10-5-09 lec 1

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MATRIX ALGEBRA

$F = \mathbb{R}$ or $\mathbb{C}$ (real / Complex numbers)

$\mathbb{P} = \{1, 2, 3, \ldots\}$ = positive integers

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ = nonnegative integers

Let $m, n \in \mathbb{P}$. $m \times n$ matrix M is

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$a_{ij} \in F$

a_{ij} = entry of M in row i, Col. j

m rows, n Columns

$1 \times n$ matrix $[a_1 \ a_2 \ \cdots \ a_n]$ - row vector

$m \times 1$ matrix $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ - Column vector

Matrix Addition M, N - $m \times n$ matrices

$M + N$ - also $m \times n$ matrix

(i,j) entry of $M+N$ = (i,j) entry of M + (i,j) entry of N

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 3 \\ 4 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3 \\ 5 & 0 & 6 \end{bmatrix}$$

NOTE: Addition defined only when matrices have same size.

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Scalar Multiplication of a matrix by a number (also called scalars)

$\alpha \in F$, M - $m \times n$ matrix

αM - also an $m \times n$ matrix

(i,j) entry of $\alpha M = \alpha \times \{ (i,j)$ entry of $M \}$

$$2 \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 4 & 6 \\ 4 & 2 \end{bmatrix}$$

Matrix Multiplication

Very important

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A - $m \times n$ matrix

B - $n \times p$ matrix

$C = AB$ - $m \times p$ matrix

NOTE: AB is defined only when

\# of Columns of A = \# of rows of B

$A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj} \text{ , i.e,}$$

(i,j) entry of C = product of i^{th} row of A with j^{th} col. of B

$$= [a_{i1}\, a_{i2} \cdots a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

Write $B = (B_1, \ldots, B_p)$, where

$$B_j = \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix} \text{ - } j^{th} \text{ col. of } B$$

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}, \quad A_i = (a_{i1} \cdots a_{in}) \text{ - } i^{th} \text{ row of } A$$

(i,j) entry of $C = A_i B_j$

$$j^{th} \text{ col. of } C = \begin{pmatrix} A_1 B_j \\ A_2 B_j \\ A_m B_j \end{pmatrix} = \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix} B_j = A B_j \quad \cancel{AB}$$

So,

$$AB = (AB_1 \quad AB_2 \quad AB_p) \quad \cancel{=A}$$

$$= \begin{pmatrix} A_1 B \\ A_2 B \\ \vdots \\ A_m B \end{pmatrix} \leftarrow \text{Exercise?}$$

Another View

$$A = \begin{pmatrix} D_1 & \cdots & D_n \end{pmatrix}$$

NOTE
Columns of A

$$B = \begin{pmatrix} B_1 & \cdots & B_p \end{pmatrix}$$

$$j^{th} \text{ Col. of } AB = AB_j = \begin{pmatrix} D_1 & & D_n \end{pmatrix} \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix}$$

$$= b_{1j} D_1 + \cdots + b_{nj} D_n$$

Exercise Get a similar expression for the i^{th} row of AB.

$$\begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2\alpha_1 + 4\alpha_2 + 6\alpha_3 \\ 3\alpha_1 + 5\alpha_2 + 7\alpha_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \alpha_3 \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

Example

Consider m varieties of chocolate bars with n ingredients each

$A = (a_{i\ell})_{m \times n}$, $a_{i\ell}$ = # of grams of ingredient ℓ in one bar of the i^{th} variety

$B = (b_{\ell j})_{n \times p}$, $b_{\ell j}$ = Cost in Rs. of 1 gm. of ingredient ℓ in year j

$C = (AB) = (c_{ij})_{m \times p}$

c_{ij} = Cost in Rs. of one bar of i^{th} variety in year j

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$$c_{ij} = (a_{i1}\ a_{i2} \cdots a_{in}) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Identities

$$\left.\begin{aligned} A(B+C) &= AB + AC \\ (B+C)A &= BA + CA \end{aligned}\right\} \text{distributive law}$$

$$(AB)C = A(BC) \quad \text{associative law}$$

$$\alpha(AB) = (\alpha A)B = A(\alpha B)$$ Scalar multiplication compatible with matrix multiplication.

NOTE: These identities hold for suitable matrix sizes.

$$AB \neq BA$$ Commutative law does not hold. even if both AB and BA are defined

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$O = O_{n\times n}$ - zero matrix

$I = I_{n\times n} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n\times n}$ identity matrix

A - $n\times n$ $\implies A I_n = I_n A = A$.

Inverse A - $n\times n$. If there is a $n\times n$ B s/t $AB = BA = I$, then B is called inverse of A and we write $B = A^{-1}$.

Inverse is Unique If B, B' are two inverses of A, then

$$B' = B'(AB) = (B'A)B = B$$

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Product of invertible matrices is invertible

$$(AB)^{-1} = B^{-1}A^{-1}, \text{ if } A^{-1}, B^{-1} \text{ both exist.}$$

Proof

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$$

Similarly $(AB)(B^{-1}A^{-1}) = I$

So, $B^{-1}A^{-1} = (AB)^{-1}$.

Similarly, $(A_1 A_2 \cdots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \cdots A_1^{-1}$, if each A_i^{-1} exists.

How to find A^{-1} (if it exists).

By solving equations (we shall see this later)

BLOCK MULTIPLICATION (useful in induction proofs)

$$
\begin{array}{c} \\ k_1\{ \\ k_2\{ \end{array}
\overset{\qquad\quad r \qquad\quad s}{\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]}
\begin{array}{c} \\ r\{ \\ s\{ \end{array}
\overset{\qquad\quad n_1 \qquad\quad n_2}{\left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right]}
$$

$$
= \begin{array}{c} \\ k_1\{ \\ k_2\{ \end{array}
\overset{\qquad\qquad n_1 \qquad\qquad\qquad n_2}{\left[\begin{array}{c|c} AA' + BC' & AB' + BD' \\ \hline CA' + DC' & CB' + DD' \end{array}\right]}
$$

CHECK

$$
\left[\begin{array}{cc|c} 1 & 0 & 5 \\ \hline 0 & 1 & 7 \end{array}\right]
\left[\begin{array}{cc|cc} 2 & 3 & 1 & 1 \\ 4 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \end{array}\right]
= \left[\begin{array}{cc|cc} 2 & 8 & 6 & 1 \\ \hline 4 & 8 & 7 & 0 \end{array}\right]
$$

ROW-REDUCTION

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A system of m linear equations in n unknowns $x_1, x_2, \dots, x_n$

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 &+ a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots & \qquad \vdots \qquad\qquad\quad \vdots \\ a_{m1}x_1 &+ a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{aligned}$$

can be written compactly as

$$Ax = b$$

where $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

or $$x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Elementary row operations

For $i, j \in \{1, \dots, m\}$, let E_{ij} be the matrix with 1 in (i,j) position and 0 elsewhere

$$E_{ij} = \begin{array}{cl} \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix}_{m \times m} & \leftarrow i \\ \uparrow j & \end{array}$$

For $m \times n$ matrix $A = \begin{pmatrix} A_1 \\ A_m \end{pmatrix}$ (rows of A)

$$E_{ij} A = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \longleftarrow & A_j & \longrightarrow \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow i \\ \\ \\ \end{array} \qquad \begin{bmatrix} 0 \cdots 0 \\ \\ \\ \end{bmatrix} \begin{bmatrix} A_1 \\ \\ A_m \end{bmatrix}$$

$$i^{th} \text{ Row of } E_{ij} A = \begin{array}{c} \begin{bmatrix} 0 \cdots 1 \; 0 \cdots 0 \end{bmatrix} \\ \uparrow \\ j \end{array} \begin{bmatrix} A_1 \\ \\ A_m \end{bmatrix}$$

Type I operation

For $i \neq j$, replace E_i (equation i)

by $E_i + a E_j$

After a type I operation

$Ax = b$ becomes

$$(I + aE_{ij}) Ax = (I + aE_{ij}) b$$

$$(I + aE_{ij})^{-1} = I - aE_{ij}. \quad \text{(why?)}$$

$i \rightarrow \quad 2x_1 + 3x_2 + 5x_3 = 6$

$j \rightarrow \quad x_1 - x_2 + x_3 = 7$

$2x_1 + 3x_2 + 5x_3$
$+ a(x_1 - x_2 + x_3) = 6 + 7a$

$x_1 - x_2 + x_3 = 7$

So, inverse of a type I operation is a type I operation.

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$$Ax = b$$

$$(I + aE_{ij})A$$

$$= A + aE_{ij}A$$

$$= \begin{bmatrix} A_1 \\ A_{i\cdot} \\ A_m \end{bmatrix} + a \begin{array}{c} \begin{bmatrix} 0 \\ A_{j\cdot} \\ 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \leftarrow i \\ \\ \end{array}$$

$$\begin{bmatrix} A_i + aA_j \\ A_j \end{bmatrix} \begin{array}{c} \leftarrow i \\ \leftarrow j \end{array}$$

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Show

$$\underline{\underline{(I + aE_{ij})^{-1}}} = \underline{\underline{I - aE_{ij}}}$$ ✓

$$(I + aE_{ij})(I - aE_{ij})$$

$$= I + aE_{ij} - aE_{ij} - a^2 E_{ij}^2$$

$$= I - a^2 (E_{ij})^2$$

$E_{ij} E_{ij}$.

$$\begin{array}{cc} & \\ (0 \cdots 1 \, 0 \cdots 0) & \begin{pmatrix} E_{ij} \\ \\ \end{pmatrix} \\ \uparrow & \uparrow \\ j & i \end{array}$$

$= 0.$

$Ax = b$ $\qquad$ $Ax = 0$

$\uparrow$

homogeneous.

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Particular solution y_p

$$Ay_p = b.$$

$$\{ x \mid Ax = b \} = \{ y_p + x' \mid Ax' = 0 \} \quad \text{Prove.}$$

$x \in LHS \implies Ax = b$

$$A(x - y_p) = Ax - Ay_p = b - b = 0.$$

$$x = y_p + \underline{\underline{(x - y_p)}}$$

$\implies x \in RHS.$

$y_p + x' \in RHS$

$\therefore Ax' = 0$

$$\text{So } A(y_p + x') = Ay_p + Ax'$$
$$= b + 0$$
$$= b.$$

$\implies y_p + x' \in LHS.$

AB , $A - m \times n$

$B - n \times p$

$$B = \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix}$$ rows of B.

$$\boxed{i^{th} \text{ row of } AB = a_{i1} R_1 + a_{i2} R_2 + \cdots + a_{in} R_n}$$

i^{th} row of $A = (a_{i1} \; a_{i2} \; \cdots \; a_{in})$