DETERMINE A C^*-CONVEX SET BY ITS NUMERICAL RANGE

Heng Sun
Department of Mathematics
University of Toronto
100 St. George St.
Toronto, Ontario
Canada M5S 1A1

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Abstract. For a C^*-convex set K, we denote $\sigma(K)$ the set $\{\lambda : \lambda \text{ is an eigenvalue of some } A \in K\}$. We prove the following proposition in this paper. If Λ is a compact convex set in the complex plane $\mathbf{C}$, then there is a unique C^*-convex set K such that $\sigma(K) = \Lambda$ if and only if Λ is a triangle (may be degenerate).

In this paper, we use the following notations:
$\mathbf{M}_n$ $(n \geq 2)$:the collection of all $n \times n$ matrices;
$\mathcal{W}(A) = \{(Ax, x) : x \in \mathbf{C}^n, \|x\| = 1\}$ for $A \in \mathbf{M}_n$;
$\sigma(A)$: the set of all eigenvalues of A (for $A \in \mathbf{M}_n$);
I_n: the identity matrix in $\mathbf{M}_n$.

We recall here two definitions concerning C^*-convex sets. A subset K of $\mathbf{M}_n$ is C^*-convex if $X_1, \cdots, X_k \in K$, and $A_1, \cdots, A_k \in \mathbf{M}_n$ with $\sum_{i=1}^k A_i^* A_i = I_n$, implies that $\sum_{i=1}^k A_i^* X_i A_i \in K$. Suppose K is a C^*-convex set. Then $Z \in K$ is a C^*-extreme point of K if whenever $Z = \sum_{i=1}^k A_i^* X_i A_i$, where $X_1, \cdots, X_k \in K$, and $A_1, \cdots, A_k \in \mathbf{M}_n$ with $\sum_{i=1}^k A_i^* A_i = I_n$ and each A_i is invertible, then each X_i is unitarily equivalent to Z. For basic properties of C^*-convex sets, refer [LP].

Define the spectrum of K to be

$$\sigma(K) = \{\lambda \in \sigma(A) : \text{for some} A \in K\} = \bigcup_{A \in K} \sigma(A),$$

and the numerical range of K to be

$$\mathcal{W}(K) = \{\lambda \in \mathcal{W}(A) : \text{for some} A \in K\} = \bigcup_{A \in K} \mathcal{W}(A).$$

It is easy to see that both sets are convex in the complex plane.

On the other hand, if Λ is a compact convex set in the complex plane, we set

$$K_0(\Lambda) = C^*\text{-convex hull of } \{\lambda I_n : \lambda \in \Lambda\},$$

and

$$K_1(\Lambda) = \{A \in M_n : \mathcal{W}(A) \subset \Lambda\}.$$

It is ready to see that $K_0(\Lambda)$ and $K_1(\Lambda)$ are C^*-convex. Furthermore, we have the following lemma.

Lemma 1. $K_0(\Lambda)$ is the smallest C^*-convex set K in M_n such that $\sigma(K) = \Lambda$. $K_1(\Lambda)$ is the largest C^*-convex set K such that $\sigma(K) = \Lambda$. Both $K_0(\Lambda)$ and $K_1(\Lambda)$ are closed.

Proof:

The first statement follows from the definition of C^*-convex hull (It is the intersection of all C^*-convex sets containing Λ). For the second statement, on the one hand,we notice that $\sigma(K_1(\Lambda)) = \Lambda$. On the other hand, suppose that $A \in K - K_1(\Lambda)$ for some C^*-convex set K with $\sigma(K) = \Lambda$. Without loss of generality, we can assume $0 \in \Lambda$, hence $O \in K$.

By the assumption for A, there is an $x = (x_1, x_2, ..., x_n) \in \mathbf{C}^n$, $\|x\| = 1$, s.t. $(Ax, x) \notin \Lambda$. Let

$$S = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 0 & \cdots & 0 \\ \cdots & & & \\ x_n & 0 & \cdots & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \cdots & \\ & & & 1 \end{pmatrix}$$

Then $S^*S + T^*T = I_n$. So

$$B = S^*AS + T^*OT \in K$$

Since (Ax, x) is an eigenvalue of B, $(Ax, x) \in \Lambda$. This contradicts that $(Ax, x) \notin \Lambda$. Hence $K \subset K_1(\Lambda)$.

That $K_1(\Lambda)$ is closed follows from the second statement immediately. That $K_0(\Lambda)$ is closed is a trivial corollary of 2.4 in [F]. *q.e.d.*

Lemma 2. For any closed C^*-convex set$K \subset \mathbf{M}_n$

$$\sigma(K) = \mathcal{W}(K)$$

Proof:
This can be seen by the following inclusion relations

$$\sigma(K) = \mathcal{W}(K_1(\sigma(K))) \supset \mathcal{W}(K)$$

$$\supset \mathcal{W}(K_0(\sigma(K))) \supset \sigma(K_0(\sigma(K))) \supset \sigma(K). \qquad q.e.d.$$

Theorem 1. If Δ is a triangle (may be degenerate) in $\mathbf{C}$, then there is only one C^*-convex set K such that $\mathcal{W}(K) = \Delta$.

This follows from Nakamura's Theorem. See [N]. The detailed proof will be given later.

Our purpose is to prove the negative of Theorem 1.

Theorem 2. If Λ is a compact convex set in $\mathbf{C}$ which is neither a triangle nor a degenerate triangle, then there are infinitely many C^*-convex sets K such that $\mathcal{W}(K) = \Lambda$.

Remark: By Theorem 2. for any Λ as stated in the theorem , we can find C^*-convex sets whose numerical ranges are Λ, and having structural elements of size $k \geq 2$. (Refer [Mo] for the definition of structural elements of size k.)

Combining Theorems 1 and 2, and Lemma 2, we get

Corollary. If Λ is a compact convex set in the complex plane $\mathbf{C}$, then there is a unique C^*-convex set K such that $\sigma(K) = \Lambda$ if and only if Λ is a triangle (may be degenerate).

We start from some transformations of C^*-convex sets.
For $a > 0$, define a transformation $\mathbf{T}_a$ of $\mathbf{M}_n$ as follow,

$$\mathbf{T}_a(A) = aRe(A) + \mathbf{i}Im(A), \quad \forall A \in \mathbf{M}_n$$

where $Re(A) = (A + A^*)/2$, $\qquad Im(A) = (A - A^*)/2\mathbf{i}$, $\qquad \mathbf{i}^2 = -1$.
If K is a C^*-convex set, we denote

$$\mathbf{T}_a(K) = \{\mathbf{T}_a(A) : A \in K\}.$$

If Λ is a convex set in $\mathbf{C}$, we also denote

$$\mathbf{T}_a(\Lambda) = \{aRe(\lambda) + \mathbf{i}Im(\lambda) : \lambda \in \Lambda\}$$

.

By direct computation, it is easy to see that if K is C^* -convex, then so is $\mathbf{T}_a(K)$. Furthermore, $\mathcal{W}(\mathbf{T}_a(K)) = \mathbf{T}_a(\mathcal{W}(K))$.

We can also define the following transformations (For convenience, we denote $A + a = A + aI_n$ for $A \in \mathbf{M}_n, a \in \mathbf{C}$.)

$$\lambda K = \{\lambda A : A \in K\}, \qquad \forall \lambda \in \mathbf{C}.$$

$$K + a = \{A + a : A \in K\}, \qquad \forall a \in \mathbf{C}.$$

$$\bar{K} = \{ReA - \mathbf{i}ImA : A \in K\}$$

Notice that any affine transformation of the complex plane is a finite composition of $\mathbf{T}_a$'s, multiplications by complex numbers, translations by complex numbers and a conjugation. So we can define an affine transformation $\mathbf{T}$ of a C^*-convex set K by

$$\mathbf{T}(K) = \mathbf{T}_1\mathbf{T}_2\cdots\mathbf{T}_k(K).$$

if $\mathbf{T} = \mathbf{T}_1\mathbf{T}_2\cdots\mathbf{T}_k$, where $\mathbf{T}_i$ are among the four types of affine transformations mentioned above. It is also not hard to see that this definition does not depend on the particular decomposition of $\mathbf{T}$. Indeed,if we write $z = x + \mathbf{i}y$, and think an affine transformation $\mathbf{T}$ as an action of $\mathbf{R}^2$ onto $\mathbf{R}^2$. Then $\mathbf{T}(x,y) = (f(x,y), g(x,y))$, where f, g are polynomials of degree 1. Hence $\mathbf{T}(A) = f(ReA, ImA) + \mathbf{i}g(ReA, ImA)$, for $A \in \mathbf{M}_n$. So if $\mathbf{T}_1, \ldots, \mathbf{T}_k$ is a decomposition of identity $\mathbf{1}$, then $\mathbf{T}_1 \ldots \mathbf{T}_k(x,y) = (x,y)$. This implies it acts on $\mathbf{M}_n$ as the identity.

Summarize what we did just now, we have the following lemma

Lemma 3. If $\mathbf{T}$ is an affine transformation and K is a C^*-convex set, the so is $\mathbf{T}(K)$. Furhtermore,

$$\mathcal{W}(\mathbf{T}(K)) = \mathbf{T}(\mathcal{W}(K)).$$

For example, if we denote $B(\alpha, r) = \{\lambda \in \mathbf{C} : |\lambda - \alpha| \le r\}$, then it is well known that $K_0(B(0,1)) = \{A \in M_n : \|A\| \le 1\}$. So by Lemma 3, $K_0(B(\alpha, r)) = \{a \in M_n : \|A - \alpha\| \le r\}$.

Proof of Theorem 1:

Suppose that $\xi_1, \xi_2, \xi_3$are the three vertices of Δ, and that $\mathbf{T}$ is an affine transformation of $\mathbf{C}$ bringing ξ_1, ξ_2, ξ_3 to three unimodule complex numbers $\zeta_1, \zeta_2, \zeta_3$. By Nakamura's Theorem (see [N]), if $A \in \mathbf{M}_n, \mathcal{W}(A) \subset \mathbf{T}(\Delta)$, then A is a C^*-combination of $\zeta_1, \zeta_2, \zeta_3$. This amounts to saying that $K_0(\mathbf{T}(\Delta)) = K_1(\mathbf{T}(\Delta))$. By Lemma 3, $K_0(\Delta) = K_1(\Delta)$. Now Theorem 1 folows from Lemma 1. *q.e.d.*

Lemma 4. If $A \in \mathbf{M}_2$, and $\mathcal{W}(A)$ is an elliptical disc inscribing a triangle which inscribes the unit disc with centre 0, then $\|A\| = 1$.

Proof:

By Mirman's Theorem [Mi](cf. [GS]), A is a compression of a normal operator U on a 3-dimensional Hilbert space $\mathcal{H}$ and U has eigenvalues on the unit disc, hence U is unitary. So $\|A\| \le \|U\| = 1$.

Choose a suitable basis of $\mathcal{H}$ such that the range of A is the subspace

$$\mathbf{C}^2 = \{x = (x_1, x_2, 0) : x_1, x_2 \in \mathbf{C}\}.$$

Then

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} U \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}.$$

We can always find a nonzero solution of

$$U\begin{pmatrix}x_1\\x_2\\0\end{pmatrix}=\begin{pmatrix}y_1\\y_2\\0\end{pmatrix},\qquad for\quad x_1,x_2,y_1,y_2.$$

Hence

$$\|A\begin{pmatrix}x_1\\x_2\end{pmatrix}\|=\|\begin{pmatrix}y_1\\y_2\end{pmatrix}\|=\|\begin{pmatrix}y_1\\y_2\\0\end{pmatrix}\|=\|U\begin{pmatrix}x_1\\x_2\\0\end{pmatrix}\|=\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\|$$

So we have $\|A\|=1$. *q.e.d.*

Lemma 5. Suppose that an elliptical disc B inscribes an equileteral triangle Δ with vertices 1, ω, ω^2, where $\omega^3=1$, and that $\Lambda\in\mathbf{C}$ is a compact convex set such that $B\subset\Lambda\subset\Delta, 1\notin\Lambda$. Then $K_1(B)\not\subset K_0(\Lambda)$.

Proof:
Suppose on the contrary, $K_1(B)\subset K_0(\Lambda)$.

Since $1\notin\Lambda$, we can choose a circular disc D_ϵ with center $-\epsilon$, passing the points ω,ω^2 and covering Λ for some $\epsilon>0$.

By assumption, $K_1(B)\subset K_0(D_\epsilon)=\{A\in\mathbf{M}_n:\|A+\epsilon\|\le|\omega+\epsilon|\}$. (See the example after Lemma 3.)

If we can find a matrix A such that $\mathcal{W}(A)=B, \|A+\epsilon\|>|\omega+\epsilon|$, then $A\in K_1(B)$, but $A\notin K_0(D_\epsilon)$. This contradiction shows that the lemma holds.

We prove $\|A+\epsilon\|>|\omega+\epsilon|$ by direct calculation. First we consider the case of $n=2$, i.e., all the C^*-convex sets considered are in $\mathbf{M}_2$. We choose A to be the following form

$$A=\begin{pmatrix}\lambda_1 & a\\ 0 & \lambda_2\end{pmatrix},\qquad(a\ge 0)$$

where λ_1,λ_2 are fosi of B [H,p.109]. By lemma 3, $\|A\|=1$.

$$(A+\epsilon)(A+\epsilon)^*=\begin{pmatrix}|\lambda_1+\epsilon\|^2+a^2 & a\left(\overline{\lambda_2}+\epsilon\right)\\ a(\lambda_2+\epsilon) & |\lambda_2+\epsilon|^2\end{pmatrix}$$

So $\|A+\epsilon\|^2$ is a root of

$$\left(x-|\lambda_1+\epsilon|^2-a^2\right)\left(x-|\lambda_2+\epsilon|^2\right)=a^2|\lambda_2+\epsilon|^2.$$

Since $\|A\|=1$, 1 is a root of the above equation when$\epsilon=0$. After differentiate both sides at $\epsilon=0$, and $x=1$, we have (Notice that $\frac{d}{d\epsilon}\left(|z+\epsilon|^2\right)=2\epsilon+2Rez$.)

$$(x'_\epsilon-2Re\lambda_1)\left(1-|\lambda_2|^2\right)+\left(1-|\lambda_1|^2-a^2\right)(x'_\epsilon-2Re\lambda_2)=2a^2Re\lambda_2$$

But $a^2=\left(1-|\lambda_1|^2\right)\left(1-|\lambda_2|^2\right)$ since $\|A\|=1$. So

$$\frac{d}{d\epsilon}\|A+\epsilon\|^2=x'_\epsilon=\frac{(2Re\lambda_1)\left(1-|\lambda_2|^2\right)+(2Re\lambda_2)\left(1-|\lambda_1|^2\right)}{1-|\lambda_1\lambda_2|^2}$$

($|\lambda_1\lambda_2|\ne 1$, otherwise B would be a line segment.)

But $\left[\frac{d}{d\epsilon}|\omega+\epsilon|^2\right]_{\epsilon=0}=-1$,

$$\frac{d}{d\epsilon}\left[\|A+\epsilon\|^2-|\omega+\epsilon|^2\right]_{\epsilon=0}=\frac{2(Re\lambda_1)\left(1-|\lambda_2|^2\right)+2(Re\lambda_2)\left(1-|\lambda_1|^2\right)}{1-|\lambda_1\lambda_2|^2}+1$$

$$=\frac{(1+2\lambda_1)(1+2Re\lambda_2)-(|\lambda_1|+2Re\lambda_1)(|\lambda_2|+2Re\lambda_2)}{1-|\lambda_1\lambda_2|^2}>0$$

But $\|A\| = |\omega| = 1$, so $\|A+\epsilon\| > |\omega+\epsilon|$ for small $\epsilon > 0$.

This is what we wanted.

If $n > 2$, suppose $\mu \in B$, and let

$$A = \begin{pmatrix} \lambda_1 & a & \\ & \lambda_2 & \\ & & \mu I_{n-2} \end{pmatrix}$$

Then $\|A+\epsilon\| > |\omega+\epsilon|$.

So the lemma holds for $n \geq 2$. *q.e.d.*

Lemma 6. Let $B = \{z \in \mathbf{C} : |z| \leq 1\}, \Lambda = \{z \in \mathbf{C} : |Rez| \leq M, |Imz| \leq 1\}$. Then $K_1(B) \not\subset K_0(\Lambda)$.

Proof:

As in the proof of last lemma, it is sufficient to prove the statement in $\mathbf{M}_2$.

Suppose the lemma is incerrect, then

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \in K_1(B) \subset K_0(\Lambda).$$

Hence

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = (M+\mathbf{i})\,A + (M-\mathbf{i})\,B + (-M+\mathbf{i})\,C + (-M-\mathbf{i})\,D, \tag{1}$$

where A, B, C, D are positive matrices with sum I_n. Suppose

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \qquad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \qquad D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}.$$

Consider the real and imaginary parts of (1,1) entry in (1). We have

$$a_{11} + b_{11} - c_{11} - d_{11} = 0, \qquad a_{11} - b_{11} + c_{11} - d_{11} = 0.$$

Besides, $a_{11} + b_{11} + c_{11} + d_{11} = 1$. Hence

$$b_{11} = \frac{1}{2} - a_{11}, \quad c_{11} = \frac{1}{2} - a_{11}, \quad d_{11} = a_{11}.$$

Similarly,

$$b_{22} = \frac{1}{2} - a_{22} \qquad c_{22} = \frac{1}{2} - a_{22} \qquad d_{22} = a_{22}.$$

Consider the (2,1) entries in (1)

$$(M+\mathbf{i})\,a_{21} + (M-\mathbf{i})\,b_{21} + (-M+\mathbf{i})\,c_{21} + (-M-\mathbf{i})\,d_{21} = 0. \tag{2}$$

Similarly,

$$(M+\mathbf{i})\,a_{12} + (M-\mathbf{i})\,b_{12} + (-M+\mathbf{i})\,c_{12} + (-M-\mathbf{i})\,d_{12} = 2. \tag{3}$$

Applying conjugation to the both sides, we have

$$(M-\mathbf{i})\,a_{21} + (M+\mathbf{i})\,b_{21} + (-M-\mathbf{i})\,c_{21} + (-M+\mathbf{i})\,d_{21} = 2 \tag{4}$$

Besides, $a_{21} + b_{21} + c_{21} + d_{21} = 0$.

(2),(3),(4) give the solution

$$b_{21} = \frac{1}{M} - a_{21}, \qquad c_{21} = -\frac{\mathbf{i}}{2} - a_{21}, \qquad d_{21} = a_{21} - \frac{1}{M} + \frac{\mathbf{i}}{2}.$$

So

$$A=\begin{pmatrix} a_{11} & * \\ a_{21} & a_{22} \end{pmatrix}; B=\begin{pmatrix} \frac{1}{2}-a_{11} & * \\ \frac{1}{M}-a_{21} & \frac{1}{2}-a_{22} \end{pmatrix}; C=\begin{pmatrix} \frac{1}{2}-a_{11} & * \\ -\frac{\mathbf{i}}{2}-a_{21} & \frac{1}{2}-a_{22} \end{pmatrix}; D=\begin{pmatrix} a_{11} & * \\ a_{21}-\frac{1}{M}+\frac{\mathbf{i}}{2} & a_{22} \end{pmatrix}.$$

A, B, C, D are positive, hence

$$a_{11}a_{22} \geq |a_{21}|^2; \tag{5}$$

$$\left(\frac{1}{2}-a_{11}\right)\left(\frac{1}{2}-a_{22}\right) \geq |\frac{1}{M}-a_{21}|^2; \tag{6}$$

$$\left(\frac{1}{2}-a_{11}\right)\left(\frac{1}{2}-a_{22}\right) \geq |1-\frac{\mathbf{i}}{2}-a_{21}|^2; \tag{7}$$

$$a_{11}a_{22} \geq |a_{21}-\frac{1}{M}+\frac{\mathbf{i}}{2}|^2; \tag{8}$$

$$a_{11} \leq \frac{1}{2}; \qquad a_{22} \leq \frac{1}{2}. \tag{9}$$

Observe that (by (5), (7))

$$|a_{21}| + |-\frac{\mathbf{i}}{2}-a_{21}|$$

$$\leq \sqrt{a_{11}a_{22}} + \sqrt{\left(\frac{1}{2}-a_{11}\right)\left(\frac{1}{2}-a_{22}\right)}$$

$$\leq \frac{1}{2}(a_{11}+a_{22}) + \frac{1}{2}(1-a_{11}-a_{22}) \leq \frac{1}{2}$$

. So the equalities must hold. Hence

$$a_{11} = a_{22} = |a_{21}|, \tag{10}$$

$$\frac{1}{2}-|a_{21}| = \frac{1}{2}-a_{11} = |-\frac{\mathbf{i}}{2}-a_{21}|, \tag{11}$$

By (6), (8),

$$\frac{1}{2} = \left(\frac{1}{2}-a_{11}\right) + a_{11} \geq |\frac{1}{M}-a_{21}| + |\frac{\mathbf{i}}{2}-\left(\frac{1}{M}-a_{21}\right)|.$$

By the triangle inequality, the equality must hold, so

$$\frac{1}{2}-|a_{21}| = \frac{1}{2}-a_{11} = |\frac{1}{M}-a_{21}|, \tag{12}$$

$$|a_{21}| = a_{11} = |a_{21}-\frac{1}{M}+\frac{\mathbf{i}}{2}|, \tag{13}$$

By (11) and (12),

$$|\frac{1}{M}-a_{21}| = |-\frac{\mathbf{i}}{2}-a_{21}|. \tag{14}$$

But (13) and (14) gives us only one solution $a_{21} = \frac{1}{2M}+\frac{\mathbf{i}}{4}$, which does not fit (11).

So (1) has no solution, which implies the lemma. *q.e.d.*

Lemma 7. Suppose K is a closed C^*-convex set. Let

$$K_t(\Lambda) = (1-t)\,K_0(\Lambda) + tK_1(\Lambda), \qquad \text{for} \quad t \in [0,1].$$

$$s = \sup\{t \in [0,1] : K_t(\mathcal{W}(K)) \subset K\}.$$

Then $K_s(\mathcal{W}(K)) \subset K$.

Proof: Suppose $A \in K_s(\mathcal{W}(K))$, then $A = (1-s)A_0 + sA_1$, where $A_0 \in K_0(\mathcal{W}(k)), A_1 \in K_1(\mathcal{W}(K))$. Since K is closed, so

$$A = \lim_{t \to s-} [(1-t)A_0 + tA_1] \in K.$$

The lemma folows. *q.e.d.*

Proof of Theorem 2:

We first prove $K_1(\Lambda) \neq K_0(\Lambda)$.

Suppose that B is the disc inscribing Λ with the maximal radius. Then two cases arise.

Case 1. There are only two points on the intersection of the boundary of Λ and the boundary of B. Then the supporting lines of Λ at these two points are parallel to each other. Since Λ is compact, we can find a rectangle M with the two supporting lines as opposite sides such that $B \subset \Lambda \subset M$. Suppose $\mathbf{T}$ is the affine transformation moving B to $\mathbf{T}(B)$ =the unit disc with center 0, and moving the two supporting lines to the lines parallel to the real axis. Then by Lemma 6

$$K_1(\mathbf{T}(B)) \not\subset K_0(\mathbf{T}(M)).$$

So

$$\mathbf{T}(K_1(B)) = K_1(\mathbf{T}(B)) \not\subset K_0(\mathbf{T}(M)) = \mathbf{T}(K_0(M)).$$

Hence, $K_1(B) \not\subset K_0(M)$.

But $\Lambda \subset M$, so $K_1(B) \not\subset K_0(\Lambda)$. Hence $K_1(B) \not\subset K_0(B)$ since $K_0(B) \subset K_0(\Lambda)$.

Case2. If Case 1 does not occur, there must be three points on the intersection of the boundary of Λ and the boundary of B, and the three supporting lines of Λ at these three points form a triangle Δ containing Λ, i.e., $B \subset \Lambda \not\subset \Delta, \Lambda \neq \Delta$. By Lemma 4 and the similar argument using affine transformations in Case 1, we have $K_0(\Lambda) \neq K_1(\Lambda)$.

So if Λ is not a triangle, then $K_1(\Lambda) \neq K_0(\Lambda)$. It follows that $K_1(\Lambda) \neq K_t(\Lambda)$ for $0 \leq t < 1$. In fact, by [Mo], there is a structural element of size $k \geq 2$, which is a direct summand of some C^*-extreme point $A \in K_1(\Lambda)$. Then it is impossible to write $A = tB + (1-t)C$ for $0 \leq t < 1, B \in K_1(\Lambda), C \in K_0(\Lambda)$. Hence $A \notin K_t(\Lambda)$.

If we denote

$$K_t(\Lambda) = tK_1(\Lambda) + (1-t)K_0(\Lambda),$$

By Lemma 1, it is a closed set. By the above argument and Lemma 7, we can partition [0,1] into right closed intervals $\{I_\alpha, \{1\}\}$ such that for all $t \in I_\alpha, K_t(\Lambda)$ are the same. Then $\{I_\alpha\}$ gives a partition of [0,1), which is impossible unless $\{\alpha\}$ is an infinite set. This proves Theorem 2. *q.e.d.*

An Example. Suppose Λ is the square $\{z \in \mathbf{C} : |Rez| \leq 1, |Imz| \leq 1\}$, $K = K_1(\Lambda)$. We claim that $J = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ is a nonnormal C^*-extreme point in K.

First we observe that if $A = \sum_{i=1}^k T_i^* B_i T_i$ with $\sum_{i=1}^k T_i^* T_i = 1$, then $\|A\| \leq \max\{\|B_i\|, i = 1, \cdots, k\}$. This can be seen by writing $A = (T_1^*, \cdots, T_k^*) \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix} \begin{pmatrix} T_1 \\ \vdots \\ T_k \end{pmatrix}$. Hence

$$\|A\| = \|(T_1^*, \cdots, T_k^*)\| \cdot \| \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix} \| \cdot \| \begin{pmatrix} T_1 \\ \vdots \\ T_k \end{pmatrix} \|$$

$$\leq \max\|B_i\| \cdot \|(T_1^*, \cdots, T_k^*)\| \cdot \| \begin{pmatrix} T_1 \\ \vdots \\ T_k \end{pmatrix} \| = \max\|B_i\|.$$

Suppose J is not a C^*-extreme point. By proposition 3.4.10 [Mo], $J = \sum_{i=1}^k S_i^* A_i S_i$ with $\sum_{i=1}^k S_i^* S_i = I$, and for each i, a compression of A_i is a structural element of K. Hence none of A_i is unitarily equivalent to J. Suppose $\|A_1\| \geq \|A_i\|$ for $i \neq 1$. Then by the above observation, $\|A_1\| \geq \|J\| = 1$.

By lemma17 [LP], we can rewrite $J = T_1^* A_1 T_1 + T_2^* B T_2$, where $T_1^* T_1 + T_2^* T_2 = I$, and $B \in K$. Hence the unit disc is in the convex hull of the ellipse $\mathcal{W}(A_1)$ and $\mathcal{W}(B)$ by lemma2.3.[HMP]. By elementary geometry argument, $\mathcal{W}(A_1)$ and $\mathcal{W}(B)$ must inscribe the square Λ.

Hence the two engenvalues of A_1 are on the line $Rez = Imz$ or the line $Rez = -Imz$. Suppose the former(The proof for the later is similar), then we can further suppose $A_1 = \begin{pmatrix} t(1+\mathbf{i}) & \mu \\ 0 & -t(1+\mathbf{i}) \end{pmatrix}$, where $\mu, t \geq 0$. Since A_1 is not unitarily equivalent to $J, t \neq 0$, so $0 < t \leq 1$.

We claim $\|A_1\| < 1$. This contradicts $\|A_1\| \geq 1$ we have already got. Hence J must be a C^*-extreme point.

By [H](p109), we can calculate the axes of the ellipse $\mathcal{W}(A_1)$. The result is

$$\text{minor axis} = \mu; \qquad \text{major axis} = \sqrt{\mu^2 + 8t^2}.$$

Hence the equation of the ellipse $\mathcal{W}(A_1)$ is (in x-y plane)

$$\frac{2(x+y)^2}{\mu^2 + 8t^2} + \frac{2(x-y)^2}{\mu^2} = 0.$$

It must inscribe the square Λ, so by simple calculations, $\mu^2 + 4t^2 = 4$.

If we calculate the norm of $A_1 = \begin{pmatrix} t(1+\mathbf{i}) & \mu \\ 0 & -t(1+\mathbf{i}) \end{pmatrix}$, we find

$$\|A_1\|^2 = \frac{1}{2}\left(\left(\mu^2 + 4t^2\right) + \sqrt{\left(\mu^2 + 4t^2\right)^2 - 16t^4} \right)$$
$$= 2 + 2\sqrt{1 - t^4} < 4.$$

So $\|A_1\| < 2$, which is what we wanted to prove.

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