Math G63.2045 Computational Methods for Finance
Final Project

P Botha, D Coluccio, C Nelson, H Sun

December 18, 2007

Contents

1 Introduction

1.1 Summary

The market standard for CDO pricing is to use a one-factor Gaussian copula to model defaults through time. This model gives prices based on a correlation parameter ρ. Unfortunately, for a single inputted ρ parameter, it becomes clear when we look at data that this model does not fit the market very well.

In its defense, however, a practitioner might say that its usefulness is that for any particular equity tranche price, we can back out at most one ρ parameter, which we call the ‘base correlation’. From these, we obtain a correlation curve which can be interpolated, like the implied volatility surface in option markets. However, as of late 2007, this practice is giving correlation values over 100% at upper attachment points, and so even this tool has become suspect.

We therefore see the need for an improved model of the loss distribution. We will try to do this by looking at one-factor models with distributions other than Gaussian, but which preserve the correlation parameter idea. Moreover, we will allow the loss given default of each particular name to be a random variable, with an interdependence structure with the random variables that cause defaults.

Broadening the scope of models like this introduces an interesting computational problem. The techniques used to implement pricing with the Gaussian one-factor model do not easily extend to other cases. Therefore, we use a fast fourier transform method, which naturally allows pricing with the broad class of models mentioned above, and also allows underlying indices which can have either homogeneous or heterogeneous weightings.

Although research has been done on all these aspects separately, we have not encountered a paper which combines them into a general treatment, so we hope we have something unique to offer. One convenient test of whether our models offer significant improvement, over the plain Gaussian model, is to compare implied base correlation curves. As a good approximation, the flatter the curve, the better the model fits.

In addition to all of this, we incorporate into our models a largely ignored problem of accounting for not only the loss given default, but also the recovery given default. Although this can be neglected for standard base correlation fitting, it is an important issue for market participants, especially when pricing senior tranches.

2 A General One Factor Model for the Loss Distribution

We have a portfolio of assets, some of which will be either completely or partially lost as a result of default. We try to model the proportion of our portfolio that is lost at some point in the future. We call this the 'portfolio loss rate', and denote it by the random variable L.

X_k is a random variable which indicates whether or not default has occurred for asset k, by taking the values 1 or 0 respectively.

LGD_k is the proportion of asset k that is lost when default occurs. This is some random number in [0,1].

s_k is the fixed weight of asset k in the portfolio.

Then,

$$L = \sum_{k=1}^{N} s_k X_k \mathrm{LGD}_k \tag{1}$$

We can model the joint distribution of the $X_k, \mathrm{LGD}_k, 1 \leq k \leq N$, using the following one-factor model:

For each asset k, there is a random variable A_k such that

$$A_k = W + Y_k \tag{2}$$

and a fixed number α_k such that

$$X_k = \begin{cases} 1 & \text{if} \quad A_k < \alpha_k \\ 0 & \text{if} \quad A_k \geq \alpha_k \end{cases} \tag{3}$$

We suppose that the 'idiosyncratic factors' Y_k are i.i.d., and that the 'common factor' W is independent of all of these. Therefore the A_k are identically distributed - but not independent. However, $A_k|W$ are i.i.d.

Then to model the individual asset loss in the event of default, we will use the same factor W as follows:

$$\mathrm{LGD}_k = G(W) + H_k \tag{4}$$

where G maps the range of W to some interval $[a, b] \subseteq [0, 1]$, and H_k are i.i.d. random variables such that $-a \leq H_k \leq 1 - b$. This formulation guarantees $0 \leq \mathrm{LGD}_k \leq 1$, and also that given W, the LGD_k are i.i.d., and independent of A_k.

We now think of H_k as the 'idiosyncratic part', and $G(W)$ as the common factor.

3 The Fourier Transform of the Loss Distribution

3.1 Deriving the Fourier Transform

It is not immediately obvious what the distribution of L is. The easiest way to figure this out is to look at the function

$$\hat{f}_L(u) = \mathrm{E}[e^{-iLu}] \tag{5}$$

where E is the expectation operator.

This is the characteristic function of L (or by most people's definition, the reflection of it around $u = 0$). It can also be thought of as the Fourier transformation of the density of L, when the density exists, and so for a general random variable X, we will call $\hat{f}_X$, the 'Fourier transform for X'. Theoretically, knowing this is equivalent to knowing the distribution.

By the tower law, and using the independence of X_k, LGD_k given W,

$$\hat{f}_L(u) = E[E[\exp(-iLu)|W]] = E[\prod_{k=1}^{N} E[\exp(-is_k X_k \mathrm{LGD}_k u)|W]] \tag{6}$$

Now, let's define $p_k(w) = P(Y_k < \alpha - w)$. Then given W, the event $X_k = 1$, which is equivalent to the event $A_k (= W + Y_k) \le \alpha$, has probability $p_k(W)$.

Then, looking at the inner expectation, and using laws of conditional expectation and $\mathrm{LGD}_k = G(W) + H_k$,

$$\begin{aligned} E[\exp(-is_k X_k \mathrm{LGD}_k u)|W] &= E[E[\exp(-is_k X_k \mathrm{LGD}_k u)|W, X_k]|W] \\ &= (1 - p_k(W)) + p_k(W) E[\exp(-is_k \mathrm{LGD}_k u)|W] \\ &= 1 + p_k(W)(\exp(-is_k G(W) u) E[\exp(-is_k H_k u)] - 1) \end{aligned} \tag{7}$$

Let $\hat{f}_H(u) = E[\exp(-iuH_k)]$ be the Fourier transform for all H_k. Then substituting (7) into (6) gives:

$$\hat{f}_L(u) = E[\prod_{k=1}^{N} (1 + p_k(W)(\hat{f}_H(s_k u) e^{-is_k G(W) u} - 1)] \tag{8}$$

3.2 Requirements to Obtain the Fourier Transform

Let us suppose that W has a density f_w. Then, we can write the expression (8) as

$$\hat{f}_L(u) = \int_{-\infty}^{\infty} \prod_{k=1}^{N} (1 + p_k(w)(\hat{f}_H(s_k u) e^{-is_k T(w) u} - 1) f_w(w) dw \tag{9}$$

In order to calculate $\hat{f}_L$, we therefore need $f_w(w)$, $p_k(w)$, $\hat{f}_H(u)$, $T(w)$ and s_k. By the definition of $p_k(w)$, we can get this by knowing $F_Y(y)$, the cdf for Y_k (remember that all the Y_k are identically distributed), and α_k.

A final point is that in general the value of the α_k is not obvious, and we will usually have some market given or modeled default probabilities q_k, where $q_k = P(X_k = 1)$. So, if we let F_A^{-1} be the inverse cdf for A_k, then looking back at (3) and solving gives

$$\alpha_k = F_A^{-1}(q_k) \tag{10}$$

4 Special Cases

Listed below are some special cases of interest. Note that they are overlapping, in the sense that each one might specify different aspects of what contributes to the distribution of L, so that later we can combine some of these cases.

4.1 Homogeneous Weighings

Let us suppose that all $q_k = q$ and $s_k = s = 1/N$ for all k. In other words assume we have a portfolio of assets such that they all have the same weighting and default probability. This is often the case with market traded indices, for example, the iTraxx.

In addition to the above assumptions, we could also say that in the event of default, the portfolio is guaranteed to lose all of the asset, so $\mathrm{LGD}_k = 1$. (Let's represent this as $H_k = 1$ and $T \equiv 0$). In this case we get a very simple form of (9):

$$\hat{f}_L(u) = \int_{-\infty}^{\infty} (1 + p(w)(e^{-\frac{iu}{N}} - 1))^N f_w(w) dw \tag{11}$$

4.2 Gaussian Loss Model

We choose a parameter ρ, and say that that $W \sim N(0, \rho)$ and $Y_k \sim N(0, 1 - \rho)$. Therefore, $A_k \sim N(0, 1)$, by additivity of normals.

This is the standard model discussed in the introduction. One advantage of this model is that we can write $W = \rho Z$, and therefore can think of the parameter ρ as specified independently of the factor Z, which we can think think of as an economic factor. We will see the relevance of this later in terms of base correlation.

4.3 Levy Loss Model

In [3], we again specify ρ, and now also specify $\mathcal{L}(t)$, which is the distribution of a given Levy process at time t. Then, $W \sim \mathcal{L}(\rho)$ and $Y_k \sim \mathcal{L}(1 - \rho)$. By additivity of increments of a Levy process, $A_k \sim L(1)$. (Note that Moody's Gaussian model is a special case, using Brownian motion as the Levy process).

Note that in this model, it is not possible in general to write W as $\rho\times$(something independent of rho), so we lose this convenience.

A nice case of the above Levy model is to use $W = \sqrt{a}\rho - G_\rho$, where G_ρ is a Gamma$(a, \sqrt{a})$ process evaluated at time ρ. Y_k and A_k are defined correspondingly. The full details of this model are spelled out on pg. 5 of [3], and is referred to as the 'Shifted Gamma Model'.

4.4 Beta Loss Given Default

Given the common factor W, a very natural and workable choice of LGD_k is

$$\mathrm{LGD}_k = 1 - F_\beta^{-1}(F_W(W)) \tag{12}$$

where F_β is the CDF of the beta distribution $\text{Beta}(a,b)$, and F_W is the CDF of the common factor W. In other words, in the notation of Section 2, we have $G = F_\beta^{-1} \circ F_W$ and $H_k = 0$.

Note that if we set $U = F_W(W)$, then $U \sim \text{Uniform}(0,1)$, and $1 - \text{LGD}_k = F_\beta^{-1}(U)$, so $1 - \text{LGD}_k \sim \text{Beta}(a,b)$. We also have $W = F_W^{-1}(U)$ and $\text{LGD}_k = 1 - F_\beta^{-1}(U)$. This is convenient, since we can then take the expectation over U to get:

$$E[\text{LGD}_k \times W] = \int_0^1 \left(1 - F_\beta^{-1}(x)\right) F_W^{-1}(x) dx \tag{13}$$

Then given the expectation and variance of W and LGD_k, we can explicitly determine the correlation between the these two variables, which should generally be negative. This is because as the common factor W decreases (which we'd can associate with worse economic conditions), not only would the default rate increase, but so would LGD_k.

To summarize the above model has the following useful properties:

- LGD_k follows a beta distribution, whose mean and standard deviation can be specified by us (according to historical data).
- LGD_k has correlation with the default rate common factor, which is generally negative.
- LGD_k results in continuous loss distribution.

Our loss distribution in (9) becomes:

$$\hat{f}_L(u) = \int_{-\infty}^{\infty} \prod_{k=1}^{N} \left(1 + p_k(w) \left(e^{-its_k\left(1 - F_\beta^{-1}(F_W(w))\right)} - 1\right)\right) f_W(w) dw \tag{14}$$

Note that we could extend this case quite easily to allow for some idiosyncratic loss for each k. To do this, we could shrink the above beta distribution by a linear transformation, so that, for example, G is constrained between 10% and 90%, and then add a H_k idiosyncratic part which could vary between -10% and $+10\%$. This would be a welcome improvement, but for initial modeling, will not offer significantly different results.

5 The Recovery Distribution

It is important to also consider the recovery distribution. The relevance of this will be explained in the pricing section. The recovery is the amount that is recovered on defaulted assets. Therefore, it is represented by the random variable:

$$R = \sum_{k=1}^{N} s_k X_k (1 - \text{LGD}_k) \tag{15}$$

We see that it has exactly the form of the loss distribution L, except that the factor in the event of a loss is $(1 - \mathrm{LGD}_k)$, instead of LGD_k. Therefore, using the above model for LGD_k, we have

$$1 - \mathrm{LGD}_k = 1 - G(w) - H_k \tag{16}$$

To derive the Fourier transform, we use the same steps as for loss and referring to (7), the

$$E[\exp(-is_k \mathrm{LGD}_k u)|W] \tag{17}$$

term is now

$$E[\exp(-is_k(1 - \mathrm{LGD}_k)u)|W] = \exp(-is_k u)\exp(is_k G(w)u)E[\exp(is_k H_k u)] \tag{18}$$

and therefore similarly to (8), we get the characteristic function

$$\hat{f}_R(u) = E[\prod_{k=1}^{N}(1 + p_k(W)(\hat{f}_H(-s_k u)e^{-is_k(1-G(w))u} - 1)] \tag{19}$$

6 From the Fourier Transform to the Probability Density

At this point, we know how to get the Fourier transforms of the distributions relevant for CDO pricing. However, in order to be able to price, we first need to convert this Fourier transform into a probability function. This is not easily done analytically, so we use numerical techniques, specifically FFT (Fast Fourier Transform). This algorithm can then be applied to either the loss distribution or the recovery distribution.

For our project, we implement two different versions of the FFT. The first case assumes that all the above functions are chosen so that the loss distribution is discrete. For example, we might restrict the loss to be a multiple of 0.1%. The second case assumes that the loss distribution can take any value between 0 and 1, and does not have mass at any point (except 0), so it is an almost continuous distribution.

The advantage of having both these implementations is that we can try a wide variety of possible distributions, and also check the results of one against the other, since we can always approximate a continuous distribution by a discrete one, and vice versa. We will now detail the implementation of these two methods.

6.1 Discrete Fourier Transform

We consider a discrete probability mass function f. The function is non-zero only at a multiple of some Δx. Suppose there are M such points on $[0, 1)$. In other words, $M = \frac{1}{\Delta x}$. Then its Fourier transform can be written in a discrete form.

$$\hat{f}(t) = \sum_{k=1}^{M+1} f_k e^{-it(k-1)\Delta x} \tag{20}$$

Let $\Delta t = 2\pi$. Then, consider any $1 \leq j \leq M$. Plugging into the above formula gives:

$$\hat{f}((j-1)\Delta t) = \sum_{k=1}^{M+1} f_k e^{-i(j-1)\Delta t(k-1)\Delta x} = \sum_{k=1}^{M+1} f_k e^{-\frac{2\pi i(j-1)(k-1)}{M}} \tag{21}$$

This equation is now in a form ready for FFT. Specifically, we can apply the matlab function `ifft()`. This means that given the values $\hat{f}((j-1)\Delta t)$ (which we will generally know, based on the previous sections), we can calculate the f_k with very little computational time.

The original distribution can then be recovered by

$$f(x) = \sum_{k=1}^{M+1} f_k \delta_{(k-1)\Delta x}(x) \tag{22}$$

Note that a large part of the speed of FFT is due to the degree to which $M+1$ can be factorized. The best case is if there some m such that $M+1 = 2^m$. The worst case is if $M+1$ is a prime number. So we should adjust M appropriately, by possibly including zero probability points past 1, to account for this.

6.2 Continuous Fourier Transform

The only obstacle to applying the FFT is from the portfolio loss rate spike at zero; however this can easily be removed. As observed in the Moody's FTM paper (page 31), we have

$$\begin{aligned} f_L(x) &= \hat{f}_\infty \delta(x) + f_{\text{BWL}}(x) & (23)\\ \hat{f}_\infty &= \int_{-\infty}^{\infty} \prod_{k=1}^{N} (1 - p_k(w))\, f_W(w) dw & (24)\\ \hat{f}_{\text{BWL}}(x) &= \hat{f}_L - \hat{f}_\infty & (25) \end{aligned}$$

where $\delta(x)$ is the Dirac function. So $\hat{f}_{\text{BWL}}$ is known function disappearing at infinity. We need to compute f_L via f_{BWL}.

To this end, assume $\hat{f}_{\text{BWL}}$ is close to 0 outside $[-T, T]$ for some number T. It then has Fourier expansion on $[-T, T]$ given by

$$\hat{f}_{\text{BWL}}(t) = \sum_{k=-\infty}^{\infty} c_k e^{-i\frac{\pi t}{T}(k-1)} \tag{26}$$

Since we assume f_{BWL} has support on $[0, V]$ for some $V \leq 1$, the above sum is actually a finite sum for k between 1 and $TV/\pi + 1$. Let $n = 2^m$ be a power of 2 that is larger than TV/π. Partition $[-T, T]$ into n intervals of equal lengths. Then we have the following discretization of the above equation.

$$\hat{f}_{\text{BWL}}\left(-T + \frac{2T(j-1)}{n}\right) = \sum_{k=1}^{n} c_k e^{-i\pi\left(-1+\frac{2(j-1)}{n}\right)(k-1)} = \sum_{k=1}^{n} c_k (-1)^{k-1} e^{-\frac{2\pi i}{n}(j-1)(k-1)} \tag{27}$$

The coefficients c_k are ready to be solved by MatLab FFT. Then we can compute f_{BWL}.

$$f_{\text{BWL}}(x) \;=\; \frac{T}{\pi}\sum_{1\le k\le \frac{TV}{\pi}+1} c_k \text{sinc}\left(\frac{Tx}{\pi}-(k-1)\right) \tag{28}$$

7 Pricing a CDO

7.1 Loss and Recovery Process

The model we developed in the previous sections was for some arbitrary time. We can therefore suppose that we have such a model for all times $t > 0$, and consider the loss and recovery processes $L(t)$ and $R(t)$ respectively. Note that all input parameters to specify the loss distribution (see Section 3.2) are now time dependent.

At time zero, the portfolio starts with a notional of $N(t) = 1$. If an asset k defaults, this notional will decrease by $s_k(t)$, the weighting of the asset at the time of default. (Remember that all parameters are now time dependent, although generally s_k remains constant through time, and as we will see later, we will assume many of our other inputs do too).

Since the default amount is divided into recovery and loss, we clearly have the relation:

$$N(t) = 1 - R(t) - L(t). \tag{29}$$

7.2 CDO Cash Flows

We price a CDO tranche with attachment point a and detachment point d. We assume that the buyer of the tranche pays a continuous coupon of c, an annualized parameter expressed as a percentage of the notional, and in return receives (immediately) any losses which occur in the specified tranche.

Note however that the notional used as a reference for the coupon payment is now tranche specific. This notional determines the amount of coupon that needs to be paid for the tranche. Before $L(t)$ hits the attachment point a, there is no reason for the coupon to change. However, once $L(t)$ has gone through a, the buyer of the CDO would be paying for less future protection (since part of the tranche has already been eaten up), so the coupon paid should decreases by the appropriate amount.

We also need to account for the recovery $R(t)$ by decreasing the notional amount outstanding. The adjustment process will begin with the senior-most trance and work its way down the capital structure of the CDO from the detachment points. The reason is that if some part of the notional $N(t)$ disappears without contributing to losses (in other words, if $R(t)$ increases), this puts a cap on how far the loss can go. If this cap is below the detachment point d, then the protection required on the tranche has decreased, so the coupon should decrease by the appropriate amount once again.

We therefore have tranche specific loss, recovery, and notional random processes, which are given by:

$$\begin{aligned}
L_{[a,d]}(t) &= (L(t)-a)^{+} - (L(t)-d)^{+} && (30)\\
&= \min(L(t),d) - \min(L(t),a) && (31)\\
R_{[a,d]}(t) &= (R(t)-(1-d))^{+} - (R(t)-(1-a))^{+} && (32)\\
&= \min(R(t),1-a) - \min(R(t),1-d) && (33)\\
N_{[a,d]}(t) &= d-a-L_{[a,d]}(t) - R_{[a,d]}(t) && (34)
\end{aligned}$$

Therefore we we have the following pricing formula

$$\begin{aligned}
\int_0^T P(0,t)E\left(dL_{[a,d]}(t)\right) &= c\int_0^T P(0,t)E\left(N_{[a,d]}(t)\right)dt && (35)\\
E\left[L_{[a,d]}(t)\right] &= \int_a^d (x-a)f_L(x)dx + (d-a)\int_d^1 f_L(x)dx && (36)\\
E\left[R_{[a,d]}(t)\right] &= \int_{1-d}^{1-a} (x-(1-d))\,f_R(x)dx + (d-a)\int_{1-a}^1 f_R(x)dx && (37)\\
E\left[N_{[a,d]}(t)\right] &= d-a-E\left[L_{[a,d]}(t)\right] - E\left[R_{[a,d]}(t)\right] && (38)
\end{aligned}$$

where $P(0,t)$ is the discount factor (the price of a zero coupon bond that matures at t). The left hand side in the first equation is total cashflow of the floating leg. The right hand side is the fixed leg. The notional amount for the tranche at time t is denoted by $N_{[a,d]}(t)$.

The price is represented by the spread, or coupon, c. We see that if we know the the $L(t)$ and $R(t)$ distributions then by using the above equations, we are able to price.

The above equations of this need to be discretized. We use time-grid $\{T_j\}_{j=1}^{Q}$ coinciding with the fixed leg coupon payment dates. The two sides of equation (35) give the floating and fixed legs of the cashflows, from which we obtain the following discretization.

$$\begin{aligned}
&\sum_{j=1}^{Q} P\left(0,\frac{T_j+T_{j-1}}{2}\right)\left(E\left[L_{[a,d]}(T_j)\right] - E\left[L_{[a,d]}(T_{j-1})\right]\right) && (39)\\
=\ &\frac{c\delta}{2}\sum_{j=1}^{Q} P(0,T_j)\left(E\left[N_{[a,d]}(T_j)\right] + E\left[N_{[a,d]}(T_{j-1})\right]\right) && (40)
\end{aligned}$$

where δ is the length, measured in years, between two coupon payment.

8 Base Correlation and Calibration

There are many ways to test how well a model fits. In general, we could consider, for example, the sum of squares of differences between the market prices for each tranche and our modeled prices. This would be standard least squares regression, and we could compare models by this sum of squares number.

However, given the importance that 'Base Correlation' has assumed over the last few years in the CDO markets, we may use this concept to provide an alternative method for comparing two models. First, base correlation will be explained, and subsequently we will outline the method of model comparison and its advantages.

8.1 Parameter ρ

We refer back to Section 2, where W and Y_k are independent random variables.

To understand base correlation, especially in our extended model which possibly mixes different types of processes, we first do some simple math.

We have

$$\begin{aligned}
\mathrm{Var}(W+Y_k) &= \mathrm{Var}(W)+\mathrm{Var}(Y_k) && (41)\\
\mathrm{Cov}(W+Y_i, W+Y_j) &= \mathrm{Var}(W) && (42)\\
\mathrm{Corr}(W+Y_i, W+Y_j) &= \frac{\mathrm{Var}(W)}{\sqrt{(\mathrm{Var}(W)+\mathrm{Var}(Y_i))\,(\mathrm{Var}(W)+\mathrm{Var}(Y_j))}} && (43)
\end{aligned}$$

Without loss of generality (by proper scaling), we can assume we require that for any k,

$$\mathrm{Var}(W)+\mathrm{Var}(Y_k) = 1 \tag{44}$$

Then our correlation formula becomes

$$\mathrm{Corr}(W+Y_i, W+Y_j) = \mathrm{Var}(W) = \rho \tag{45}$$

8.2 Implied Correlation for a Tranche

Therefore, we see one of the main advantages of considering the framework in Section 2 is that no matter what distribution we choose for W, we obtain a ρ parameter. Moreover, referring to Section 4.2 and Section 4.3, this specializes to the ρ parameter mentioned in these cases and therefore with regard to the former, is the same ρ as the implied correlation parameter that is quoted in the markets, such as we might find on Bloomberg.

Now, the idea is that given a model and a market price of a particular tranche and holding constant all parameters which do not depend on ρ, we hope to vary ρ so that our model gives the same price as the market price. The solution will be referred to as the 'implied correlation'. This is not always so easy, and there might be no ρ or more than one ρ which satisfies this.

Therefore, we focus on prices of equity tranches (those trances with attachment points 0). The intuition is that the price of a tranche over an interval $[0, d]$ will decrease as ρ is increased, since there will be some instances when many assets default together, and push L above d, which means less protection is required on the equity tranche.

When we fit such a ρ we call it the 'implied base correlation', and we will get a different number for each available detachment point d, and therefore will obtain a 'base correlation curve'. Note that it is not always guaranteed that we will find such a ρ, and so it is important that our model and its other parameter inputs allow for this.

8.3 Calibration Algorithm

For a detachment point d, let

$$V_{\text{flt}}(\rho, d) = \sum_{j=1}^{Q} P\left(0, \frac{T_j + T_{j-1}}{2}\right)\left(E\left[L^{\rho}_{[0,d]}(T_j)\right] - E\left[L^{\rho}_{[0,d]}(T_{j-1})\right]\right) \tag{46}$$

$$U_{\text{fix}}(\rho, d) = \frac{1}{2}\sum_{j=1}^{Q} P(0, T_j)\left(E\left[N^{\rho}_{[0,d]}(T_j)\right] + E\left[N^{\rho}_{[0,d]}(T_{j-1})\right]\right) \tag{47}$$

be the floating leg value and fixed leg principal of the CDO respectively. Note that we use $L^{\rho}_{[0,d]}$ to indicate that our computation of the loss depends on base correlation ρ.

Let ρ_j be the base correlation for tranche j. Let $[d_{j-1}, d_j]$ be the tranche j, with convention $d_0 = 0$. Let s_j be the spread (CDO index quote) of tranche j. The CDO price formula (39) says

$$V_{\text{flt}}\left(\rho_{j+1}, d_{j+1}\right) - V_{\text{flt}}\left(\rho_j, d_j\right) = s_{j+1}\delta\left(U_{\text{fix}}\left(\rho_{j+1}, d_{j+1}\right) - U_{\text{fix}}\left(\rho_j, d_j\right)\right) \tag{48}$$

Now we can summarize the calibration process, from lower tranches to higher tranches, as follows. Assume we have finished calibrating base correlations for tranches up to j. We would like to calibrate tranche $j + 1$.

1. Given base correlation ρ_{j+1}, we can compute the loss distribution for the base tranche $[0, d_{j+1}]$ by FFT using (9).
2. Using (36), we can compute the expected loss for the base tranche $[0, d_{j+1}]$.
3. Use (48) to compute the spread s_{j+1}.

4. By a root search algorithm, find the base correlation ρ_{j+1} so that the above spread s_{j+1} matches the market quote.

The above description is the recipe to calibrate base correlations by market quotes. Note that here we assume market quotes are given by break-even spreads, which are typically usually used for mezzanine and senior tranches. Equity tranche are quoted with both an upfront payment and a running spread, and the the above formulas need to be adjusted slightly.

8.4 Measuring Model Strength in Terms of Base Correlation

We now understand how to find a base correlation curve. So what is it's usefulness? Firstly, it is an easy and informative way to monitor the CDO markets through time. But this would occur, regardless of which model was chosen. So the question should be rephrased as: What do we expect from a good model in terms of the base correlation curve?

Consider first a linear regression on a sequence of data points (x_i, y_i). In general we will get a slope parameter β and an intercept α, and use this to plot ϵ_i, the error between our linear model $\alpha + \beta x_i$ and the values y_i. The smaller the errors, the better the model. This is the standard approach. Alternatively, we could choose some α, and plot $\beta_i = \frac{y_i - \alpha}{x_i}$. For a perfect model, $\beta_i = \beta$, a constant slope, and in general the flatter our β_i plot, the better the model. Given that we have not fitted the α, we could then try different value to see which best fits the data.

This is analogous to the case of base correlations. Instead of picking particular parameters for each of our models, and then looking at errors (like the standard regression approach), we choose a model and all parameters except for ρ. We then plot ρ as a function of the data points (like the β_i approach above). Therefore, the flatter our base correlation curve, the better our choice of model and parameters.

As mentioned above, it is important that we choose a model which actually allows a base correlation curve to be constructed. For many models, this may be impossible, so these models must be discarded before the comparison step. (Whatever ρ value we put in, between 0% and 100%, we are not able to generate market prices).

A further practical point is that since the interval $[0, 1]$ is generally only divided up into about 6 tranches, we will only have the base correlation curve known for a specific set of detachment points. So, if we wanted to price an off market tranche, base correlation would be a convenient way to accomplish this. By interpolating to the desired detachment point, and feeding this new base correlation into our model, we could obtain the price of any equity tranche, and from this any more senior tranche. As a rule of thumb we could say that the flatter the base correlation the better the accuracy of this interpolation. This, of course, is not a mathematical fact, but is a likely benefit of a flatter curve.

9 Modeling Assumptions in Preparation for Implementation

We now have all the tools to go from choosing a model to fitting a base correlation curve. Next is to see an example of how this works. Remember that referring to Section 3.2, there are many parameters we need to specify, and as mentioned in Section 7, these parameters need to be specified through time. The following sections describe how this is done.

9.1 Stationarity Through Time

We assume that W and Y_k, the variables causing default, have the same distribution at any time t. Note that because of the way we are evaluating the CDO, we do not need to a define a joint distribution between $L(t)$ at different times, only to know that distribution of $L(t)$ at set times. Therefore we will simply specify a single distribution for W, and a separate distribution for Y_k. We will do likewise for $G(W)$ and H_k, with regard to LGD_k.

This means that we have all the required functions from Section 3.2, except the $s_k(t)$ and $\alpha_k(t)$.

9.2 Weightings

Our model allows for any kind of weightings $s_1, ..., s_N$ of the N names in the index. In fact, this is one of our tweaks, since the more common recursive methods for pricing do not allows these 'heterogeneous' weightings to be easily included, whereas using fast fourier transforms, this is easily done. This is important for bespoke CDOs.

However, given that the synthetic CDO market is still relatively new, the only traded indices with easily available quotes generally have homogeneous weightings. Therefore, we can assume for testing purposes that $s_k = \frac{1}{N}$, as in the special case mentioned earlier in Section 4.1

9.3 Default Threshold Changes with Time

In section Section 3.2, we see the default threshold $\alpha_k(t)$ can be solved by inputting the default probability $q_k(t)$. Theoretically, we would be able to extract these from market instruments for each name and each time. However, there is always missing data, and of course we would need to interpolate to get all possible times, or at least the discrete times we require for CDO pricing.

We therefore try a simpler way to understand how $q_k(t)$ changes with time. Assume the default has a constant hazard rate, which means there is a constant $\lambda_k > 0$ such that

$$q_k(t) = 1 - e^{-\lambda_k t} \tag{49}$$

Further, assume we can obtain the default probability $q_k(t_0)$ at time t_0. We then see that

$$q_k(t) = 1 - (1 - q_k(t_0))^{\frac{t}{t_0}} \tag{50}$$

The $\alpha_k(t)$ can then be solved as before.

For project purposes, given significant time constraints, we chose an approach somewhere. Instead of inputting default probabilities, we combine them in such a way so as to produced a single weighted average hazard rate λ, which we assume for all assets. Therefore, we obtain just one function $q(t)$, and use it for all names.

For a discussion of how λ was calculates, please see Appendix 2.

9.4 Specific Distribution Assumptions

Now, let us go a level further, and specify the particular distributions that we had time to implement:

- A shifted gamma model, described in Section 4.3, and assumption of a set constant recovery. Thus, we hope to get improvements in base correlation similar to that presented in [3]. This is our most robust implementation, and most of the results presented later will be based on this.
- Again use shifted gamma model, but now combined with an LGD_k following an independent distribution. In other words, $G = 0$, and the idiosyncratic variable H_k has a beta distribution. This method requires a particular implementation method, which is discussed in the next section.
- Improvement on the above points by letting LGD_k and W, the common factor, be correlated. This directly combines the special cases in Section 4.3 and Section 4.4, and is our most complex model.

9.5 Independent Beta LGD

This section describes how to compute the Fourier transform of LGD assuming LGD follows a beta distribution, independent of the common factor of the default process. We use the notation system on pages 18-19 in [2]. That paper discretizes the Fourier transform of beta LGD by a plain Riemann sum:

$$\hat{f}_H(u) = \frac{1}{B(a,b)M} \sum_{k=1}^{M-1} \left(\frac{k}{M}\right)^{a-1} \left(1 - \frac{k}{M}\right)^{b-1} e^{-iu\frac{k}{M}} \tag{51}$$

where M is the number of partitions we need.

The above discretization is what we implemented in the code. Actually, we can improve performance by using FFT, which is described as follows. Since we prefer to use MatLab `IFFT` function, we have to take care of the nuances due to different definitions of the Fourier transform between [2] and MatLab.

We would like to compute the Fourier transform of the beta PDF by FFT.

$$\begin{aligned} f_{\mathrm{LGD}}(x) &= x^{n-1}(1-x)^{p-1} \qquad (52) \\ \hat{f}_{\mathrm{LGD}}(t) &= \frac{1}{B(n,p)}\int_0^1 x^{n-1}(1-x)^{p-1}e^{-ixt}dx \qquad (53) \end{aligned}$$

We know that $\hat{f}_{\mathrm{LGD}}(t) \to 0$, as $t \to \infty$. Assume we have an integer m_1 such that $\hat{f}$ is computationally deemed as 0 outside the interval $[-2\pi m_1, 2\pi m_1]$. Observe that

$$\hat{f}(-t) = \overline{\hat{f}(t)} \qquad (54)$$

So we only need to compute $\hat{f}(t)$ for $t \in [0, 2\pi m_1]$. To this end, we partition $[0, 2\pi m_1]$ into $m_1 m_2$ subintervals, each of length $\Delta t = 2\pi/m_2$, where m_2 is some integer. Integers m_1 and m_2 will both be determined later for the convenience of applying FFT. Partition $[0,1]$ into m_1 intervals, each of length $\Delta x = 1/m_1$. Denote $m = m_1 m_2$. Let

$$A(k) = \begin{cases} 0 & \text{if } k = 1 \text{ or } m_1 < k \le m \\ ((k-1)\Delta x)^{n-1}\,((1-(k-1)\Delta x)^{p-1} & \text{if } 1 < k \le m_1 \end{cases} \qquad (55)$$

We have the following discretization of the Fourier transform of (53). For $1 \le j \le m$,

$$\begin{aligned} \hat{f}((j-1)\Delta t) &= \frac{1}{B(n,p)}\sum_{k=1}^{m_1} A(k)e^{-i(k-1)\Delta x(j-1)\Delta t}\Delta x \qquad (56) \\ &= \frac{2\pi m_2}{B(n,p)}\cdot\frac{1}{m}\sum_{k=1}^{m} A(k)e^{-\frac{2\pi i}{m}(k-1)(j-1)} \qquad (57) \end{aligned}$$

The last expression is MatLab's `IFFT` on $A(k), 1 \le k \le m$ multiplied by $\frac{2\pi m_2}{B(n,p)}$. To use FFT efficiently, we should choose m to be a power of 2.

10 Results

10.1 Calibration Assuming Fixed LGD

We managed to do a full base correlation run on all available data for the standard market case of one-factor Gaussian copula with fixed LGD, and for the Shifted Gamma version of this model.

In this section, we briefly show a couple of these outputs. Please see Section 12 below for the full base correlation fitting, with accompanying three-dimensional graphs to understand how base correlation curves evolve through time.

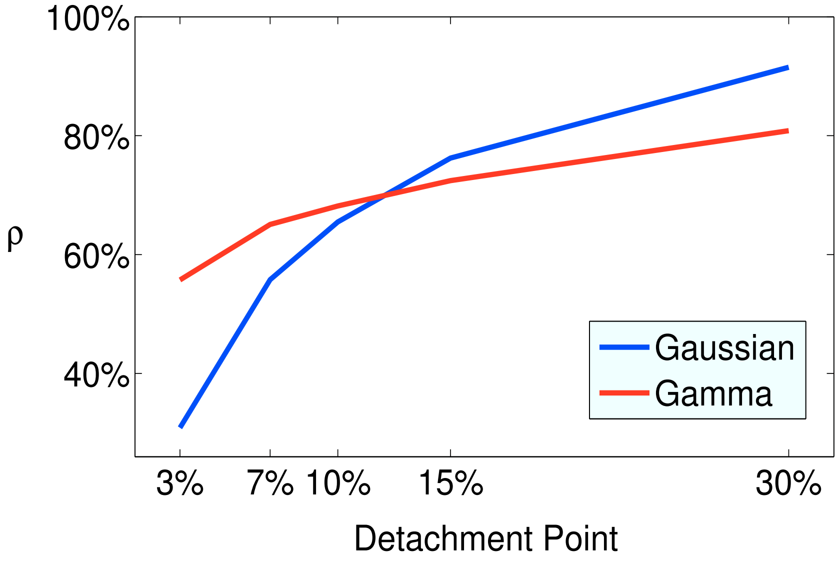


Figure 1: Base Correlation curves for CDX.IG, 5-year maturity, Dec 12 2007

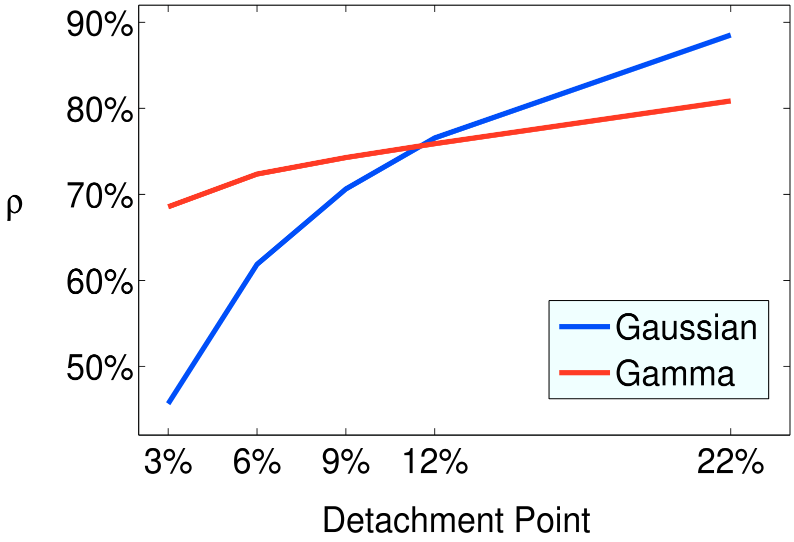


Figure 2: Base Correlation curves for iTraxx, 7-year maturity, Dec. 12 2007

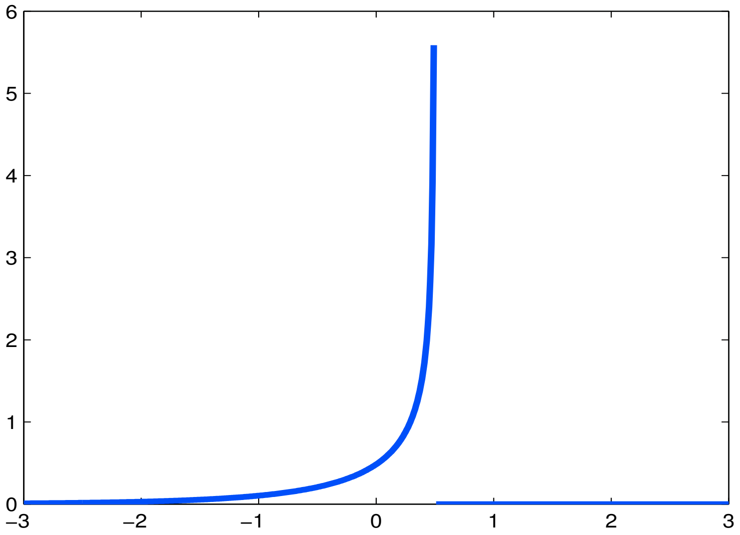


Figure 3: Example of the pdf of a shifted gamma

We label the standard market model, from Section 4.2, as 'Gaussian', and the Shifted Gamma model from Section 4.3 as 'Gamma'. We see that across indices and for different maturities, the Shifted Gamma provides a consistently flatter base correlation.

To understand the intuition behind why this works, let us first consider the Gaussian case. We need very high correlations to correctly price at higher detachment points. This is because the market pricing is presumably accounting for the fact that sometimes many names might default together, which would push L even past this detachment point. This lowers the price of protection for this equity tranche. This can only be achieved in the Gaussian case using a high correlation.

At the lower attachment points, the correlation is very low. To understand this, consider the following: In the case of completely independent names, equity tranches will be priced at their maximum. This is because there should usually be some names, but not all, defaulting, so protection is always required. On the other hand, if defaults are perfectly correlated, then sometimes no names will default, and at other times, when all names default together, protection is capped at the detachment point. Therefore, on average, much less protection is required.

We see that to price the lower attachment points, we are assuming the situation is nearer to the former case i.e. that the defaults are close to being independent/uncorrelated (these concepts are equivalent under Gaussian copula). In other words, the market believes there will almost always be some defaults.

Now, we investigate how the shifted gamma fixes both of these problems. Figure 3 is an example of the shape of the density of a shifted gamma process (taken at time $t = .5$). Firstly we see a long tail towards $-\infty$. This means that even if correlation is not very high, we still get many defaults occurring together (since defaults occur in the left tail of A_k), thanks to the idiosyncratic factor Y_k. So the base correlation does not have to be as high as for Gaussian at higher detachment points.

Next, notice that on the right, there is a large amount of density leading up to the point $t = .5$. This is a region safe from default, and this means that in contrast with the extreme case above, most of the time there are only a small proportion of defaults, for a reasonably large range of common factor W. Therefore, we do not need a low base correlation, which is like a low weight on W, to produce an effect similar to independence most of the time.

To summarize, essentially the Shifted Gamma model gives a more sensible distribution. With high probability, there are a small proportion of defaults. With a small but noticeable probability, there are a very large number of defaults. The Gaussian model, on the other hand, gives too much probability to events between these two extremes.

10.2 Portfolio Loss Rate Assuming Random LGD

We now broaden our horizons, and draw four graphs depicting the distribution of L, the portfolio loss rate, for different base correlation parameter ρ inputs. We will always assume LGD follows a beta distribution. There are four cases:

- A_k is Gaussian, LGD_k is independent of W (Sections 4.2 and 9.5).
- A_k is Shifted Gamma, LGD_k is independent of W (Sections 4.2 and 9.5).
- A_k is Gaussian, LGD_k is determined by W (Sections 4.2 and 4.4).
- A_k is Shifted Gamma, LGD_k is determined by W (Sections 4.3 and 4.4).

For demonstation, we will use the following parameters: LGD_k has mean 60% with volatility 20%. There are 125 names in the portfolio. The default threshold is taken to be 0.

The four graphs are given by Figures 4, 5, 6, and 7.

We observe that using either random LGD or shifted gamma gives a greater probability to possible higher losses.

We do not give calibration, and the reason is as follows: From the previous section, we see the market practitioners tend to price CDOs using fixed LGD. To calibrate using random LGD, we find we must set the LGD variance to be very small. When we compute the Fourier transform of LGD, we use a uniform discretization on interval [0,1]. Small variances therefore produce very inaccurate loss distributions and hence it is difficult to match the market quotes. Therefore, we see that we need to add an improvement here, and compute the Fourier transform of LGD using a non-uniform discretization.

On the other hand, statistics show LGD does have not a small variance. This means the market should probably consider this variance, instead of using the fixed recovery approach which, from the data, is clearly prevalent.

11 Conclusions and Further Work

From our experiment, we see that base correlation curves using Shifted Gamma are certainly flatter. So even in the case of a fixed LGD, there is definite improvement by using a general one-factor framework, and this improvement has been successfully implemented. Therefore, our framework has passed its first hurdle, and there is now much more experimentation to be done with distributions for common and idiosyncratic factors, W and Y_k, and the loss given default LGD_k, in the hopes of finding even better market fits and flatter base correlations.

On the computational side, we have encountered and dealt with many technical issues, but as we expand our choice of models, many more appear. For example, different choices of W causes different numerical problems in computing (9), and letting LGD_k be random introduces complications for how to compute its fourier transform for a given accuracy. Therefore there is much work still to be done to implement different models.

To understand further the results that we have, we examine carefully the graphs in Section 12 below. First we observe that the base correlation curve does not change much from day to day for any CDO product of a given maturity, regardless of our distribution.

For any fixed tranche, base correlation does not change significantly across tenors using the Gaussian model. However, this is not the case for the Shifted Gamma and would seem to imply that practitioners are still using some form of Gaussian model. Any significant base correlation inconsistencies induced by the Gaussian model across maturities should not exist for long, especially in light of potential credit relative value trading strategies.

Therefore our general one-factor model offers improvements not only from a theoretical standpoint, but in practice could help to make the CDO market more stable, and allow pricing to better reflect the true underlying risk.

12 Complete Output for Fixed LGD Case

We present the complete output from our base correlation run with the fixed LGD model, comparing Gaussian and Shifted Gamma cases. For full details of how this was implemented, please see Section 13.

12.1 Numbers

Output consists of computed base correlations for 5-year, 7-year, and 10-year CDX and iTraxx on the following days: 2007-12-06, 2007-12-07, 2007-12-08, 2007-12-10, 2007-12-11 and 2007-12-12. Please see section 13.5 for more details.

The following results are from MatLab outputs. In each matrix, each column gives base correlations for one day, from 2007-12-06 to 2007-12-12. Each row gives base correlations of a tranche, in the order of tranche detachment point. For iTraxx, the tranches detach at 3%, 6%, 9%, 12%, and 22%. For CDX, the tranches detach at 3%, 7%, 10%, 15%, 30%, and 30%.

```
iTraxx using Gaussian model:

5-year

    0.4352    0.4357    0.4357    0.4364    0.4590    0.4580
    0.5974    0.5980    0.5980    0.5982    0.6245    0.6215
    0.6847    0.6852    0.6852    0.6849    0.7109    0.7065
    0.7425    0.7431    0.7430    0.7428    0.7684    0.7639
    0.8588    0.8598    0.8598    0.8593    0.8817    0.8784

7-year

    0.4371    0.4367    0.4367    0.4368    0.4503    0.4561
    0.5994    0.5993    0.5993    0.5994    0.6162    0.6184
    0.6876    0.6872    0.6872    0.6871    0.7039    0.7052
    0.7478    0.7476    0.7476    0.7472    0.7639    0.7650
    0.8701    0.8697    0.8698    0.8694    0.8848    0.8855

10-year

    0.4395    0.5633    0.4393    0.4376    0.4500    0.4483
    0.5551    0.6321    0.5561    0.5529    0.5652    0.5609
    0.6492    0.7109    0.6499    0.6471    0.6620    0.6543
    0.7215    0.7733    0.7222    0.7191    0.7337    0.7247
    0.8634    0.8965    0.8645    0.8623    0.8764    0.8702

iTraxx using shifted gamma model:

5-year

    0.6266    0.6271    0.6271    0.6276    0.6483    0.6473
    0.6727    0.6733    0.6733    0.6735    0.6973    0.6946
```

```
    0.6965    0.6970    0.6970    0.6967    0.7218    0.7176
    0.7139    0.7146    0.7145    0.7143    0.7406    0.7360
    0.7610    0.7623    0.7622    0.7616    0.7903    0.7861

7-year

    0.6701    0.6697    0.6697    0.6698    0.6812    0.6859
    0.7086    0.7084    0.7084    0.7086    0.7218    0.7236
    0.7276    0.7272    0.7272    0.7271    0.7415    0.7427
    0.7430    0.7428    0.7428    0.7425    0.7578    0.7589
    0.7904    0.7899    0.7900    0.7895    0.8077    0.8086

10-year

    0.7086    0.7955    0.7085    0.7072    0.7166    0.7153
    0.7067    0.7662    0.7074    0.7049    0.7145    0.7112
    0.7233    0.7750    0.7239    0.7215    0.7324    0.7277
    0.7414    0.7893    0.7420    0.7394    0.7516    0.7458
    0.7939    0.8369    0.7950    0.7927    0.8101    0.8027

CDX using Gaussian model:

5-year

    0.3093    0.3202    0.3237    0.3230    0.3330    0.3393
    0.5591    0.5703    0.5749    0.5751    0.5864    0.5920
    0.6561    0.6652    0.6697    0.6698    0.6810    0.6849
    0.7637    0.7722    0.7763    0.7765    0.7869    0.7900
    0.9160    0.9218    0.9251    0.9256    0.9318    0.9319

7-tear

    0.2999    0.3093    0.3134    0.3179    0.3226    0.3297
    0.5402    0.5514    0.5564    0.5625    0.5687    0.5736
    0.6352    0.6479    0.6526    0.6591    0.6657    0.6692
    0.7459    0.7576    0.7621    0.7684    0.7746    0.7771
    0.9127    0.9207    0.9243    0.9268    0.9313    0.9317

10-year

    0.3117    0.3208    0.3239    0.3277    0.3314    0.3380
    0.4799    0.4903    0.4947    0.5001    0.5055    0.5106
    0.5860    0.5960    0.6001    0.6049    0.6097    0.6125
    0.7106    0.7209    0.7251    0.7298    0.7345    0.7351
```

```
    0.9067    0.9148    0.9179    0.9195    0.9231    0.9220

CDX using shifted gamma model:

5-year

    0.5567    0.5676    0.5709    0.5702    0.5799    0.5859
    0.6504    0.6603    0.6643    0.6645    0.6745    0.6794
    0.6813    0.6912    0.6952    0.6953    0.7054    0.7090
    0.7242    0.7336    0.7374    0.7376    0.7495    0.7525
    0.8080    0.8172    0.8215    0.8222    0.8349    0.8351

7-year

    0.5859    0.5953    0.5993    0.6037    0.6082    0.6151
    0.6679    0.6767    0.6806    0.6854    0.6902    0.6940
    0.6920    0.7011    0.7051    0.7107    0.7162    0.7192
    0.7289    0.7384    0.7426    0.7485    0.7543    0.7567
    0.8148    0.8241    0.8288    0.8349    0.8409    0.8416

10-year

    0.6268    0.6364    0.6395    0.6433    0.6469    0.6514
    0.6542    0.6627    0.6663    0.6706    0.6750    0.6782
    0.6790    0.6875    0.6909    0.6949    0.6989    0.7006
    0.7170    0.7264    0.7302    0.7344    0.7386    0.7389
    0.8129    0.8234    0.8274    0.8323    0.8370    0.8353
```

12.2 Graphs

The graphic representations of the above data are given by Figures 8, 9, 10, 11 12, 13.

13 Implementation

13.1 Terminology

There are only a finite number of names in an CDO portfolio. If we assume a fix LGD (Loss Given Default), it can be easily seen that the loss distribution is discrete. So we call this the "discrete case". On the other hand, if LGD is a random variable following some continuous distribution, then the loss function is continuous, and so we call this the "continuous case".

13.2 Implementation Strategy

We strictly implement the mathematical models described above. There are quite a few loss/recovery distribution models to select, and quite a few parameters that can be changed to facilitate testing, and so we wrap all these parameters into a single global variable.

The main driver file sets up the above global parameters and runs the calibration code. The CDO discounted cash flows, and hence the CDO pricing, are implemented as separate functions. The calibration code use root finding algorithm to find a base correlation that matches the above CDO price with the market quote.

When CDO cash flows are computed, we need loss and recovery distributions. The distribution code implements the single framework using FFT, whose central formula is (9). The default probability function, the Fourier transform of LGD conditional on a common factor, the mapping function from a common factor to expected LGD, and various PDF/CDF's are all implemented as individual functions. Each model registers its specific implementation via the above functions.

13.3 Implementation Recipe

This section gives detailed recipe for computing the base correlation curve, assuming the same family of distributions for both the default common factor and the default idiosyncratic factor. Furthermore, it assumes a beta distribution for LGD, and that we compute CDO cashflows at each coupon payment date. Following the conventions in the previous sections, f_X and F_X denote the PDF and CDF of a random variable X, respectively.

1. Assume the following information is given at time $t = 0$.
    - The CDO portfolio consists N assets. Each asset weighs s_k for $1 \leq k \leq N$.
    - We have the prediction of the default probability q_k at some time t_0 for each asset k.
    - Fix a set of tranches with detachment points $d_0 = 0, d_1, d_2, \cdots$.
    - The maturity of the CDO contract is T.
    - There are Q coupons of equal amounts paid at $t_1, t_2, \cdots, t_Q = T$. For simplicity, we assume the time intervals between coupons are of the same length δ. This means, $t_{j+1} - t_j = \delta$.
2. Fix a set of discretization parameters.

- For discrete case, choose the smallest $n_L = 2^m$ a power of 2 bigger than N, the number of names in the CDO portfolio.
- For the continuous case:
  - Let $V_{\max} < 1$ such that PDFs of all loss/recovery rates are virtually zero outside $[0, V_{\max}]$.
  - Take $T_{\max}$ such that the Fourier transforms of all loss/recovery PDFs are virtually constant outside $[-T_{\max}, T_{\max}]$.
  - Take $n_L = 2^m$ a power of 2 larger than $T_{\max} V_{\max}/\pi$.

3. Fix a set of model parameters $\Theta = (X, T, H, r)$ where

- X is a Levi process. There are two cases implemented.
  - X is Brownian motion. So $W = X(\rho)$ follows the normal distribution with mean 0 and standard deviation $\sqrt{t}$.
  - X is shifted gamma, which is determined by two parameters a and b in the Gamma distribution. This means for $t \in [0, 1]$,

    $$f_{X(t)}(x) \quad = \quad \frac{b^{at}}{\Gamma(at)} \left(\frac{at}{b} - x \right)^{at-1} e^{bx-at} \tag{58}$$

    if $at > bx$, 0 otherwise. We choose $b = \sqrt{a}$ so that the standard deviation of $X(1)$ is 1.
- T is a function that maps each value of $W = X(\rho)$ into $[0, 1]$. Two special cases are given below
  - $T = 0$. In this case, LGD and loss are independent.
  - Assume LGD follows a beta distribution. Assume the distribution of common factor $W = X(\rho)$.

    $$G(w) = 1 - F_\beta^{-1}\left(F_W(w)\right) \tag{59}$$
- H is the distribution of loss given default and common factor w. It is assumed H is independent of W. Two special cases are given below
  - $H = c$ is a constant. In this case,

    $$\hat{f}_H(t) = e^{-itc} \tag{60}$$

    If this LGD assumption is used, there are finite number of possible loss rates. The later Fourier transform will be discrete. So we refer this as "discrete case".
  - H follows a beta distribution. μ and σ are the mean and deviation of the LGD beta distribution. This means

    $$f_H(x) \quad = \quad \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1} \tag{61}$$

    $$a \quad = \quad \mu \left(\frac{\mu(1-\mu)}{\sigma^2} - 1 \right) \tag{62}$$

    $$b \quad = \quad (1-\mu) \left(\frac{\mu(1-\mu)}{\sigma^2} - 1 \right) \tag{63}$$

This is referred as continuous case as the loss/recovery distributions are continuous functions except point 0 when no default occurs. Its Fourier transform will be computed by Riemann sum as shown in section 9.5:

$$\hat{f}_H(u) = \frac{1}{B(a,b)M} \sum_{k=1}^{M-1} \left(\frac{k}{M}\right)^{a-1} \left(1 - \frac{k}{M}\right)^{b-1} e^{-iu\frac{k}{M}} \tag{64}$$

where M is the number of partitions we need.

- r is the constant risk-free interest rate.

4. For each coupon payment date t_j, find the default probability and threshold for each asset k. Refer to (50) and (10).

$$\begin{aligned} q_k(t_j) &= 1 - (1 - q_k)^{\frac{t_j}{t_0}} && (65) \\ \alpha_k &= F^{-1}_{X(1)}(q_k(t_j)) && (66) \end{aligned}$$

5. (LOOP LEVEL THREE) From the equity tranche to the most senior tranche, use a root search algorithm, solve the base correlation ρ_n for each base tranche $[0, d_n]$ as follows. For a fixed ρ, compute the following steps for tranche $[0, d_n]$. The root search should find $\rho = \rho_n$ that equals the coupon rate computed in step 5c to the market price quote. Remark each of the following steps depends on ρ. Also, the results for tranche $[0, d_{n-1}]$ are assumed to be known. For a more efficient computing, use ρ_{n-1} as the starting value of ρ.

   (a) (LOOP LEVEL FOUR) For each coupon payment date t_j, carry out the following steps for tranche $[0, d] = [0, d_n]$.

      i. Find default probability conditional to common factor w.

$$p_k(w) = F_{X(1-\rho)}(\alpha_k - w) \tag{67}$$

      ii. Compute the Fourier transforms of portfolio loss rate and recovery rate. Refer to (14). We use trapezoidal rule to compute the outer integral.

$$\begin{aligned} \hat{f}_L(u) &= \int_{-\infty}^{\infty} \prod_{k=1}^{N} \left(1 + p_k(w)\left(\hat{f}_H(us_k)e^{-ius_kG(w)} - 1\right)\right) f_{X(\rho)}(w)dw && (68) \\ \hat{f}_R(u) &= \int_{-\infty}^{\infty} \prod_{k=1}^{N} \left(1 + p_k(w)\left(\hat{f}_H(-us_k)e^{-ius_k(1-G(w))} - 1\right)\right) f_{X(\rho)}(w)dw && (69) \end{aligned}$$

      iii. For continuous case, compute the probability of no loss. Refer to (24).

$$\hat{f}_\infty = \int_{-\infty}^{\infty} \prod_{k=1}^{N} (1 - p_k(w)) f_{X(\rho)}(w)dw \tag{70}$$

      For discrete case, set $\hat{f}_\infty$ to 0.

iv. Compute the inverse Fourier transform of the bandwidth limited functions for loss and recovery distribution by applying FFT to solve for $c_k, 1 \leq k \leq m$. Refer to (27).

$$\hat{f}_L\left(-T + \frac{2T(j-1)}{n_L}\right) - \hat{f}_\infty = \sum_{k=1}^{n_L} c_k (-1)^{k-1} e^{-\frac{2\pi i}{n_L}(j-1)(k-1)} \tag{71}$$

$$\hat{f}_R\left(-T + \frac{2T(j-1)}{n_L}\right) - \hat{f}_\infty = \sum_{k=1}^{n_L} c'_k (-1)^{k-1} e^{-\frac{2\pi i}{n_L}(j-1)(k-1)} \tag{72}$$

where $1 \leq j \leq n_L$. Note that the above form is ready to plug into MatLab IFFT. Also remark that $\hat{f}(-t) = \overline{\hat{f}(t)}$. So we only need to compute $\hat{(f)}$ on half of the points to be valued, say, at negative t points.

v. Bandwidth limited functions for both loss and recovery can be computed. For continuous case, refer to (28).

$$f_{\mathrm{BWL}}(x) = \frac{T_{\max}}{\pi} \sum_{1 \leq k \leq \frac{T_{\max} V_{\max}}{\pi} + 1} c_k \,\mathrm{sinc}\left(\frac{T_{\max} x}{\pi} - (k-1)\right) \tag{73}$$

$$f'_{\mathrm{BWL}}(x) = \frac{T_{\max}}{\pi} \sum_{1 \leq k \leq \frac{T_{\max} V_{\max}}{\pi} + 1} c'_k \,\mathrm{sinc}\left(\frac{T_{\max} x}{\pi} - (k-1)\right) \tag{74}$$

For discrete case, both f_{BWL} and f'_{BWL} can be obtained by direct application of IFFT on c_k's.

vi. Compute the expected loss, recovery, and notional amounts. Refer to formulas from (36) to (38). (In practice, the expected recovery is only needed for the senior tranche.)

$$E\left[L_{[0,d]}(t_j)\right] = \int_0^d x f_{\mathrm{BWL}}(x)dx + d\int_d^1 f_{\mathrm{BWL}}(x)dx \tag{75}$$

$$E\left[R_{[0,d]}(t_j)\right] = \int_d^1 (x + d - 1) f'_{\mathrm{BWL}}(x)dx \tag{76}$$

$$E\left[N_{[0,d]}(t_j)\right] = d - E\left[L_{[0,d_n]}(t_j)\right] - E\left[R_{[0,d_n]}(t_j)\right] \tag{77}$$

(b) Compute the present values of the floating leg value and fixed leg principal. Refer to (46) and (47).

$$V_{\mathrm{flt}}(\rho, d_n) = \sum_{j=1}^{Q} e^{-\frac{r\left(t_j + t_{j-1}\right)}{2}} \left(E\left[L^{\rho}_{[0,d_n]}(t_j)\right] - E\left[L^{\rho}_{[0,d_n]}(t_{j-1})\right]\right) \tag{78}$$

$$U_{\mathrm{fix}}(\rho, d_n) = \frac{1}{2}\sum_{j=1}^{Q} e^{-rt_j} \left(E\left[N^{\rho}_{[a,d]}(t_j)\right] + E\left[N^{\rho}_{[a,d]}(t_{j-1})\right]\right) \tag{79}$$

(c) Compute the values of float leg and fixed leg for the tranche using market quotes (coupon rate or spread). Refer to (48).

$$V_{\mathrm{flt}}(\rho, a_n, d_n) = V_{\mathrm{flt}}(\rho, d_n) - V_{\mathrm{flt}}(\rho_{n-1}, d_{n-1}) \tag{80}$$

$$V_{\mathrm{fix}}(\rho, a_n, d_n) = \mathrm{Prem} + c\delta\left(U_{\mathrm{fix}}(\rho, d_n) - U_{\mathrm{fix}}(\rho_{n-1}, d_{n-1})\right) \tag{81}$$

For the equity tranche, Prem is the market quoted up front payment, and c is the fixed spread by market convention. For other tranches, Prem is 0 and c is the market quoted spread.

6. For information only, compute the correlation between LGD and common factor for each base tranche $[0, d_n]$.

$$\mathrm{Corr}(\mathrm{LGD}, X(\rho_n)) = \frac{1}{\sigma\rho}\int_0^1 \left(1 - F_\beta^{-1}(x)\right) F_{X(\rho_n)}^{-1}(x)dx \tag{82}$$

7. (LOOP LEVEL TWO) Use different parameters Θ to repeat steps 3 to 5.
8. (LOOP LEVEL ONE) Repeat all the above steps for other CDO products or indices.

13.4 Code Description

This MatLab package calibrates CDX according the recipe in the project PDF, for the following combinations.

- Continuous FT method for:
  - normal default, beta LGD independent of common factor.
  - normal default, beta LGD determined by common factor.
  - shifted gamma default, beta LGD independent of common factor.
  - shifted gamma default, beta LGD determined by common factor.
- Discrete FT method for:
  - normal default distribution, fixed LGD.
  - shifted gamma default, fixed LGD.

We only implemented for homogeneous portfolios. And each name in the portfolio has the same hazard rate.

After the program successfully finished, the output of calibration results is contained in file results.txt.

List of program files and their descriptions.

- Files to compute loss and recovery distributions

  cf3_ft_lr8.m Compute FT of loss and recovery distributions and their limit at infinity. It uses trapz to compute integrals.

cf3_ft_lr8_quad.m Compute FT of loss and recovery distributions and their limit at infinity. It uses quad to compute integrals.

cf3_f_hat_H.m Compute FT of H variable in project PDF file. (the part of LGD independent of common factor).

cf3_T.m Compute the transform T on common factor. It is the part of LGD that is determined by the common factor. See project PDF for details.

cf3_pdf_X.m Compute the default PDF.

cf3_cdf_X.m Compute the default CDF.

cf3_icdf_X.m Compute the inverse of the default CDF.

cf3_pdf_limits.m Discretize the domain of Fourier transform of the loss/recovery distribution. It is a preparation for the trapezoidal method in the computing of FT integral.

cf3_dist_fft.m Compute the loss/recovery distributions. It is the Fourier inverse process after cf3_ft_lr8.m.

- Files to compute CDO cashflows and to calibrate base correlations:

calibrate.m It performs the actual calibration from tranche to tranche.

calibtest.m It connects CDO calibration and loss/recovery distribution.

quotes.m It obtains the market quotes, together with hazard rates.

getTrancheExpValues.m It compute the expected values (both floating and fixed logs) for a tranche.

computeTranchePrice.m It computes the price of a tranche.

getDiscFactor.m Compute the discount factor for any given future time.

time_stamp.m Log messages together with time stamp.

caliOneFamily.m The function to calibrate one family CDO (either CDX or iTraxx) over different days and maturities.

main.m The main driver file. All fields of the global struct `par` are defined there.

- Test Files

cf3_unit_test.m Implement a function performing 12 unit tests. It has test cases for various functions mentioned above.

cf3_test_drive.m The driver file to perform unit tests. Either a single unit test can be performed, or hundreds of extensive unit tests can be performed.

unit_test_main.m Unit test of functions not related to loss distribution calculations.

test_main_drive.m Test driver file to the above unit test cases.

draw_graphs.m Draw 3-D graphs of base correlation curve changing with time.

The global variable (MatLab structure) `par` is used throughout all functions. It has the following fields:

par.dflt_type The default distribution type. 1: Gaussian 2: shifted gamma

par.dflt_gamma_a If the default type is shifted gamma, this is the parameter a in the gamma distribution. The parameter b is set to be sqrt(a). For Gaussian default type, this parameter is not used.

par.lgd_type The LGD distribution type. 1: LGD independent of common factor. 2: LGD determined by common factor

par.lgd_mean The mean of LGD.

par.lgd_std The standard deviation of LGD. Set LGD standard deviation to 0 to have constant LGD. Once LGD is constant, par.lgd_type is ignored.

par.n_names Number of names in the CDO portfolio.

par.n_hazardRate The default hazard rate for each name in the portfolio.

par.notional The notional amount of a CDO security.

par.maturity The maturity of a CDO security.

par.tol The tolerance level in root searching algorithm.

par.rate The constant risk free interest rate.

par.verbose Set this to 1 if more detailed info should be output into the screen, 0 otherwise.

The full run of the program will take about 2.5 hours on a Windows XP machine with 3 GHz Celeron CPU and 1 GB Memory.

13.5 Data

All our data are obtained from Bloomberg terminals. Specifically, we use these data:

- A recent week (from Dec 6 to Dec 12, 2007) of iTraxx quotes. For each day, there are three maturities: 5-year, and 7-year, 10-year. Each maturity has quotes on 5 tranches: 0-3%, 3-6%, 6-9%, 9-12%, 12-22%.
- A recent week (from Dec 6 to Dec 12, 2007) of iTraxx quotes. For each day, there are three maturities: 5-year, and 7-year, 10-year. Each maturity has quotes on 5 tranches: 0-3%, 3-7%, 7-10%, 10-15%, 15-30%.
- Recent hazard rates for iTraxx and CDX. There is a hazard rate for each maturity in a CDO family. Please see Appendices I & II on how we compute hazard rates.

References

[1] Leif Andersen. *NYU Course: Interest Rate and Credit Models.* Lecture Notes. Spring 2007.

[2] Alain Debuysscher, Marco Szego, Marc Freydefront. *The Fourier Transform Method - Technical Document.* International Structured Finance Working Paper. January 30, 2003.

[3] Joao Garcia, Serge Goossens, Viktoriya Masol, Wim Schoutens. *Levy Base Correlation.* Draft. September 04, 2007.

[4] Ali Hirsa *NYU Course: Computational Methods in Finance.* Lecture Notes. Fall 2007

[5] Dominic O'Kane, Matthew Livesey. *Base Correlation Explained.* Fixed Income, Quantiative Credit Research, Lehman Brothers. November 15, 2004.

[6] Soren Willemann. *An Evaluation of the Base Correlation Framework for Synthetic CDOs.* Draft. December 20, 2004.

14 Appendices

Appendices have been merged onto the end of the document.

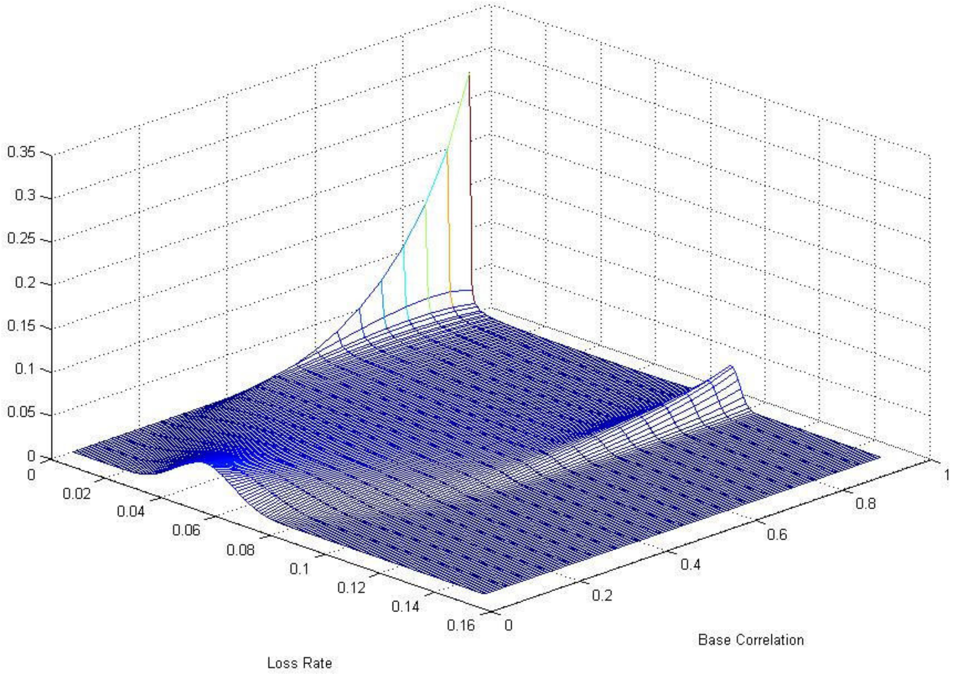


Figure 4: Portfolio Loss Distribution of Base Correlation: Gaussian Case with Independent LGD

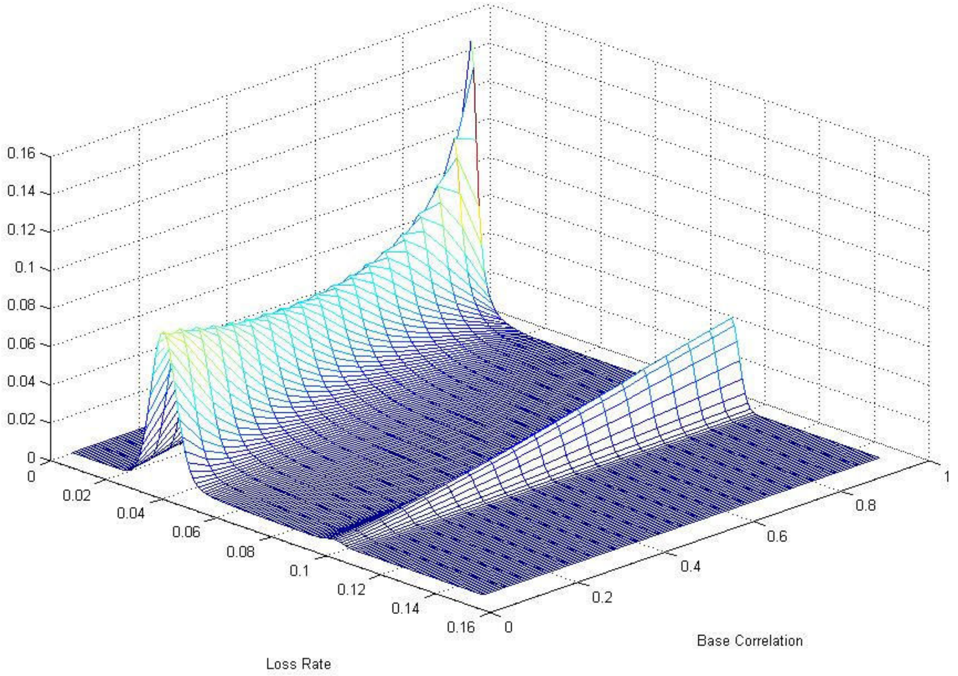


Figure 5: Portfolio Loss Distribution of Base Correlation: Gamma Case with Independent LGD

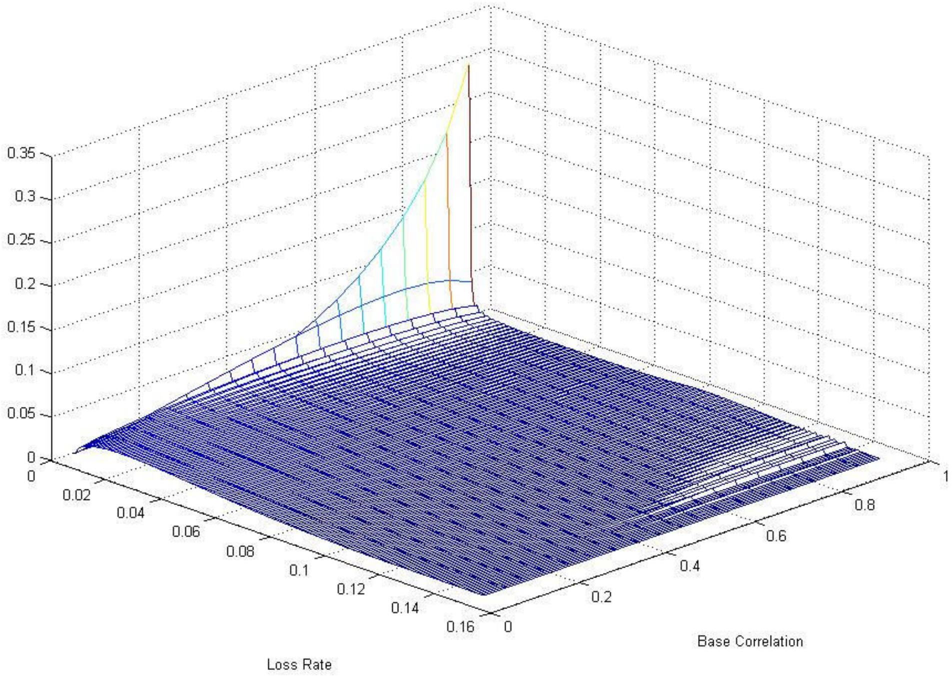


Figure 6: Portfolio Loss Distribution of Base Correlation: Gaussian Case with LGD Determined by Common Factor

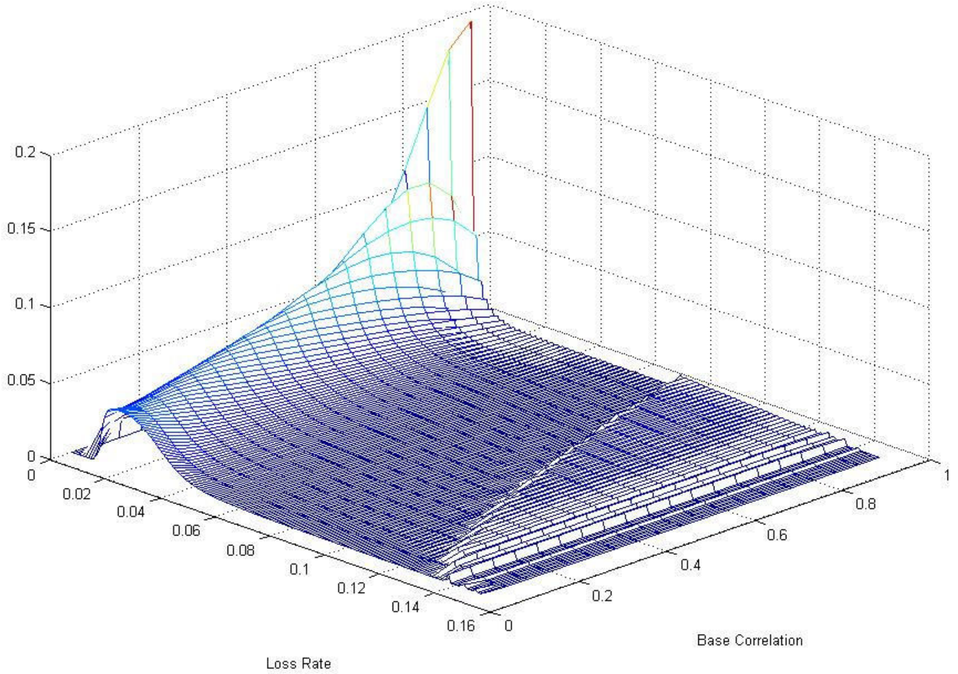


Figure 7: Portfolio Loss Distribution of Base Correlation: Gaussian Case with LGD Determined by Common Factor

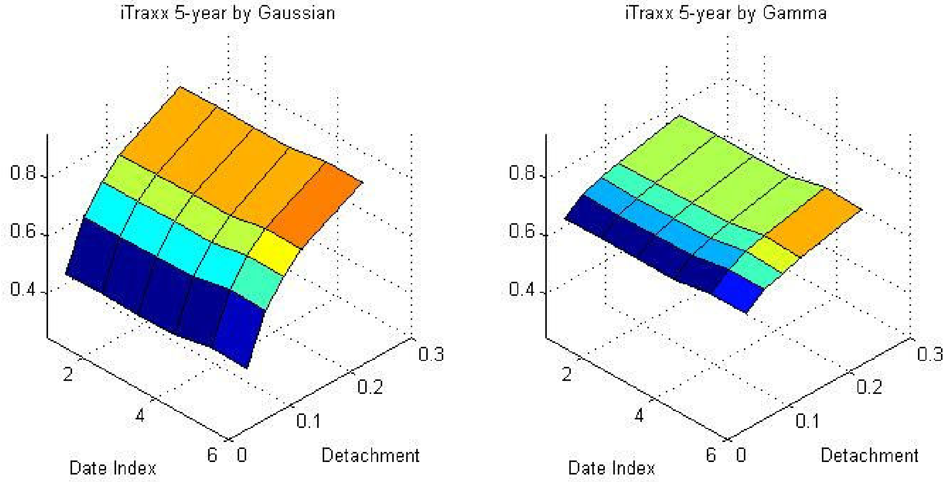
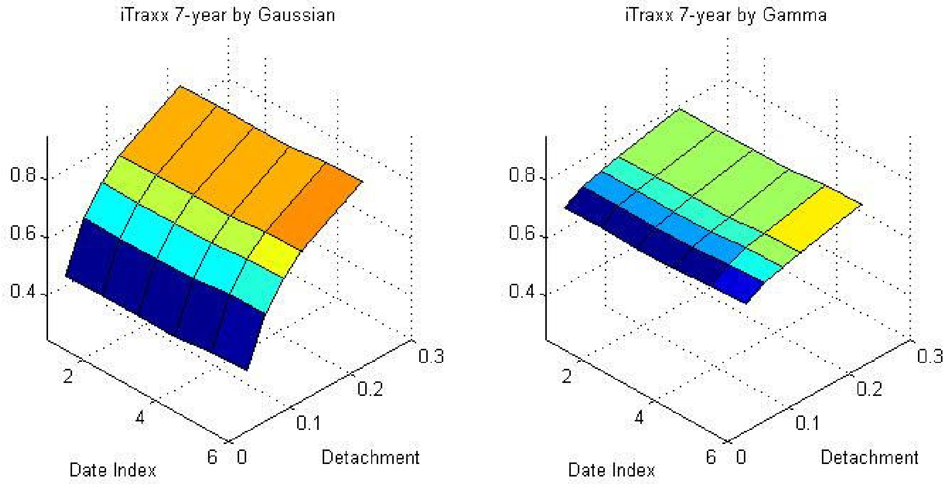
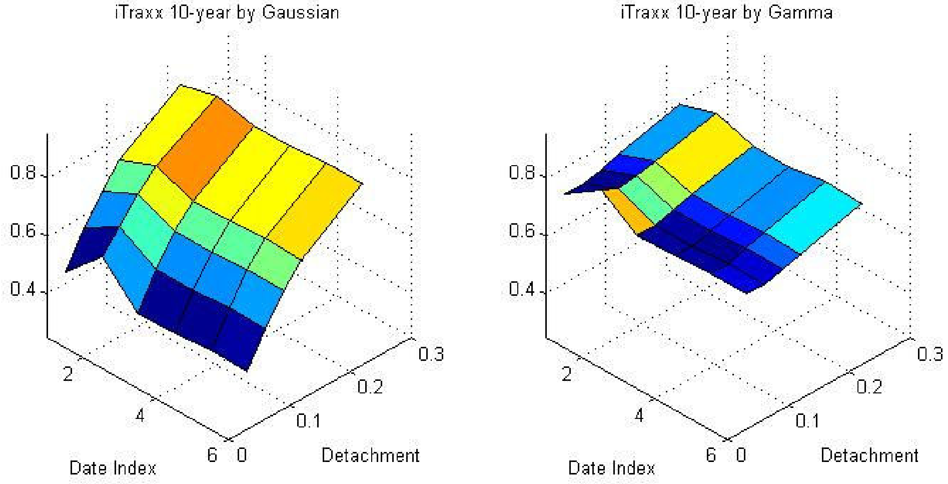


Figure 8: Base Correlation for iTraxx


iTraxx 7-year by Gaussian

iTraxx 7-year by Gamma

Figure 9: Base Correlation for iTraxx


iTraxx 10-year by Gaussian

iTraxx 10-year by Gamma

Figure 10: Base Correlation for iTraxx

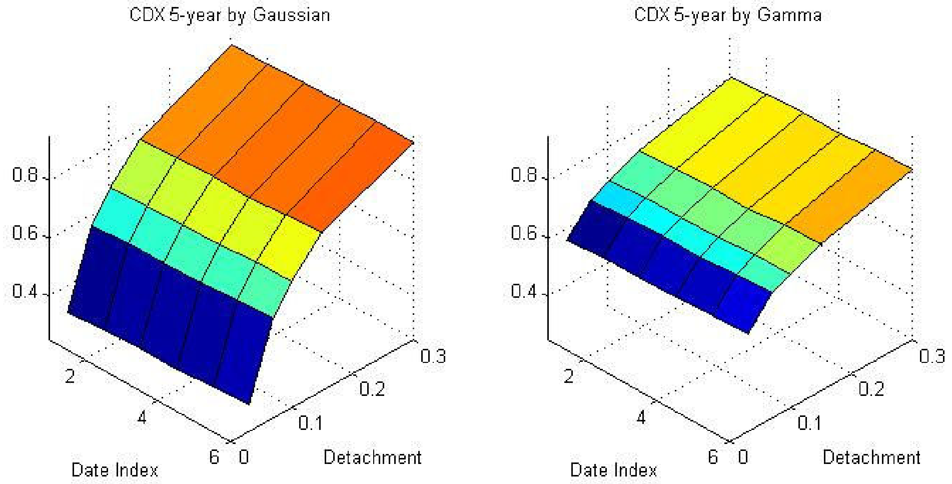


Figure 11: Base Correlation for CDX

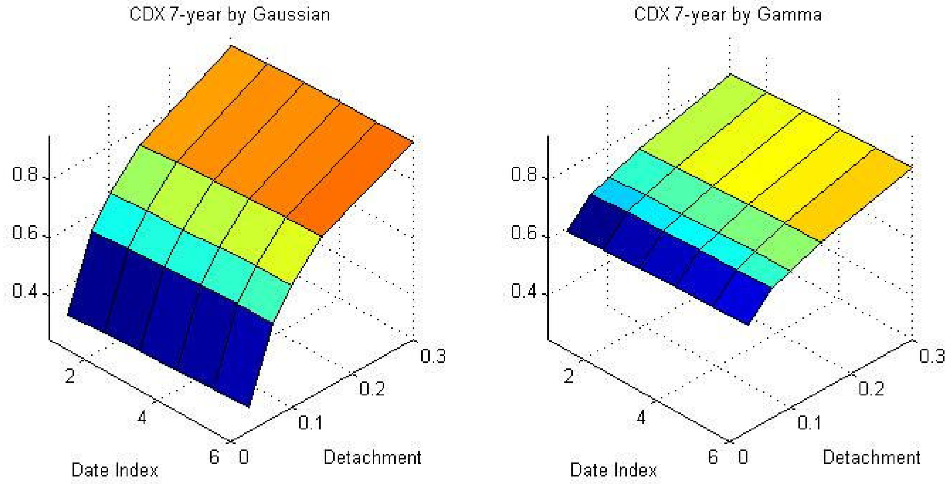


Figure 12: Base Correlation for CDX

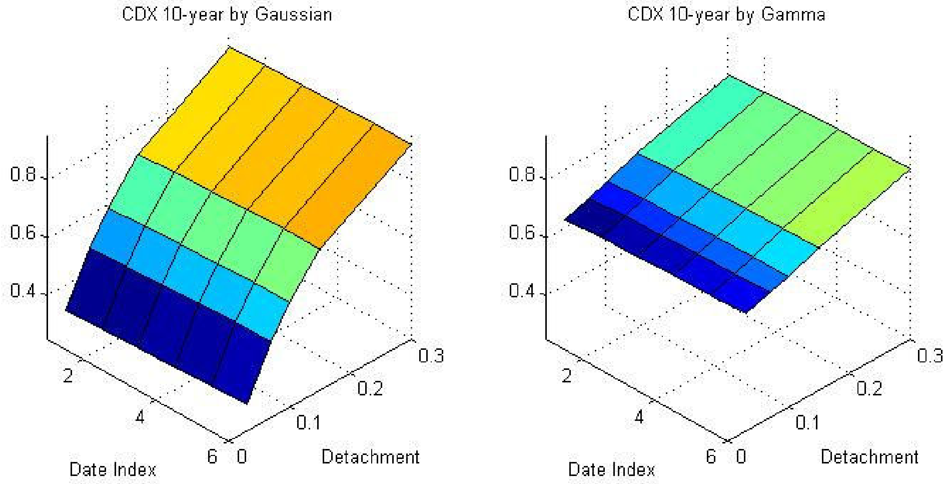


Figure 13: Base Correlation for CDX

Appendix I (Tranche Index Description and Membership)

iTraxx Europe Series 8

The iTraxx Europe Index is composed of 125 investment grade entities, distributed among 9 sub-indices: Autos, Consumers, Energy, Industrials, TMT, Financials (Senior & Subordinated), Non-Financials, HiVol. The composition of each iTraxx index is determined by the Index Rules. iTraxx indices roll every 6 months in March & September. Further details on can be found on www.indexco.com.

The current index membership for Series 9 is as follows:

No.	Name	Wgt	5yr CDS Spread
1	ABN Amro Bank NV	0.8	46.264
2	Accor SA	0.8	46.335
3	Adecco SA	0.8	58.741
4	Aegon NV	0.8	52.838
5	Electrolux AB	0.8	52.924
6	Volvo AB	0.8	52.736
7	Akzo Nobel NV	0.8	37.835
8	Allianz SE	0.8	43.995
9	Altadis SA	0.8	64.005
10	Arcelor Finance SCA	0.8	86.997
11	Assicurazioni Generali SpA	0.8	44.594
12	Aviva PLC	0.8	46.833
13	AXA SA	0.8	53.67
14	BASF AG	0.8	28.538
15	Banca Monte dei Paschi di Siena SpA	0.8	41.827
16	Banco Bilbao Vizcaya Argentaria SA	0.8	37.667
17	Banco Espirito Santo SA	0.8	47.833
18	Banco Santander SA	0.8	39
19	Barclays Bank PLC	0.8	41.644
20	Bayer AG	0.8	40.998
21	Bayerische Motoren Werke AG	0.8	43.066
22	Bertelsmann AG	0.8	50.901
23	BNP Paribas	0.8	25.134
24	British American Tobacco PLC	0.8	45.071
25	British Telecommunications PLC	0.8	54.106

No.	Name	Wgt	5yr CDS Spread
26	Cadbury Schweppes PLC	0.8	48.731
27	Carrefour SA	0.8	30.934
28	Casino Guichard Perrachon SA	0.8	69.934
29	Centrica PLC	0.8	40
30	Ciba Specialty Chemicals Holding Inc	0.8	73.83
31	Commerzbank AG	0.8	39.129

32	Cie de Saint-Gobain	0.8	81.911
33	Compagnie Financiere Michelin	0.8	66.736
34	Compass Group PLC	0.8	37
35	Continental AG	0.8	73.609
36	Credit Agricole SA	0.8	35.898
37	Credit Suisse Group	0.8	39.931
38	Daimler AG	0.8	44.231
39	Deutsche Bank AG	0.8	38.167
40	Deutsche Lufthansa AG	0.8	60.731
41	Deutsche Post AG	0.8	37.18
42	Deutsche Telekom AG	0.8	57.429
43	Diageo PLC	0.8	47.769
44	DSG International PLC	0.8	108.901
45	E.ON AG	0.8	43.005
46	Edison SpA	0.8	49.043
47	Energias de Portugal SA	0.8	47.5
48	Electricite de France	0.8	30.566
49	EnBW Energie Baden-Wuerttemberg AG	0.8	40.299
50	Endesa SA	0.8	70.033

No.	**Name**	**Wgt**	**5yr CDS Spread**
51	Enel SpA	0.8	70.264
52	European Aeronautic Defence and Space Co NV	0.8	55.609
53	Experian Finance PLC	0.8	52.396
54	Fortum Oyj	0.8	42.269
55	France Telecom SA	0.8	44.363
56	Gallaher Group PLC	0.8	28.165
57	Gas Natural SDG SA	0.8	34.896
58	Gaz de France SA	0.8	34.599
59	GKN Holdings PLC	0.8	71.34
60	Glencore International AG	0.8	94.812
61	Groupe Auchan SA	0.8	31.939
62	Groupe Danone	0.8	51.066
63	Hannover Rueckversicherung AG	0.8	32.835
64	Hellenic Telecommunications Organization SA	0.8	49.771
65	Henkel KGaA	0.8	34.835
66	Iberdrola SA	0.8	61.401
67	Imperial Chemical Industries PLC	0.8	22.911
68	Intesa Sanpaolo SpA	0.8	34.766
69	Kingfisher PLC	0.8	130.345
70	Koninklijke DSM NV	0.8	37.667
71	Royal KPN NV	0.8	57.33
72	Koninklijke Philips Electronics NV	0.8	42.167
73	Lafarge SA	0.8	90.16
74	Linde AG	0.8	41.835
75	LVMH Moet Hennessy Louis Vuitton SA	0.8	49.071

No.	Name	Wgt	5yr CDS Spread
76	Marks & Spencer PLC	0.8	62.198
77	Metro AG	0.8	51.434
78	Muenchener Rueckversicherungs AG	0.8	31.667
79	National Grid PLC	0.8	52.17
80	Next PLC	0.8	73.929
81	Pearson PLC	0.8	35.601
82	Peugeot SA	0.8	59.934
83	Portugal Telecom International Finance BV	0.8	61.802
84	PPR	0.8	89.901
85	Reed Elsevier PLC	0.8	31.835
86	Renault SA	0.8	64.665
87	Repsol YPF SA	0.8	63.901
88	Reuters Group PLC	0.8	25.939
89	RWE AG	0.8	34.533
90	Safeway Ltd	0.8	60.533
91	Sanofi-Aventis SA	0.8	40.335
92	Siemens AG	0.8	38.604
93	Sodexho Alliance SA	0.8	37.769
94	Solvay SA	0.8	38.995
95	STMicroelectronics NV	0.8	38.071
96	Suedzucker International Finance BV	0.8	48.767
97	Suez SA	0.8	46.825
98	Svenska Cellulosa AB	0.8	51.173
99	Swiss Reinsurance	0.8	45.66

No.	Name	Wgt	5yr CDS Spread
100	Tate & Lyle PLC	0.8	60.002
101	Telecom Italia SpA	0.8	61.599
102	Telefonica SA	0.8	59.729
103	Telekom Austria AG	0.8	49
104	Telenor ASA	0.8	51.038
105	TeliaSonera AB	0.8	41.535
106	Tesco PLC	0.8	30.868
107	Royal Bank of Scotland PLC/The	0.8	53.167
108	ThyssenKrupp AG	0.8	61.34
109	UBS AG	0.8	41.766
110	UniCredito Italiano SpA	0.8	42.838
111	Unilever NV	0.8	26.901
112	Union Fenosa SA	0.8	50.835
113	United Utilities PLC	0.8	58.467
114	UPM-Kymmene Oyj	0.8	111.279
115	Valeo SA	0.8	81.033
116	Vattenfall AB	0.8	37.5

117	Veolia Environnement	0.8	55.401
118	Vinci SA	0.8	73.873
119	Vivendi	0.8	61.769
120	Vodafone Group PLC	0.8	51.766
121	Volkswagen AG	0.8	60.797
122	Wolters Kluwer NV	0.8	34.101
123	WPP 2005 Ltd	0.8	61.543
124	Zurich Insurance Co	0.8	45.165
125	UniCredito Italiano SpA	0.8	42.838

CDX.NA.IG Series 9

The CDX North America Investment Grade Index is composed of 125 investment grade entities, distributed among 6 sub-indices: High Volatility, Consumer, Energy, Financial, Industrial, and Technology, Media & Tele-communications. The composition of CDX Indices is determined by a consortium of 16 member banks. CDX indices roll every 6 months in March & September. Further details on can be found on www.markit.com.

No.	Name	Wgt	5yr CDS Spread
1	ACE Ltd	0.8	38.903
2	Aetna Inc	0.8	49.127
3	Alcan Inc	0.8	25.731
4	Alcoa Inc	0.8	47.83
5	Altria Group Inc	0.8	33.002
6	American Electric Power Co Inc	0.8	41.998
7	American Express Co	0.8	75.894
8	American International Group Inc	0.8	57.998
9	Amgen Inc	0.8	36.495
10	Anadarko Petroleum Corp	0.8	52.264
11	Arrow Electronics Inc	0.8	41.766
12	AT&T Inc	0.8	43.005
13	AT&T Mobility LLC	0.8	24.995
14	Autozone Inc	0.8	44.109
15	Baxter International Inc	0.8	20.83
16	Belo Corp	0.8	100
17	Boeing Capital Corp Ltd	0.8	19.16
18	Bristol-Myers Squibb Co	0.8	24
19	Burlington Northern Santa Fe Corp	0.8	31.736
20	Campbell Soup Co	0.8	22.566
21	Capital One Bank	0.8	217.5
22	Cardinal Health Inc	0.8	32.5
23	Carnival Corp	0.8	54.243
24	Caterpillar Inc	0.8	35.667
25	CBS Corp	0.8	67.761

No.	Name	Wgt	5yr CDS Spread
26	Centex Corp	0.8	360.705
27	CenturyTel Inc	0.8	56.398
28	Cigna Corp	0.8	48.325
29	CIT Group Inc	0.8	341.675
30	Comcast Cable Communications LLC	0.8	57.594
31	Computer Sciences Corp	0.8	38.233
32	ConAgra Foods Inc	0.8	29.833
33	ConocoPhillips	0.8	37.495
34	Constellation Energy Group Inc	0.8	68.17

35	Countrywide Home Loans Inc	0.8	904.64
36	COX Communications Inc	0.8	60.5
37	CSX Corp	0.8	61.84
38	CVS Caremark Corp	0.8	41.264
39	Darden Restaurants Inc	0.8	100
40	Deere & Co	0.8	34.165
41	Devon Energy Corp	0.8	33.264
42	Dominion Resources Inc/VA	0.8	41.495
43	Duke Energy Corp	0.8	42.799
44	EI Du Pont de Nemours & Co	0.8	24.924
45	Eastman Chemical Co	0.8	41.83
46	Embarq Corp	0.8	100
47	Freddie Mac	0.8	37.774
48	Federal National Mortgage Association	0.8	37.774
49	FirstEnergy Corp	0.8	67.731
50	Fortune Brands Inc	0.8	100

No.	**Name**	**Wgt**	**5yr CDS Spread**
51	Gannett Co Inc	0.8	82.995
52	General Electric Capital Corp	0.8	62.655
53	General Mills Inc	0.8	39.723
54	Goodrich Corp	0.8	24.163
55	Halliburton Co	0.8	27.16
56	Hewlett-Packard Co	0.8	27.068
57	Honeywell International Inc	0.8	24.83
58	IAC/InterActiveCorp	0.8	100
59	Ingersoll-Rand Co	0.8	32.099
60	International Business Machines Corp	0.8	27.063
61	International Lease Finance Corp	0.8	54.498
62	International Paper Co	0.8	46.101
63	iStar Financial Inc	0.8	418.873
64	JC Penney Co Inc	0.8	151.563
65	Jones Apparel Group Inc	0.8	221.939
66	Kraft Foods Inc	0.8	51.127
67	Lennar Corp	0.8	578.898
68	Ltd Brands Inc	0.8	193.83
69	Liz Claiborne Inc	0.8	100
70	Lockheed Martin Corp	0.8	22.231
71	Loews Corp	0.8	26.03
72	Macy's Inc	0.8	133.878
73	Marriott International Inc/DE	0.8	64.782
74	Marsh & McLennan Cos Inc	0.8	55.584
75	MBIA Insurance Corp	0.8	208.746

No.	Name	Wgt	5yr CDS Spread
76	McDonald's Corp	0.8	22.568
77	McKesson Corp	0.8	24.066
78	MeadWestvaco Corp	0.8	70.134
79	MetLife Inc	0.8	47.071
80	Motorola Inc	0.8	73.465
81	National Rural Utilities Cooperative Finance Corp	0.8	31.231
82	Newell Rubbermaid Inc	0.8	31.589
83	News America Inc	0.8	37.254
84	Nordstrom Inc	0.8	56.571
85	Norfolk Southern Corp	0.8	32.604
86	Northrop Grumman Corp	0.8	21.084
87	Omnicom Group Inc	0.8	35.66
88	Progress Energy Inc	0.8	37.231
89	Pulte Homes Inc	0.8	365.903
90	Quest Diagnostics Inc	0.8	100
91	RR Donnelley & Sons Co	0.8	100
92	Radian Group Inc	0.8	580.825
93	Raytheon Co	0.8	22.663
94	Rohm & Haas Co	0.8	39.825
95	Safeway Inc	0.8	34.934
96	Sara Lee Corp	0.8	33.7
97	Sempra Energy	0.8	37.495
98	Simon Property Group LP	0.8	121.025
99	Southwest Airlines Co	0.8	61
100	Sprint Nextel Corp	0.8	100

No.	Name	Wgt	5yr CDS Spread
101	Starwood Hotels & Resorts Worldwide Inc	0.8	129.122
102	Target Corp	0.8	43
103	Textron Financial Corp	0.8	100
104	Allstate Corp/The	0.8	30.634
105	Chubb Corp	0.8	32.401
106	Dow Chemical Co/The	0.8	41.198
107	Hartford Financial Services Group Inc	0.8	42.51
108	Home Depot Inc	0.8	87.934
109	Kroger Co/The	0.8	37
110	Sherwin-Williams Co/The	0.8	42.863
111	Walt Disney Co/The	0.8	28.228
112	Time Warner Inc	0.8	70.997
113	Toll Brothers Inc	0.8	232.638
114	Transocean Inc	0.8	68.929
115	Union Pacific Corp	0.8	35.203
116	Universal Health Services Inc	0.8	100
117	Valero Energy Corp	0.8	63.82

118	Verizon Communications Inc	0.8	100
119	Wal-Mart Stores Inc	0.8	26.667
120	Washington Mutual Inc	0.8	339.538
121	Wells Fargo & Co	0.8	53.639
122	Weyerhaeuser Co	0.8	89.003
123	Whirlpool Corp	0.8	69.16
124	Wyeth	0.8	20.323
125	XL Capital Ltd	0.8	106.592

Appendix II (Derivation of Hazard Rate)

Preliminaries

Assume the r is the risk free interest rate, s is the spread rate, and let PD be the default rate and R the recovery rate.

The present value of the bond if no default happens is:

$$(1-PD)*\exp(-r*t) \qquad \textbf{(1)}$$

And if there is default, the present value would be:

$$PD*(1-R)*\exp(-r*t) \qquad \textbf{(2)}$$

Against (1) and (2) is the present value of a risky cash flow defined as follows:

$$\exp(-(r+s)*t) \qquad \textbf{(3)}$$

Adding (1) and (2) and equating to (3) gives:

$$(1-PD)*\exp(-r*t) + PD*(1-R)*\exp(-r*t) = \exp(-(r+s)*t) \qquad \textbf{(4)}$$

Rearranging (4) yields:

$$PD = (1-\exp(-s*t))/(1-R) \qquad \textbf{(5)}$$

Applying (5) against a par credit default swap curve allows one to express credit spreads in terms of default probability, which in turn can be used to back out the implied hazard rate λ. Lambda, or hazard rate, is defined as the default likelihood per unit time.

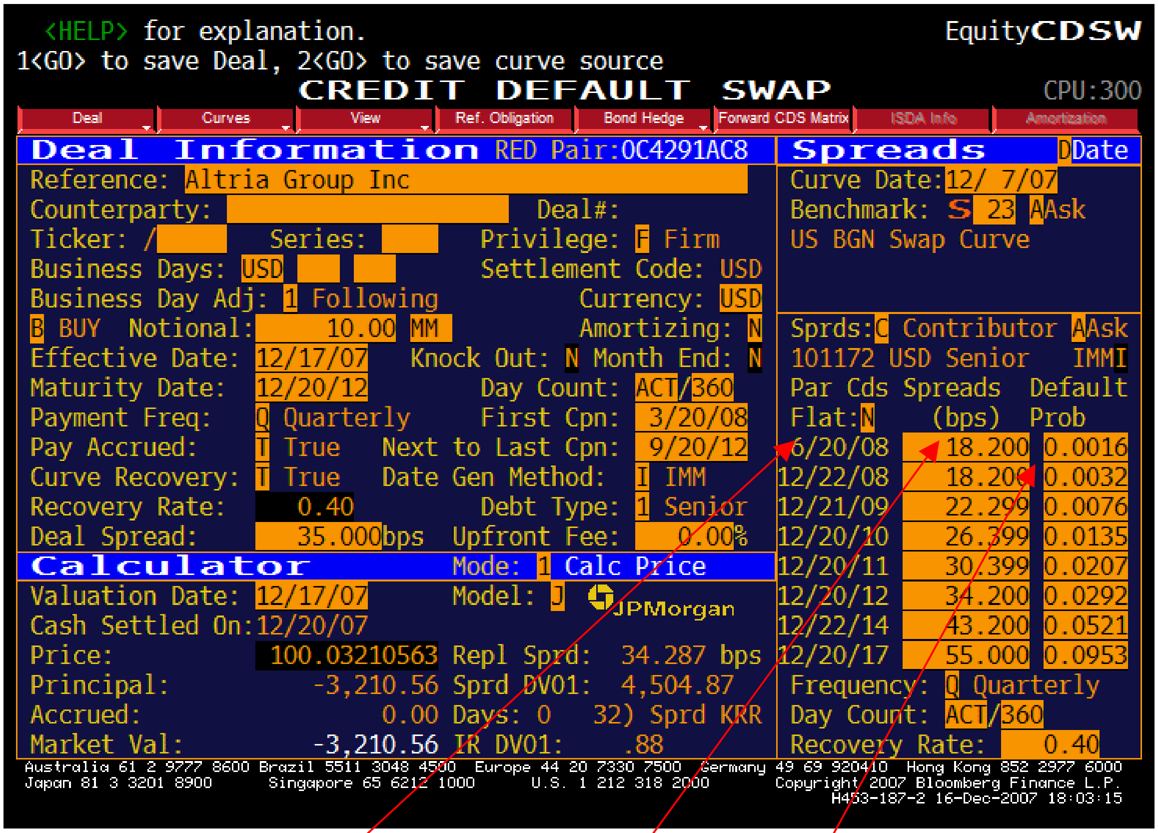
We can relate the default probability to the hazard rate as follows:

$$PD = 1 - \exp(-\lambda*t) \qquad \textbf{(6)}$$

In the next section a worked out example is provided which applies (5) and (6) to par cds spreads to arrive at implied default probabilities. The resulting default probabilities can then be used to for deriving implied hazard rates, which is the approach we took.

Note that our preferred method would have been to invert a regular CDS pricer to derive the implied default probabilities, but this would have taken substantially more time to code relative to the accuracy required for this project.

Example



```
 <HELP> for explanation.                                          EquityCDSW
1<GO> to save Deal, 2<GO> to save curve source
                         CREDIT  DEFAULT  SWAP                       CPU:300
Deal  Information RED Pair:0C4291AC8     Spreads             DDate
 Reference: Altria Group Inc              Curve Date:12/ 7/07
 Counterparty:             Deal#:         Benchmark:  S 23 AAsk
 Ticker: /        Series:      Privilege: F Firm   US BGN Swap Curve
 Business Days: USD            Settlement Code: USD
 Business Day Adj: 1 Following        Currency: USD
 B BUY  Notional:    10.00 MM        Amortizing: N  Sprds:C Contributor AAsk
 Effective Date: 12/17/07  Knock Out: N Month End: N  101172 USD Senior   IMMI
 Maturity Date:  12/20/12        Day Count: ACT/360  Par Cds Spreads  Default
 Payment Freq:   Q Quarterly     First Cpn:  3/20/08  Flat:N    (bps)    Prob
 Pay Accrued:    T True    Next to Last Cpn:  9/20/12  6/20/08   18.200  0.0016
 Curve Recovery: T True    Date Gen Method: I IMM     12/22/08   18.200  0.0032
 Recovery Rate:     0.40         Debt Type: 1 Senior  12/21/09   22.299  0.0076
 Deal Spread:     35.000bps  Upfront Fee:    0.00%   12/20/10   26.399  0.0135
Calculator                 Mode: 1 Calc Price       12/20/11   30.399  0.0207
 Valuation Date: 12/17/07     Model: J  JPMorgan    12/20/12   34.200  0.0292
 Cash Settled On:12/20/07                           12/22/14   43.200  0.0521
 Price:          100.03210563 Repl Sprd:  34.287 bps 12/20/17   55.000  0.0953
 Principal:         -3,210.56 Sprd DV01:  4,504.87  Frequency: Q Quarterly
 Accrued:                0.00 Days: 0    32) Sprd KRR Day Count: ACT/360
 Market Val:        -3,210.56 IR DV01:      .88      Recovery Rate:    0.40
Australia 61 2 9777 8600 Brazil 5511 3048 4500  Europe 44 20 7330 7500  Germany 49 69 920410  Hong Kong 852 2977 6000
Japan 81 3 3201 8900      Singapore 65 6212 1000      U.S. 1 212 318 2000      Copyright 2007 Bloomberg Finance L.P.
                                                                       H453-187-2 16-Dec-2007 18:03:15
```

Maturity	Years	Par CDS Spread	PD	Bloomberg Default Prob	Diff
6/20/2008	0.53611	0.00182	0.00165	0.00160	0.00005
12/22/2008	1.04167	0.00182	0.00320	0.00320	0.00000
12/21/2009	2.03889	0.00223	0.00765	0.00760	0.00005
12/20/2010	3.03611	0.00264	0.01345	0.01350	-0.00005
12/20/2011	4.03611	0.00304	0.02052	0.02070	-0.00018
12/20/2012	5.03611	0.00342	0.02868	0.02920	-0.00052
12/22/2014	7.04167	0.00432	0.05009	0.05210	-0.00201
12/20/2017	10.03611	0.00550	0.08902	0.09530	-0.00628

And from the PD's above we can derive the hazard rate…

Maturity	Implied Hazard Rate
5 Years	0.0052
7 Years	0.0062
10 Years	0.0077