \cdots \otimes \pi||^{\frac{1-N}{2n}}\right)$.

A more precise description:

Define a $\overline{G}$-invariant inner product, induced from $\|\cdot\|$ on $\mathrm{Ind}_{M}^{\overline{G}}\left(J_{\underline{p}}[\underline{z}]\right)$, $\underline{z} \in Z\left(\underline{p}\right)$ by

$$\left\|N\left(w_{\underline{p}}, \underline{s}\right) f\right\| = \left\langle N\left(w_{\underline{p}}, \underline{s}\right) f, f\right\rangle ,$$

where $f \in \mathrm{Ind}_{H}^{\overline{G}} \pi_{\underline{p}}[\underline{s}]$,
$\underline{s} = \left(z_1 + \frac{p_1 - 1}{2n}, \cdots, z_1 + \frac{1-p_1}{2n}, \cdots, z_r + \frac{1-p_r}{2n}\right).$

Then $L_{\left\{\pi_{\underline{p}}\right\}}$ is the completion of the space of collections (denote $\left\{\underline{p}\right\} = \left\{\tau\underline{p} : \tau \in \mathfrak{S}_r\right\}$)

$$\Phi_{\{\underline{p}\}} = \left\{\Phi_{\underline{p}} \in \mathrm{Ind}_{M}^{\overline{G}}\left(J_{\underline{p}}[\underline{z}]\right) : \underline{p} \in \left\{\underline{p}\right\}\right\}$$

such that for any $\tau \in \mathfrak{S}_r$

$$\Phi_{\tau\underline{p}}(\tau\underline{z}) = M\left(\tau, J_{\underline{p}}, \underline{z}\right) \Phi_{\underline{p}}(\underline{z});$$

$$\left\|\Phi_{\{\underline{p}\}}\right\| = \textstyle\sum_{\underline{p} \in \{\underline{p}\}} c_{\underline{p}} \int_{Z(\underline{p})} \left\|\Phi_{\underline{p}}\right\| d_{\underline{p}}\underline{z} < \infty.$$

The inner product on $L_{\left\{\pi_{\underline{p}}\right\}}$ is given by the above norm.

THE PROOF

1. (local) Normalization of intertwining operators (Kazhdan and Patterson): If ρ° is an irreducible component of $\rho|_{\overline{H_v^\circ}}$, $\rho_{ij} = \left(\rho_i^\circ / \rho_j^\circ\right)^n$ is a character on $F_v^\times$. The operator $N(\sigma, \rho, \underline{s}) = r^{-1}(\sigma, \rho, \underline{s}) M(\sigma, \rho, \underline{s})$ is normalized, where $r(\sigma, \rho, \underline{s})$ equals

$$\prod_{i,j \in \mathrm{inv}(\sigma)} \frac{|n|_v^{1/2} L\left(n\left(s_i - s_j\right), \rho_{ij}\right)}{L\left(n\left(s_i - s_j\right) + 1, \rho_{ij}\right) \varepsilon\left(n\left(s_i - s_j\right), \rho_{ij}\right)}.$$

2. (local) Langlands quotient: If the exponent of ρ_{ij} is not less than 0 for any $i < j$, $\mathrm{Ind}_{\overline{H}}^{\overline{G}} \rho$ has a unique irreducible quotient which is isomorphic to the unique irreducible subrepresentation of $\mathrm{Ind}_{\overline{H}}^{\overline{G}} w(\rho)$. The quotient has multiplicity one in $\mathbf{JH}\left(\mathrm{Ind}_{\overline{H}}^{\overline{G}} \rho\right)$.

3. (local) Technical lemmas (Mœglin and Waldspurger): Study the meromorphic properties and images of certain normalized intertwining operators (combinatorial arguments).

4. (global) Representations of the diagonal group: The space $L^2\left(\overline{H}, \chi_H\right)$ is irreducible and isomorphic to $\mathrm{Ind}_{\overline{H^\circ}}^{\overline{H}}\rho^\circ$, where ρ° is the extension of $\chi_H|_{\overline{H^n}Z}$ to a maximal abelian subgroup $\overline{H^\circ} = \prod_v \overline{H_v^\circ}/*$. Check the isomorphism matches with the intertwining operators and the inner product (1). Reduce (1) to the study of the representations induced from $\overline{H^\circ}$.

5. (global) Contour integration:

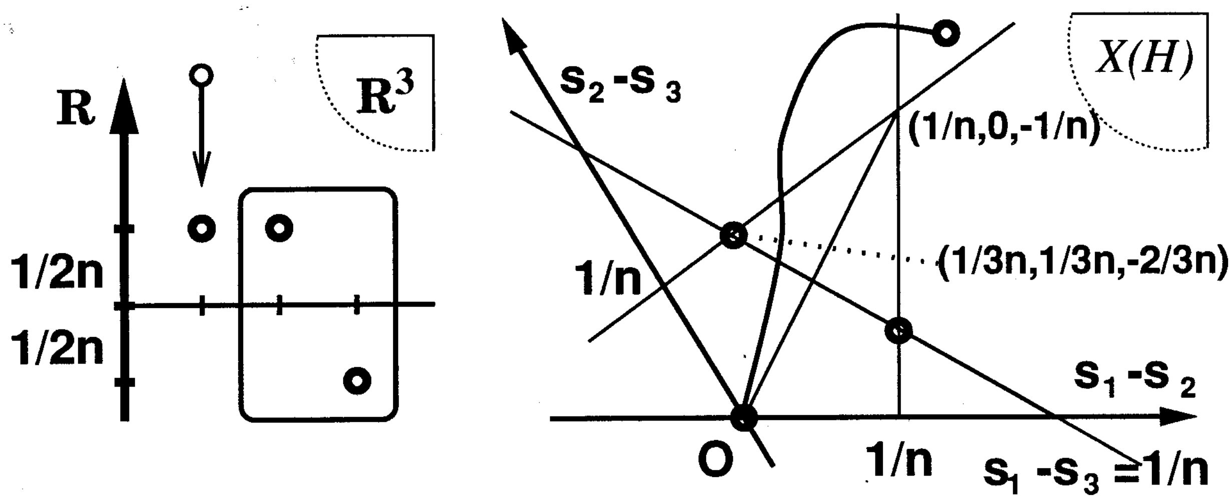
Preparations:

1) Singular hyper-planes are among the subspaces defined by $s_i - s_j = c \in \frac{1}{2n}\mathbf{Z}$, $i < j$.

2) The sum in (1) and its residues at the singular hyper-planes are analytic at the origins of these singular hyper-planes.

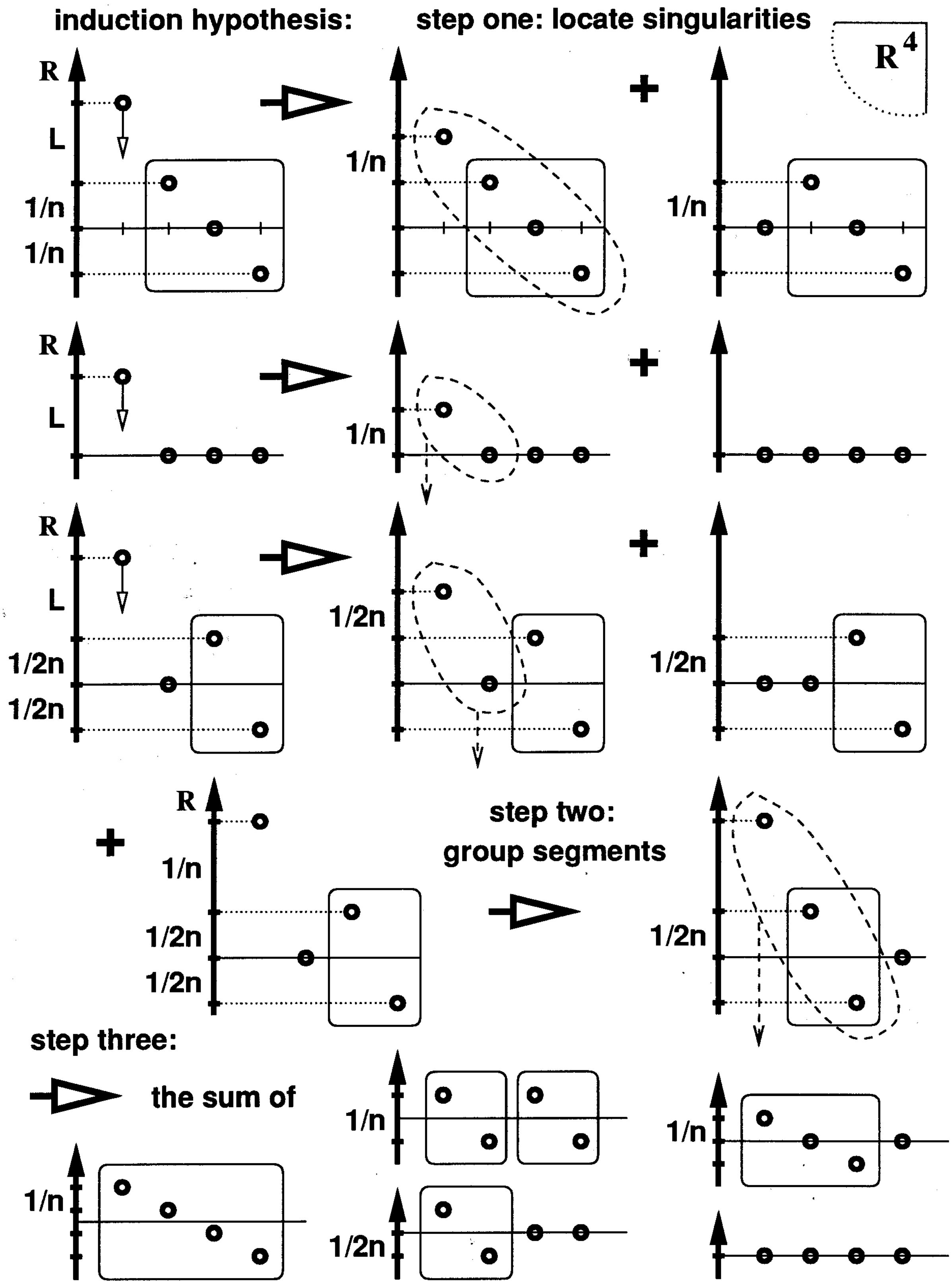
Both properties follow from properties of the L-function. They are predicted by Langlands' general theory on Eisenstein series.

3) They are analytic at the singular hyperplanes defined by non-linked segments.
We call an ordered set $\left\{\pi||^{a}, \pi||^{a-1/n}, \cdots, \pi||^{b}\right\}$ a segment ($a-b \in \frac{1}{2n}\mathbf{Z}$). Two such segments are called linked if neither contains the other and the union of them is again a segment. Item 3) follows from that if τ is a simple reflection, and $\rho^{\tau}=\rho$, then $M(\tau,\rho,\underline{0})=-1$.
EX: the situation for $\overline{GL(3)}$:



The figures illustrate the situation of two segments $\{\pi||^{s_1}\}$ and $\left\{\pi||^{1/2n}, \pi||^{-1/2n}\right\}$ when s_1 is pushed down at $1/2n$.

The Contour Integration by Induction:


induction hypothesis:
step one: locate singularities
R^4
R
L
1/n
1/n
1/2n
1/2n
step two:
group segments
step three:
the sum of