On optimal linear codes over $\mathbb{F}_5$

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Abstract

Let $n_q(k,d)$ be the smallest integer n for which there exists a linear code of length n, dimension k and minimum distance d over $\mathbb{F}_q$, the field of q elements. We determine $n_5(5,d)$ for d =476-479, 491-530, 538-540, 542-560, 563-625. We also show that $n_5(5,d) \geq g_5(5,d)+1$ for d =70-120, 144-150, 268-275, 280-290, 293-300, 394,395, 398-400 and that $n_5(5,d) \geq g_5(5,d)+2$ for d =373-375 and so on, where $g_q(k,d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil$.

MSC: 94B05, 94B15, 51E20

Keywords: Optimal linear codes; Extension of linear codes; Projective geometry

1 Introduction

We denote by $\mathbb{F}_q$ the field of q elements. Let $V(n,q)$ denote the vector space of n-tuples over $\mathbb{F}_q$. A q-ary linear code $\mathcal{C}$ of length n and dimension k, called an $[n,k]_q$ code, is a k-dimensional subspace of $V(n,q)$. An $[n,k]_q$ code $\mathcal{C}$ with minimum (Hamming) distance $d = d(\mathcal{C})$ is referred to as an $[n,k,d]_q$ code.

A fundamental problem in coding theory is to optimize one of the parameters n, k, d for given the other two (over a given field $\mathbb{F}_q$). Two versions of the problem are:

Problem 1. Find $n_q(k,d)$, the smallest value of n for which an $[n,k,d]_q$ code exists.

Problem 2. Find $d_q(n,k)$, the largest value of d for which an $[n,k,d]_q$ code exists.

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†This research has been partially supported by Grant-in-Aid for Scientific Research of Japan Society for the Promotion of Science under Contract Number 17540129

We mainly deal with Problem 1 for linear codes over $\mathbb{F}_5$. We note for fixed q that solving Problem 1 for all k, d is equivalent to solving Problem 2 for all n, k. See [8] for the updated known results on $d_q(n,k)$. See also [25] for optimal parameters of linear codes. An $[n,k,d]_q$ code is called *optimal* if $n = n_q(k,d)$ or $d = d_q(n,k)$. There is a natural lower bound on $n_q(k,d)$ called the Griesmer bound [9],[27]:

$$n_q(k,d) \geq g_q(k,d) = \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. The values of $n_q(k,d)$ are determined for all d only for some small values of q and k. See [22] for the known results on $n_q(k,d)$ for some small q and k. See also [4] for $q = 2$, [28] for $q = 3$ and [3] for $q = 4$. For $q = 5$, the problem of finding $n_5(k,d)$ has been solved for $k \leq 4$ for all d except only eight cases.

Theorem 1.1 ([12])**.** (1) $n_5(3,d) = g_5(3,d)$ *for* $d \in \{1\text{–}4, 6, 7, 8, 11, 12\}$ *and for* $d \geq 16$.
(2) $n_5(3,d) = g_5(3,d) + 1$ *for* $d \in \{5, 9, 10, 13, 14, 15\}$.

Theorem 1.2 ([18])**.** (1) $n_5(4,d) = g_5(4,d)$ *for* $d \in \{1, 2, 3, 6, 7, 8, 11, 16\text{–}20, 26\text{–}30, 51\text{–}60, 76\text{–}80, 96\text{–}145, 151\text{–}160\}$ *and for* $d \geq 176$.
(2) $n_5(4,d) = g_5(4,d)+1$ *for* $d \in \{4, 5, 9, 10, 12\text{–}15, 21\text{–}24, 33, 34, 35, 38\text{–}50, 61\text{–}75, 83\text{–}95, 146\text{–}150, 163\text{–}175\}$.
(3) $n_5(4,d) = g_5(4,d) + 2$ *for* $d = 25$.
(4) $n_5(4,d) = g_5(4,d)$ *or* $g_5(4,d) + 1$ *for* $d \in \{31, 32, 36, 37, 81, 82, 161, 162\}$.

We tackle the problem of finding $n_5(5,d)$ for $d \leq 5^4$. Such a code is necessarily projective if it attains the Griesmer bound. See [8] for the known results for $d \leq 100$. Our results are summarized as follows.

Theorem 1.3. (1) $n_5(5,d) = g_5(5,d)$ *for* $d \in \{496\text{–}520, 526\text{–}530, 551\text{–}560, 576\text{–}625\}$.
(2) $n_5(5,d) = g_5(5,d)+1$ *for* $d \in \{476\text{–}479, 491\text{–}495, 521\text{–}525, 538\text{–}540, 542\text{–}550, 563\text{–}575\}$.
(3) $n_5(5,d) \geq g_5(5,d)+1$ *for* $d \in \{70\text{–}120, 144\text{–}150, 161\text{–}175, 186\text{–}250, 268\text{–}275, 288\text{–}290, 293\text{–}372, 394, 395, 398\text{–}400, 411\text{–}475, 480\text{–}490\}$.
(4) $n_5(5,d) \geq g_5(5,d) + 2$ *for* $d \in \{121\text{–}125, 373\text{–}375\}$.
(5) $n_5(5,d) = g_5(5,d)$ *or* $g_5(5,d) + 1$ *for* $d \in \{531\text{–}537, 541, 561, 562\}$.
(6) $n_5(5,d) \leq g_5(5,d)+2$ *for* $d \in \{451\text{–}454, 456\text{–}462, 466\text{–}468, 471\text{–}474, 480\text{–}484, 486\text{–}490\}$.

We summarize the lower and upper bounds for $n_5(5,d)$, $d \leq 5^4$ as Table 2 except for some values of d where the Griesmer bound is attained. Table 2 contains known results from [8] and (rather weak) existence results obtained by our computer search.

Since it holds that $g_q(k,d) \geq g_q(k-1,d) + 1$, shortening and Theorems 1.2, 1.3 yield the following:

Theorem 1.4. *For $k \geq 5$,*
(1) $n_5(k,d) \geq g_5(k,d)+1$ *for* $d \in \{4,5,9,10,12-15,21-24,33,34,35,38-50,61-120, 144-150,161-175,186-250,268-275,288-290,293-372,394,395,398-400,411-495, 521-525,538-540,542-550,563-575\}$;
(2) $n_5(k,d) \geq g_5(k,d)+2$ *for* $d \in \{25,121-125,373-375\}$.

A linear code is *projective* if it has a generator matrix any two columns of which are linearly independent. We also prove the following theorems:

Theorem 1.5. *If there exists a projective* $[n,k,d]_q$ *code, then so does a projective* $[n+q^k-q-1,k+1,d+q^k-q^{k-1}-q]_q$ *code.*

Theorem 1.6. $n_q(5,d) = g_q(5,d)+1$ *for*
(1) $q^4-q^3-q^2+1 \leq d \leq q^4-q^3-q^2+q-1$ *for odd* q,
(2) $q^4-q^3-q^2+1 \leq d \leq q^4-q^3-q^2+q$ *for even* q.

Theorem 1.7. $g_q(5,d)+1 \leq n_q(5,d) \leq g_q(5,d)+2$ *for*
(1) $q^4-q^3-q^2-q+1 \leq d \leq q^4-q^3-q^2-1$ *for odd* q,
(2) $q^4-q^3-q^2-q+1 \leq d \leq q^4-q^3-q^2$ *for even* q.

It is known only for $q=2,3,4$ that $n_q(5,d) = g_q(5,d)+1$ holds for $q^4-q^3-q^2+1 \leq d \leq q^4-q^3-q^2+q$ and for $q^4-q^3-q^2-q+1 \leq d \leq q^4-q^3-q^2$, see [8],[22].

In Section 2, we describe the geometric method from projective geometry. In Section 3, many good codes are constructed. The proofs of the nonexistence results are given in Section 5. Some results on 4-dimensional codes are given in Section 4, which are needed in the last section. The results in this paper include work for the second author's Master's thesis [26].

2 Preliminary results

We first note that the following result has been already known for $n_q(5,d)$, $d \leq q^4$.

Theorem 2.1 ([6],[19],[20],[23]). (1) $n_q(5,d) = g_q(5,d)$ *for* $q^4-q^3-q+1 \leq d \leq q^4-q^3+q^2-q$ *for all* q.
(2) $n_q(5,d) = g_q(5,d)$ *for* $q^4-2q^2+1 \leq d \leq q^4$ *for all* q.
(3) $n_q(5,d) = g_q(5,d)+1$ *for* $q^4-q^3-2q+1 \leq d \leq q^4-q^3-q$ *for* $q \geq 3$.
(4) $n_q(5,d) \geq g_q(5,d)+1$ *for* $q^4-q^3-q^2+1 \leq d \leq q^4-q^3-2q$ *for* $q \geq 3$.
(5) $n_q(5,d) = g_q(5,d)+1$ *for* $q^4-2q^2-q+1 \leq d \leq q^4-2q^2$ *for* $q \geq 3$.
(6) $n_q(5,d) = g_q(5,d)+1$ *for* $q^4-2q^2-2q+1 \leq d \leq q^4-2q^2-q$ *for* $q \geq 4$.

The above (4) implies $n_q(k,d) \geq g_q(k,d)+1$ in Theorem 1.6.

Remark 1. There is a misprint in Theorem 2.4(ii) of [23]: "$n_q(k,d) = g_q(k,d)+1$" must be "$n_q(k,d) \geq g_q(k,d)+1$" although the equality holds when $s=1$ from (i).

The code obtained by deleting the same coordinate from each codeword of $\mathcal{C}$ is called a *punctured code* of $\mathcal{C}$. If there exists an $[n+1,k,d+1]_q$ code $\mathcal{C}'$ which gives $\mathcal{C}$ as a punctured code, $\mathcal{C}$ is called *extendable* and $\mathcal{C}'$ is an *extension* of $\mathcal{C}$. We use the following extension theorems for proving the nonexistence of some codes.

Theorem 2.2 ([13],[14]). *Let $\mathcal{C}$ be an $[n,k,d]_q$ code with gcd$(d,q)=1$ whose weights are congruent to 0 or d (mod q). Then $\mathcal{C}$ is extendable.*

Theorem 2.3 ([21]). *Let $\mathcal{C}$ be an $[n,k,d]_q$ code with odd $q \geq 5, d \equiv -2$ (mod q)$, k \geq 3$. Then $\mathcal{C}$ is extendable if $A_i = 0$* for all $i \not\equiv 0,-1,-2$ *(mod q).*

An $[n,k,d]_q$ code $\mathcal{C}$ is called *divisible by m* if all codewords have weights divisible by an integer $m > 1$. The following is known as Ward's divisibility theorem.

Theorem 2.4 ([29]). *Let $\mathcal{C}$ be an $[n,k,d]_p$ code, p a prime, attaining the Griesmer bound. If p^e divides d, then p^e is a divisor of all nonzero weights of $\mathcal{C}$.*

We denote by PG(r,q) the projective geometry of dimension r over GF(q). A *j-flat* is a projective subspace of dimension j in PG(r,q). 0-flats, 1-flats, 2-flats, 3-flats, $(r-2)$-flats and $(r-1)$-flats are called *points, lines, planes, solids, secundums* and *hyperplanes* respectively. We denote by $\mathcal{F}_j$ the set of j-flats of PG(r,q) and denote by θ_j the number of points in a j-flat, i.e. $\theta_j = (q^{j+1}-1)/(q-1)$. We set $\theta_j = 0$ for $j < 0$.

Let $\mathcal{C}$ be an $[n,k,d]_q$ code which does not have any coordinate position in which all the codewords have a zero entry. We always consider such codes throughout this paper. The columns of a generator matrix of $\mathcal{C}$ can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k-1,q)$ denoted also by $\mathcal{C}$. We see linear codes from this geometrical point of view. An *i-point* is a point of Σ which has multiplicity i in $\mathcal{C}$. Denote by γ_0 the maximum multiplicity of a point from Σ in $\mathcal{C}$ and let C_i be the set of i-points in Σ, $0 \leq i \leq \gamma_0$. For any subset S of Σ we define *the multiplicity of S with respect to $\mathcal{C}$*, denoted by $m_{\mathcal{C}}(S)$, as

$$m_{\mathcal{C}}(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|,$$

where $|T|$ denotes the number of points in T for a subset T of Σ. When the code is projective, i.e. when $\gamma_0 = 1$, the multiset $\mathcal{C}$ forms an n-set in Σ and the above $m_{\mathcal{C}}(S)$ is equal to $|\mathcal{C} \cap S|$. A line l with $t = m_{\mathcal{C}}(l)$ is called a *t-line.* A *t-plane*, a *t-solid* and so on are defined similarly. Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ such that

$$\begin{aligned} n &= m_{\mathcal{C}}(\Sigma), \\ n-d &= \max\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\}. \end{aligned}$$

Conversely such a partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ as above gives an $[n,k,d]_q$ code in the natural manner. For an m-flat Π in Σ we define

$$\gamma_j(\Pi) = max\{m_{\mathcal{C}}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\},\ 0 \leq j \leq m.$$

We denote simply by γ_j instead of $\gamma_j(\Sigma)$. It holds that $\gamma_{k-2} = n-d,\ \gamma_{k-1} = n$.

Lemma 2.5. (1) *Let Π be an $(s-1)$-flat in Σ, $2 \le s \le k-1$, with $m_{\mathcal{C}}(\Pi) = w$. For any $(s-2)$-flat δ in Π, we have*

$$m_{\mathcal{C}}(\delta) \le \gamma_{s-1} - \frac{n-w}{\theta_{k-s}-1}.$$

In particular for $0 \le j \le k-3$,

$$\gamma_j \le \gamma_{j+1} - \frac{n-\gamma_{j+1}}{\theta_{k-2-j}-1}.$$

(2) *Let δ_1 and δ_2 be distinct t-flats in a fixed $(t+1)$-flat Δ in Σ, $1 \le t \le k-2$. Then*

$$m_{\mathcal{C}}(\delta_1) + m_{\mathcal{C}}(\delta_2) \ge m_{\mathcal{C}}(\Delta) - (q-1)\gamma_t + q{\cdot}m_{\mathcal{C}}(\delta_1 \cap \delta_2).$$

Proof. (1) Considering the $(s-1)$-flats in Σ through δ, we have

$$n \le (\gamma_{s-1} - m_{\mathcal{C}}(\delta))(\theta_{k-s}-1) + w.$$

(2) Considering the t-flats in Δ through $\delta_1 \cap \delta_2$, we have

$$m_{\mathcal{C}}(\Delta) \le m_{\mathcal{C}}(\delta_1) + m_{\mathcal{C}}(\delta_2) - m_{\mathcal{C}}(\delta_1 \cap \delta_2) + (\gamma_t - m_{\mathcal{C}}(\delta_1 \cap \delta_2))(q-1). \quad \square$$

Corollary 2.6. *Setting $t = k-2$, $q = 5$, $a = m_{\mathcal{C}}(\delta_1)$, $b = m_{\mathcal{C}}(\delta_2)$ and $c = m_{\mathcal{C}}(\delta_1 \cap \delta_2)$ in Lemma 2.5(2), it holds that*

$$a + b \ge 4d - 3n + 5c. \tag{2.1}$$

When $\mathcal{C}$ attains the Griesmer bound, γ_j's are uniquely determined as follows.

Lemma 2.7 ([20])**.** *For an $[n,k,d]_q$ code attaining the Griesmer bound, it holds that*

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \quad \text{for } 0 \le j \le k-1.$$

By Lemma 2.7 every $[n,k,d]_q$ code attaining the Griesmer bound is projective if $d \le q^{k-1}$. Denote by a_i the number of hyperplanes Π in Σ with $m_{\mathcal{C}}(\Pi) = i$ and by λ_s the number of s-points in Σ. Note that we have $\lambda_2 = \lambda_0 + n - \theta_{k-1}$ when $\gamma_0 = 2$. The list of a_i's is called the *spectrum* of $\mathcal{C}$. We usually use τ_j's for the spectrum of a hyperplane of Σ to distinguish from the spectrum of $\mathcal{C}$. Simple counting arguments yield the following.

Lemma 2.8. (1) $\sum_{i=0}^{n-d} a_i = \theta_{k-1}$. (2) $\sum_{i=1}^{n-d} i a_i = n\theta_{k-2}$.

(3) $\sum_{i=2}^{n-d} i(i-1)a_i = n(n-1)\theta_{k-3} + q^{k-2}\sum_{s=2}^{\gamma_0} s(s-1)\lambda_s$.

When $\gamma_0 = 1$ we get the following from the three equalities of Lemma 2.8:

$$\sum_{i=0}^{n-d-2} \binom{n-d-i}{2} a_i = \binom{n-d}{2}\theta_{k-1} - n(n-d-1)\theta_{k-2} + \binom{n}{2}\theta_{k-3}. \tag{2.2}$$

Lemma 2.9. *Let Π be an i-hyperplane through a t-secundum δ with $t = \gamma_{k-3}(\Pi)$. Then*
(1) $t \le \gamma_{k-2} - \dfrac{n-i}{q} = \dfrac{i + q\gamma_{k-2} - n}{q}$.
(2) $a_i = 0$ *if an* $[i, k-1, d_0]_q$ *code with* $d_0 \ge i - \left\lfloor \dfrac{i + q\gamma_{k-2} - n}{q} \right\rfloor$ *does not exist, where* $\lfloor x \rfloor$ *denotes the largest integer less than or equal to* x.
(3) $t = \left\lfloor \dfrac{i + q\gamma_{k-2} - n}{q} \right\rfloor$ *if an* $[i, k-1, d_1]_q$ *code with* $d_1 \ge i - \left\lfloor \dfrac{i + q\gamma_{k-2} - n}{q} \right\rfloor + 1$ *does not exist.*
(4) *Let c_j be the number of j-hyperplanes through δ other than Π. Then the following equality holds:*

$$\sum_j (\gamma_{k-2} - j)c_j = i + q\gamma_{k-2} - n - qt. \tag{2.3}$$

(5) *For a γ_{k-2}-hyperplane Π_0 with spectrum $(\tau_0, \cdots, \tau_{\gamma_{k-3}})$, $\tau_t > 0$ holds if $i + q\gamma_{k-2} - n - qt < q$.*

Proof. (1) Straightforward from Lemma 2.5.
(2) Π gives an $[i, k-1, d_0]_q$ code with $d_0 \ge i - \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor$ by (1).
(3) If $t \le \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor - 1$, then Π gives an $[i, k-1, d_1]_q$ code with $d_1 \ge i - \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor + 1$. Hence our assertion follows from (1).
(4) (2.3) follows from $\sum_j c_j = q$ and $\sum_j (j-t)c_j = n - i$.
(5) It holds that $c_{\gamma_{k-2}} > 0$ when the right hand side of (2.3) is at most $q - 1$. □

An f-set F in PG(r, q) satisfying

$$m = \min\{|F \cap \pi| \mid \pi \in \mathcal{F}_{r-1}\}$$

is called an $\{f, m; r, q\}$-*minihyper*. When an $[n, k, d]_q$ code is projective (i.e. $\gamma_0 = 1$), the set of 0-points C_0 forms a $\{\theta_{k-1} - n, \theta_{k-2} - (n-d); k-1, q\}$-minihyper, and vice versa.

Theorem 2.10 ([10],[11]). (1) *Every $\{\theta_\alpha, \theta_{\alpha-1}; r, q\}$-minihyper with $1 \le \alpha < r$ is an α-flat.*
(2) *Every $\{\theta_\alpha + \theta_\beta, \theta_{\alpha-1} + \theta_{\beta-1}; r, q\}$-minihyper with $0 \le \beta < \alpha < r$ satisfies $r \ge \alpha + \beta + 1$ and consists of mutually disjoint an α-flat and a β-flat.*
(3) *Every $\{\theta_\alpha + \theta_\beta + \theta_\gamma, \theta_{\alpha-1} + \theta_{\beta-1} + \theta_{\gamma-1}; r, q\}$-minihyper with $r \ge 2$, $q \ge 5$, $0 \le \gamma \le \beta \le \alpha < r$ satisfies $r \ge \alpha + \beta + 1$ and consists of mutually disjoint an α-flat, a β-flat and a γ-flat.*
(4) *Every $\{2\theta_\alpha + 2\theta_\beta, 2\theta_{\alpha-1} + 2\theta_{\beta-1}; r, q\}$-minihyper with $q \ge 5$, $0 \le \beta < \alpha < r$ satisfies $r \ge 2\alpha + 1$ and consists of mutually disjoint two α-flats and two β-flats.*

3 Upper bounds on $n_5(5, d)$

In this section we give the upper bounds on $n_5(5, d)$ in Theorem 1.3. From Theorem 2.1, we have $n_5(5, d) = g_5(5, d)$ for $d \in$ {496-520,576-625} and $n_5(5, d) = g_5(5, d) + 1$ for $d \in$ {491-495,566-575}. Taking the points of PG(4,5) out of a fixed hyperplane as the columns of a generator matrix, we can construct a $[625, 5, 500]_5$ code with $A_{625} > 0$. Applying the following lemma with a $[625, 5, 500]_5$ code as $\mathcal{C}_1$ and some optimal 5-ary linear codes of dimension 4 as $\mathcal{C}_2$, one can get codes with parameters $[664, 5, 530]_5$, $[671, 5, 535]_5$, $[677, 5, 540]_5$, $[683, 5, 545]_5$, $[689, 5, 550]_5$, $[695, 5, 555]_5$, $[701, 5, 560]_5$, $[708, 5, 565]_5$, which yield that $n_5(5, d) = g_5(5, d)$ for $d \in$ {526-530,551-560} and that $n_5(5, d) \le g_5(5, d) + 1$ for $d \in$ {531-550,561-565}.

Lemma 3.1. *Let $\mathcal{C}_1$ be an $[n_1, k, d_1]_q$ code and $\mathcal{C}_2$ be an $[n_2, k-1, d_2]_q$ code. Assume that $\mathcal{C}_1$ has a codeword c with $wt(c) \ge d_1 + d_2$. Then an $[n_1 + n_2, k, d_1 + d_2]_q$ code exists.*

Proof. The assertion follows from Theorem 14.1 of [2] (called *Construction X*) taking the 1-dimensional subcode of $\mathcal{C}_1$ generated by c as $\mathcal{D}$ and $\mathcal{C}_2$ as the auxiliary code. □

From now on, we construct some linear codes mainly by means of minihypers.

Proof of Theorem 1.5. Let Π be a hyperplane of $\Sigma = PG(k, q)$ and let $\Pi = C_0 \cup C_1$ be a partition corresponding to a given projective $[n, k, d]_q$ code. Take a line l of Σ meeting Π in a point of C_1 and put $F = C_0 \cup l$. Then F forms a $\{\theta_{k-1} - n + \theta_1, \theta_{k-2} - (n-d) + 1; k, q\}$-minihyper, which gives a projective $[n + q^k - q - 1, k + 1, d + q^k - q^{k-1} - q]_q$ code. □

Applying Theorem 1.5 to a $[39, 4, 30]_5$ code, we get a $[658, 5, 525]_5$ code. Hence $n_5(5, d) \le g_5(5, d) + 1$ for $521 \le d \le 525$.

An *n-arc* in PG$(2, q)$ is an n-set of PG$(2, q)$ no three of which are collinear. It is well known that the maximum size n for which an n-arc exists is $q + 1$ when q is odd and is $q + 2$ when q is even, see [16]. The upper bounds on $n_q(5, d)$ in Theorems 1.6, 1.7 are straightforward from the following theorems.

Theorem 3.2. (1) *There exists a projective $[g_q(5, d) + 2, 5, d = q^4 - q^3 - q^2 - 1]_q$ code for odd q.*
(2) *There exists a projective $[g_q(5, d) + 2, 5, d = q^4 - q^3 - q^2]_q$ code for even q.*

Proof. (1) Take a solid Δ in PG$(4, q)$ and a plane δ in Δ. Let $\Delta_1, \cdots, \Delta_q$ be the solids through δ other than Δ. Let $K = \{P_0, P_1, \cdots, P_q\}$ be a $(q + 1)$-arc in δ. We take lines $l_1, l_2, \cdots, l_q$ so that l_i is a line in Δ_i meeting Δ in P_i for $i = 1, ..., q$ and take a line l_0 in Δ_1 meeting Δ in P_0 such that $l_0 \cap l_1 = \emptyset$. Setting $F = \Delta \cup l_0 \cup l_1 \cup \cdots \cup l_q$, F forms a $\{\theta_3 + q^2 + q, \theta_2 + \theta_1 - 2; 4, q\}$-minihyper, which gives a $[q^4 - q^2 - q, 5, q^4 - q^3 - q^2 - 1]_q$ code.
(2) Take Δ, δ, Δ_i's, P_i's, l_i's as in (1). It is known that a $(q + 1)$-arc K is contained in a $(q + 2)$-arc K' in δ when q is even ([16]). Put $K' \setminus K = \{P\}$. Setting $F' =$

$(\Delta \cup l_0 \cup l_1 \cup \cdots \cup l_q) \setminus \{P\}$, F' forms a $\{\theta_3 + q^2 + q - 1, \theta_2 + \theta_1 - 2; 4, q\}$-minihyper, which gives a $[q^4 - q^2 - q + 1, 5, q^4 - q^3 - q^2]_q$ code. □

Theorem 3.3. (1) *There exists a projective* $[g_q(5,d) + 1, 5, d = q^4 - q^3 - q^2 + q - 1]_q$ *code for odd* q.
(2) *There exists a projective* $[g_q(5,d) + 1, 5, d = q^4 - q^3 - q^2 + q]_q$ *code for even* q.

Proof. Removing the line l_0 from F and F' in the proof of Theorem 3.2 give a $\{\theta_3 + q^2 - 1, \theta_2 + q - 2; 4, q\}$-minihyper for q odd and a $\{\theta_3 + q^2 - 2, \theta_2 + q - 2; 4, q\}$-minihyper for q even, which correspond to a $[q^4 - q^2 + 1, 5, q^4 - q^3 - q^2 + q - 1]_q$ code and a $[q^4 - q^2 + 2, 5, q^4 - q^3 - q^2 + q]_q$ code, respectively. □

The above two theorems yield that $n_5(5,d) \le g_q(5,d) + 2$ for $471 \le d \le 474$ and that $n_5(5,d) \le g_q(5,d) + 1$ for $476 \le d \le 479$.

Now it remains to show the existence of codes with parameters $[571, 5, 454]_5$, $[578, 5, 460]_5$, $[581, 5, 462]_5$, $[588, 5, 468]_5$, $[603, 5, 480]_5$, $[608, 5, 484]_5$.

A *blocking* b*-set* in $\mathrm{PG}(2,q)$ is a b-set of $\mathrm{PG}(2,q)$ meeting every line but containing no line completely. In PG(2,5), every blocking set of smallest size is a blocking 9-set which is a projective triangle of side four ([16]).

Theorem 3.4. *There exists a* $[608, 5, 484]_5$ *code.*

Proof. Let Δ and Δ' be distinct solids in PG(4,5) with $\delta = \Delta \cap \Delta'$. We first construct a $\{\theta_2 + 13, \theta_1 + 1; 3, 5\}$-minihyper in Δ containing δ. Let δ_1 be a plane ($\neq \delta$) in Δ with $l_0 = \delta \cap \delta_1$. Take a blocking 9-set $\mathcal{B}$, which is a projective triangle of side four, in δ_1 so that l_0 is a tangent line of $\mathcal{B}$ at one of the vertices of the triangle, say P. Let l_1 be another tangent line of $\mathcal{B}$ at P in δ_1 and take a point $P_1 (\neq P)$ on l_1. Take a line l_0 in Δ meeting δ_1 in P_1. Then $F_0 = \delta \cup \mathcal{B} \cup l_0$ forms a $\{\theta_2 + 13, \theta_1 + 1; 3, 5\}$-minihyper in Δ.

Now, take a line l which is skew to δ and let $l^* = l \setminus \Delta$. Setting $F = F_0 \cup l^* \cup \Delta'$, F forms a $\{\theta_3 + 17, \theta_2 + 13; 4, 5\}$-minihyper, giving the desired code. □

Theorem 3.5. *There exist codes with parameters* $[571, 5, 454]_5$, $[578, 5, 460]_5$, $[581, 5, 462]_5$, $[588, 5, 468]_5$, $[603, 5, 480]_5$.

Proof. These codes were found with the aid of a computer (cf. [24]). Let T be the companion matrix of $f(x) = x^5 - (1 + 2x + x^2 + 3x^3)$ and let τ be the projectivity of PG(4,5) defined by T. Then τ is cyclic, i.e., $\mathrm{PG}(4,5) = \{P, TP, T^2P, \cdots, T^{780}P\}$, where $P = (1,0,0,0,0)^{\mathrm{T}}$. We denote the point T^iP by i and the solid defined by $aX_0 + bX_1 + cX_2 + dX_3 + eX_4 = 0$ by $V(abcde)$.
(1) $V(14120) \setminus \{408\} \cup \{$42,44,53,65,89,94,108,116,136,137,138,149,169,181,185,208,286,320, 342,346,350,352,353,358,376,397,405,410,438,450,451,474,501,514,522,527,546,551,569, 573,588,626,634,643,662,671,677,678,680,699,700,705,728,743,780$\}$ forms a $\{210, 39; 4, 5\}$-minihyper, giving a $[571, 5, 454]_5$ code with the weight distribution

$$0^1 454^{824} 455^{432} 456^{800} 459^{564} 460^{152} 461^{200} 464^{100} 465^{36} 469^{12} 570^4.$$

(2) Let $C_0 = V(14301) \setminus \{15, 274, 684\} \cup \{$38,82,89,94,120,121,169,208,219,225,250,272,290, 299,303,309,313,314,318,345,352,353,371,379,389,397,400,410,446,450,454,455,467,485,496, 501,505,521,535,569,588,589,601,619,646,653,661,680,696,698,699,700,705,707,728,773$\}$ and let $C_2 = \{30, 206, 234, 502, 536, 718\}$, $C_1 = \mathrm{PG}(3,4) \setminus (C_0 \cup C_2)$. Setting C_i be the set of i-points, we get a $[578, 5, 460]_5$ code with the weight distribution

$$0^1 460^{1832} 465^{1084} 470^{196} 475^8 575^4.$$

(3) Let $C_0 = V(14141) \cup \{$0,78,88,112,116,173,177,202,207,209,219,223,264,275,282,297,304, 341,346,365,369,371,376,390,397,414,417,437,483,490,503,521,528,537,541,574,590,594,611, 650,682,702,746,747,754,780$\}$ and let $C_2 = \{318, 668\}$, $C_1 = \mathrm{PG}(3,4) \setminus (C_0 \cup C_2)$. Setting C_i be the set of i-points, we get a $[581, 5, 462]_5$ code with the weight distribution

$$0^1 462^{636} 463^{672} 464^{504} 465^{348} 466^{120} 467^{280} 468^{236} 469^{148} 470^{68} 471^{16} 472^{44} 473^{32} 474^8 475^8 581^4.$$

(4) $V(14141) \cup \{$0,65,70,85,95,113,125,191,216,251,340,341,391,412,426,433,444,447,459, 464,468,493,516,520,546,576,611,641,655,682,704,705,736,748,754,758,780$\}$ forms a $\{\theta_3 + 37, \theta_2 + 5; 4, 5\}$-minihyper, giving a $[588, 5, 468]_5$ code with the weight distribution

$$0^1 468^{840} 469^{536} 470^{516} 471^{328} 472^{292} 473^{284} 474^{188} 475^{36} 476^{12} 477^{48} 478^{12} 479^{16} 480^{12} 588^4.$$

(5) $V(12431) \cup \{$0,1,24,48,65,66,94,130,196,210,377,456,469,499,503,653,661,673,728,729, 760,765$\}$ forms a $\{\theta_3 + 22, \theta_2 + 2; 4, 5\}$-minihyper, giving a $[603, 5, 480]_5$ code with the weight distribution

$$0^1 480^{472} 481^{652} 482^{672} 483^{508} 484^{428} 485^{232} 486^{76} 487^{48} 488^8 489^{12} 491^{12} 603^4.$$

□

4 The spectra of some $[n, 4, d]_5$ codes

We give the information about the spectra (defined in Section 2) of some $[n, 4, d]_5$ codes, which are needed in Section 5. See Table 1 for the possible spectra of some $[n, 4, d]_5$ codes. We refer to [1],[15] for $[26 - i, 4, 20 - i]_5$ codes with $0 \le i \le 4$ and [5] for $[39, 4, 30]_5$, $[70, 4, 55]_5$, $[101, 4, 80]_5$ codes. The spectra of $[123, 4, 98]_5$, $[124, 4, 99]_5$, $[125, 4, 100]_5$, $[138, 4, 110]_5$ and $[142, 4, 113]_5$ codes are obtained from Theorem 2.10. The spectra of $[76, 4, 60]_5$ and $[132, 4, 105]_5$ codes can be derived using Theorem 2.4 and Lemmas 2.8, 2.9.

Now let $\mathcal{C}$ be a $[137, 4, 109]_5$ code. Then the set C_0 of 0-points forms a $\{19, 3; 3, 5\}$-minihyper. By Lemma 2.9, we have $a_i = 0$ for all $i \notin \{$22-25,27,28$\}$.
Suppose $a_{25} > 0$ and let δ be a 25-plane. By Theorem 2.10(1), $\delta \cap C_0$ is a line, and δ has spectrum $(\tau_0, \tau_5) = (1, 30)$. From (2.2) we get

$$15a_{22} + 10a_{23} + 6a_{24} + 3a_{25} = 195. \tag{4.1}$$

For $i = 25$, the maximum possible contributions of c_j's in (2.3) to the LHS of (4.1) are $(c_{22}, c_{24}) = (4, 1)$ for $t = 0$ and $(c_{25}, c_{28}) = (1, 4)$ for $t = 5$. Estimating the LHS of (4.1) we get $195 \leq 66 \cdot 1 + 3 \cdot 30 + 3 = 159$, a contradiction. Hence $a_{25} = 0$. Similarly we can prove $a_{24} = 0$. Thus

$$a_i = 0 \text{ for all } i \notin \{22, 23, 27, 28\}.$$

Applying Theorem 2.2, $\mathcal{C}$ is extendable. Hence C_0 is a disjoint union of three skew lines and a point in PG(3,5). This yields the spectrum of a $[137, 4, 109]_5$ code in Table 1. The spectra of $[136, 4, 108]_5$ codes can be obtained similarly by showing that C_0 is a disjoint union of three skew lines and two points in PG(3,5) or C_0 is a $\{20, 3; 2, 5\}$-minihyper in a 11-plane.

Table 1: The spectra of some $[n, 4, d]_5$ codes.

parameters	possible spectra
$[22, 4, 16]_5$	$(a_0, a_1, a_2, a_4, a_5, a_6) = (4, 22, 1, 30, 56, 43)$
	$(a_0, a_1, a_3, a_4, a_5, a_6) = (4, 22, 4, 24, 60, 42)$
$[23, 4, 17]_5$	$(a_0, a_1, a_3, a_4, a_5, a_6) = (3, 23, 1, 15, 57, 57)$
$[24, 4, 18]_5$	$(a_0, a_1, a_4, a_5, a_6) = (2, 24, 6, 48, 76)$
$[25, 4, 19]_5$	$(a_0, a_1, a_5, a_6) = (1, 25, 30, 100)$
$[26, 4, 20]_5$	$(a_1, a_6) = (26, 130)$
$[39, 4, 30]_5$	$(a_4, a_9) = (39, 117)$
$[70, 4, 55]_5$	$(a_5, a_{10}, a_{15}) = (6, 22, 128)$
$[76, 4, 60]_5$	$(a_6, a_{11}, a_{16}) = (8, 12, 136)$
$[101, 4, 80]_5$	$(a_1, a_{16}, a_{21}) = (1, 25, 130)$
	$(a_6, a_{11}, a_{16}, a_{21}) = (1, 3, 20, 132)$
$[123, 4, 98]_5$	$(a_0, a_{23}, a_{24}, a_{25}) = (1, 6, 50, 99)$
$[124, 4, 99]_5$	$(a_0, a_{24}, a_{25}) = (1, 31, 124)$
$[125, 4, 100]_5$	$(a_0, a_{25}) = (1, 155)$
$[132, 4, 105]_5$	$(a_{22}, a_{27}) = (24, 132)$
$[136, 4, 108]_5$	$(a_{11}, a_{25}, a_{26}, a_{27}, a_{28}) = (1, 20, 20, 35, 80)$
	$(a_{11}, a_{25}, a_{26}, a_{27}, a_{28}) = (1, 25, 5, 50, 75)$
	$(a_{22}, a_{23}, a_{26}, a_{27}, a_{28}) = (6, 12, 6, 44, 88)$
	$(a_{21}, a_{22}, a_{23}, a_{26}, a_{27}, a_{28}) = (1, 4, 13, 5, 46, 87)$
	$(a_{21}, a_{22}, a_{23}, a_{26}, a_{27}, a_{28}) = (2, 2, 14, 4, 48, 86)$
	$(a_{21}, a_{23}, a_{26}, a_{27}, a_{28}) = (3, 15, 3, 50, 85)$
$[137, 4, 109]_5$	$(a_{22}, a_{23}, a_{27}, a_{28}) = (3, 15, 28, 110)$
$[138, 4, 110]_5$	$(a_{23}, a_{28}) = (18, 138)$
$[142, 4, 113]_5$	$(a_{22}, a_{24}, a_{27}, a_{28}, a_{29}) = (2, 10, 4, 50, 90)$
	$(a_{22}, a_{23}, a_{24}, a_{27}, a_{28}, a_{29}) = (1, 2, 9, 5, 48, 91)$
	$(a_{23}, a_{24}, a_{27}, a_{28}, a_{29}) = (4, 8, 6, 46, 92)$

Lemma 4.1. (1) *The spectrum of a* $[38, 4, 29]_5$ *code satisfies* $a_i = 0$ *for all* $i \notin \{0, 1, 3\text{-}6, 8, 9\}$.
(2) *The spectrum of a* $[69, 4, 54]_5$ *code satisfies* $a_i = 0$ *for all* $i \notin \{0, 1, 4\text{-}6, 9\text{-}11, 14, 15\}$.

(3) *The spectrum of a* $[75, 4, 59]_5$ *code satisfies* $a_i = 0$ *for all* $i \notin \{5, 6, 10, 11, 15, 16\}$.
(4) *The spectrum of a* $[74, 4, 58]_5$ *code satisfies* $a_i = 0$ *for all* $i \notin \{4, 5, 6, 9, 10, 11, 14, 15, 16\}$.
(5) *The spectrum of a* $[95, 4, 75]_5$ *code satisfies* $a_i = 0$ *for all* $i \notin \{0, 5, 10, 15, 20\}$.

Proof. (1) and (2) are straightforward from Lemma 2.9. (3) and (4) can be obtained from the spectrum of a $[76, 4, 60]_5$ code (Table 1) by Lemma 2.9 applying Theorems 2.2 and 2.3, respectively.
(5) Let $\mathcal{C}$ be a $[95, 4, 75]_5$ code. By Lemma 2.9, we obtain

$$a_i = 0 \text{ for all } i \notin \{0, 1, 5, 6, 10, 11, 15, 16, 20\}.$$

We have $a_1 = 0$, for the equality (2.3) has no solution for $i = 1$, $t = 1$. Similarly it can be proved that $a_6 = a_{11} = a_{16} = 0$. □

Lemma 4.2. *The spectrum of a* $[100, 4, 79]_5$ *code is one of the following:*
(a) $(a_0, a_{16}, a_{20}, a_{21}) = (1, 25, 30, 100)$.
(b) $(a_1, a_{15}, a_{16}, a_{20}, a_{21}) = (1, a, 25 - a, 31 - a, 99 + a)$ *for some* a *with* $0 \le a \le 25$.
(c) $(a_5, a_{11}, a_{15}, a_{16}, a_{20}, a_{21}) = (1, 3, b, 20-b, 30-b, 102+b)$ *for some* b *with* $0 \le b \le 20$.
(d) $(a_6, a_{10}, a_{11}, a_{15}, a_{16}, a_{20}, a_{21}) = (1, c, 3 - c, d, 20 - d, 31 - c - d, 101 + c + d)$ *for some* c, d *with* $0 \le c \le 3$, $0 \le d \le 20$.

Proof. Let $\mathcal{C}$ be a $[100, 4, 79]_5$ code. By Lemma 2.9, we have

$$a_i = 0 \text{ for all } i \notin \{0, 1, 5, 6, 10, 11, 15, 16, 20, 21\}.$$

Applying Theorem 2.2, $\mathcal{C}$ is extendable. From Table 1, there are two possible spectra of the extension $\mathcal{C}'$:

$$\text{(i) } (a_1, a_{16}, a_{21}) = (1, 25, 130), \qquad \text{(ii) } (a_6, a_{11}, a_{16}, a_{21}) = (1, 3, 20, 132).$$

We first assume that the spectrum of $\mathcal{C}'$ is (i). Then the spectrum of $\mathcal{C}$ satisfies

$$a_i = 0 \text{ for all } i \notin \{0, 1, 15, 16, 20, 21\}.$$

Assume $a_0 > 0$. Setting $a = 0$ in (2.1), we have $b \ge 16$. So $a_0 = 1, a_1 = a_{15} = 0$. Hence $a_{16} = 25$. For $i = 0, t = 0$, the equality (2.3) has the solutions $(c_{16}, c_{21}) = (1, 4)$ and $c_{20} = 5$. Hence $a_{20} = 30$, $a_{21} = 100$, giving (a). Assuming $a_1 > 0$ yields (b) similarly.
Next, assume that the spectrum of $\mathcal{C}'$ is (ii). Then the spectrum of $\mathcal{C}$ satisfies

$$a_i = 0 \text{ for all } i \notin \{5, 6, 10, 11, 15, 16, 20, 21\}.$$

Assume $a_5 > 0$. Setting $a = 5$ in (2.1), we have $b \ge 11$. So $a_5 = 1$. Hence $a_6 = a_{10} = 0$, $a_{11} = 3$. For $i = 5$, the solutions of the equality (2.3) are $(c_{11}, c_{21}) = (1, 4)$, $(c_{15}, c_{20}) = (1, 4)$, $(c_{16}, c_{21}) = (2, 3)$ for $t = 0$; $(c_{16}, c_{21}) = (1, 4)$ or $c_{20} = 5$ for $t = 1$ and $c_{21} = 5$ for $t = 2$. Since the spectrum of a 5-plane is $(\tau_0, \tau_1, \tau_2) = (11, 10, 10)$, we obtain (c) with $a_{15} = b$ $(0 \le b \le 20)$. Assuming $a_6 > 0$ yields (d) similarly. □

5 Lower bounds on $n_5(5,d)$

Theorem 5.1. $n_5(5,d) \geq g_5(5,d)+1$ *for* $70 \leq d \leq 125$.

Proof. Let $\mathcal{C}$ be an $[n,5,d]_5$ code attaining the Griesmer bound. Suppose $70 \leq d \leq 125$. Then $\gamma_1 = 2$ by Lemma 2.7, and C_1, the set of 1-points, forms an *n-cap* no three of which are collinear. But it is known [7] that n-cap in PG(4,5) satisfies $n \leq 88$. This contradicts that $n = g_5(5,d) > 88$ for $d \geq 70$. □

Lemma 5.2 ([2])**.** *If there exists an* $[n,k,d]_q$ *code, then so does an* $[n-d,k-1,\lceil d/q \rceil]_q$ *code.*

Suppose $\mathcal{C}$ is a $[g_q(5,d),5,q^4-q^3-q^2-q+1]$ code. Then there exists a $[q^3-q-1,4,q^3-q^2-q]_q$ code $\mathcal{C}'$ by Lemma 5.2. $\mathcal{C}'$ corresponds to a $\{\theta_2+\theta_1,\theta_1+\theta_0;3,q\}$-minihyper, which does not exist by Theorem 2.10(2). Hence we obtain the lower bound on $n_q(5,d)$ in Theorem 1.7.

If there exists a $[g_5(5,d)=129,5,d=101]_5$ code, then so does a $[28,4,21]_5$ code by the above lemma, contradicting Theorem 1.2(2). Similarly, we get $n_5(5,d) \geq g_5(5,d)+1$ for $d \in \{$102-120,161-175,186-250,301-375,411-475$\}$ and that $n_5(5,d) \geq g_5(5,d)+2$ for $121 \leq d \leq 125$ by Lemma 5.2. We also know that $n_5(5,d) \geq g_5(5,d)+1$ for $d \in \{$476-495,566-575$\}$ from Theorem 2.1. Hence, to give the lower bounds on $n_5(5,d)$ in Theorem 1.3, it suffices to prove the following:

Theorem 5.3. *There exists no* $[g_5(5,d),5,d]_5$ *code for* $d \in \{144-150, 268-275, 288-290, 293-300, 394, 395, 398-400, 521-525, 538-540, 542-550, 563-565\}$.

Theorem 5.4. *There exists no* $[g_5(5,d)+1,5,d]_5$ *code for* $373 \leq d \leq 375$.

To prove Theorems 5.3 and 5.4, we show the nonexistence of $[g_5(5,d),5,d]_5$ codes for $d \in \{$78,96,144,146,268,271,288,293,296,394,398,521,538,542,546,563$\}$ and the nonexistence of a $[g_5(5,d)+1,5,d]_5$ code for $d = 373$.

Lemma 5.5. *There exists no* $[706,5,564]_5$ *code.*

Proof. We first note that a $[707,5,565]_5$ code does not exist. Otherwise, such a code is projective and the set of 0-points C_0 forms a $\{2\theta_2+2\theta_1,2\theta_1+2\theta_0;4,5\}$-minihyper in PG(4,5), which does not exist by Theorem 2.10(4).

Suppose a $[706,5,564]_5$ code $\mathcal{C}$ exists. Let Δ be a γ_3-solid. Then Δ has no j-planes for $j \notin \{$22-24,27-29$\}$ by Table 1, so $a_i = 0$ for all $i < 106$ by Lemma 2.9. Hence $a_i = 0$ for all $i \notin \{$131,132,136-138,141,142$\}$. From (2.2) we get

$$55a_{131}+45a_{132}+15a_{136}+10a_{137}+6a_{138} = 4230. \tag{5.1}$$

For $i = 138$, the maximum possible contributions of c_j's in (2.3) to the LHS of (5.1) are $(c_{131}, c_{137}, c_{142}) = (2, 1, 2)$ for $t = 23$ and $(c_{141}, c_{142}) = (2, 3)$ for $t = 28$. Estimating the LHS of (5.1) we get $4230 \leq 120 \cdot 18 + 0 \cdot 138 + 6 = 2166$, a contradiction. Hence $a_{138} = 0$. It follows that $a_i = 0$ for all $i \not\equiv 1, 2 \pmod 5$. Applying Theorem 2.2, $\mathcal{C}$ is extendable, which contradicts the nonexistence of its extension. This completes the proof. □

Theorem 5.6. *There exists no* $[705, 5, 563]_5$ *code.*

Proof. Suppose $\mathcal{C}$ is a putative $[705, 5, 563]_5$ code and let Δ be a γ_3-solid. Then Δ has no j-planes for $j \notin \{$22-24,27-29$\}$ by Table 1, so $a_i = 0$ for all $i < 105$ by Lemma 2.9. Hence $a_i = 0$ for all $i \notin \{$125,130-132,135-138,140-142$\}$. From an argument similar to that in the proof of Lemma 5.5, it can be shown that $a_{138} = 0$. Now, we have $a_i = 0$ for all $i \not\equiv 0, 1, 2 \pmod 5$. Applying Theorem 2.3, C is extendable, which contradicts Lemma 5.5. This completes the proof. □

Theorem 5.7. *There exists no* $[684, 5, 546]_5$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[684, 5, 546]_5$ code. A γ_3-solid Δ has the spectrum $(\tau_{23}, \tau_{28}) = (18, 138)$ by Table 1, so $a_i = 0$ for all $i < 109$ by Table 1. Hence

$$a_i = 0 \text{ for all } i \notin \{124, 125, 134{-}138\}.$$

Let Π be an i-solid. If $i = 124$, then Π has no 23-planes and 28-planes by Table 1, which contradicts the spectrum of Δ. Hence $a_{124} = 0$. Similarly $a_{125} = 0$ by Table 1. For $i = 138$, $t = 23$, the equality (2.3) is

$$4c_{134} + 3c_{135} + 2c_{136} + c_{137} = 29$$

which has no solution, a contradiction. This completes the proof. □

Theorem 5.8. *There exists no* $[679, 5, 542]_5$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[679, 5, 542]_5$ code. By Table 1, the spectrum of a γ_3-solid Δ is $(\tau_{22}, \tau_{23}, \tau_{27}, \tau_{28}) = (3, 15, 28, 110)$, so $a_i = 0$ for all $i < 104$ by Lemma 2.9. Hence $a_i = 0$ for all $i \notin \{$104,124,125,129-132,134-137$\}$.
Let Π be an i-solid. If $i = 124$, then Π has only 0-planes, 24-planes and 25-planes by Table 1, contradicting the spectrum of Δ. Hence $a_{124} = 0$. Similarly one can prove $a_{125} = 0$. For $i = 137$, the maximum possible contributions of c_j's in (2.3) to the LHS of (2.2) are $(c_{104}, c_{137}) = (1, 4)$ for $t = 22$; $(c_{129}, c_{134}, c_{136}) = (3, 1, 1)$ for $t = 23$; $(c_{129}, c_{137}) = (1, 4)$ for $t = 27$ and $(c_{134}, c_{137}) = (1, 4)$ for $t = 28$. Estimating the LHS of (2.2) we get

$$\text{RHS of (2.2)} = 5743 \leq 528 \cdot 3 + 87 \cdot 15 + 28 \cdot 28 + 3 \cdot 110 = 4003,$$

a contradiction. □

Lemma 5.9. *There exists no* $[676, 5, 540]_5$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[676, 5, 540]_5$ code. Then a γ_3-solid Δ has no j-planes for $j \notin \{$21-23, 26-28$\}$ by Table 1, so $a_i = 0$ for all $i < 51$ by Lemma 2.9. It follows from Theorem 2.4 and Lemma 2.9 that

$$a_i = 0 \text{ for all } i \notin \{101, 121, 126, 131, 136\}.$$

Let Π be a 101-solid. Setting $i = 101$, the maximum possible contributions of c_j's in (2.3) to the LHS of (2.2) are $(c_{101}, c_{121}, c_{126}, c_{131}) = (2, 1, 1, 1)$ for $t = 1$; $(c_{101}, c_{121}, c_{131}) = (1, 2, 2)$ for $t = 6$; $(c_{101}, c_{121}, c_{136}) = (1, 1, 3)$ for $t = 11$; $c_{131} = 5$ for $t = 16$ and $c_{136} = 5$ for $t = 21$. Estimating the LHS of (2.2) according to each of the two spectra of 101-solids in Table 1, we get

(1) $166 \leq 46 \cdot 1 + 0 \cdot 25 + 0 \cdot 130 + 21 = 67$,

(2) $166 \leq 27 \cdot 1 + 24 \cdot 3 + 0 \cdot 20 + 0 \cdot 132 + 21 = 120$,

giving contradictions. Hence $a_{101} = 0$. From Lemma 2.8, we get $a_{126} + a_{131} = -14$, a contradiction. This completes the proof. □

Nonexistence of a $[675, 5, 539]_5$ code and a $[674, 5, 538]_5$ code can be proved by applying Theorems 2.2 and 2.3 similarly to the proofs of Lemma 5.5 and Theorem 5.6, so we omit them here.

Theorem 5.10. *There exists no* $[674, 5, 538]_5$ *code.*

Theorem 5.11. *There exists no* $[653, 5, 521]_5$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[653, 5, 521]_5$ code. Then a γ_3-solid Δ has the spectrum $(\tau_{22}, \tau_{27}) = (24, 132)$ by Table 1, so $a_i = 0$ for all $i < 103$ by Lemma 2.9. Hence

$$a_i = 0 \text{ for all } i \notin \{103, 104, 123, 124, 125, 128{-}132\}.$$

Let Π be an i-solid. If $i = 123$, then Π has no 22-planes and no 27-planes by Table 1, contradicting the spectrum of Δ. Hence $a_{123} = 0$. Similarly we get $a_{124} = a_{125} = 0$. Setting $i = 132$, the minimum possible contributions of c_j's in (2.3) to the LHS of (2.2) are $(c_{104}, c_{131}, c_{132}) = (1, 1, 3)$ for $t = 22$ and $(c_{131}, c_{132}) = (4, 1)$ for $t = 27$. Estimating the LHS of (2.2) we get

$$7036 \geq 378 \cdot 24 + 0 \cdot 132 = 9072,$$

a contradiction. This completes the proof. □

Lemma 5.12. *There exists no* $[501, 5, 400]_5$ *code.*

Proof. Let $\mathcal{C}$ be a $[501,5,400]_5$ code. By Theorem 2.4 and Lemma 2.9, we have

$$a_i = 0 \text{ for all } i \notin \{1, 26, 76, 101\}.$$

It follows from the three equalities of Lemma 2.8 that $6a_1 + 3a_{26} = 4$, which has no solution. This completes the proof. □

Applying Theorems 2.2 and 2.3 again, we get the nonexistence of a $[500,5,399]_5$ code and a $[499,5,398]_5$ code.

Theorem 5.13. *There exists no* $[499,5,398]_5$ *code.*

Lemma 5.14. *There exists no* $[495,5,395]_5$ *code.*

Proof. Let $\mathcal{C}$ be a $[495,5,395]_5$ code. Then a γ_3-solid Δ has no j-planes for $j \notin \{0,1,5,6,10,11,15,16,20,21\}$ by Lemma 4.2. It follows from Theorem 2.4 and Lemma 2.9 that

$$a_i = 0 \text{ for all } i \notin \{0, 25, 70, 75, 95, 100\}.$$

Suppose $a_0 > 0$. Setting $i = 0$ in (3.1), we have $j \geq 95$. Then the equality (2.2) gives $38a_0 = 77$, a contradiction. Hence $a_0 = 0$.
Estimating the LHS of (2.2) with the spectra in Table 1 for $i = 25, 70$, we can get contradictions. So $a_{25} = a_{70} = 0$. Then the equalities of Lemma 2.8 yields $2a_{75} = 77$, a contradiction. This completes the proof. □

Theorem 5.15. *There exists no* $[494,5,394]_5$ *code.*

Proof. Let $\mathcal{C}$ be a $[494,5,394]_5$ code. Then a γ_3-solid Δ has no j-planes for $j \notin \{0,1,5,6,10,11,15,16,20,21\}$ by Lemma 4.2. By Lemma 2.9, we have

$$a_i = 0 \text{ for all } i \notin \{0, 1, 24, 25, 26, 69, 70, 74, 75, 76, 94, 95, 99, 100\}.$$

Suppose $a_1 > 0$. Setting $i = 1$ in (3.1), we have $j \geq 93$. Then the equality (2.2) gives $15a_{94} + 10a_{95} = 6664$, which has no solution. Hence $a_1 = 0$.
Estimating the LHS of (2.2) with the spectra in Table 1 for $i = 26, 76$, we can get contradictions. So $a_{26} = a_{76} = 0$.
Now, we have $a_i = 0$ for all $i \not\equiv 4, 0 \pmod 5$. Applying Theorem 2.2, $\mathcal{C}$ is extendable, which contradicts Lemma 5.14. This completes the proof. □

Lemma 5.16. *There exists no* $[470,5,375]_5$ *code.*

Proof. Let $\mathcal{C}$ be a $[470,5,375]_5$ code. Then, by Lemma 4.1(5), a γ_3-solid Δ has no j-planes for $j \notin \{0,5,10,15,20\}$. By Lemma 2.9, we have

$$a_i = 0 \text{ for all } i \notin \{0, 70, 95\}.$$

Note that $\gamma_0 \le 2$ by Lemma 2.5(1). From Lemma 2.8, we get

$$133a_0 = 50 + 5\lambda_2. \tag{5.2}$$

Suppose $a_0 > 0$. Setting $a = 0$ in (2.1), we have $b \ge 90$. So $a_0 = 1$, which contradicts (5.2). Hence $a_0 = 0$. Then (5.2) yields $\lambda_2 = -10$, a contradiction. □

Applying Theorems 2.2, 2.3, we get the nonexistence of a $[469, 5, 374]_5$ code and a $[468, 5, 373]_5$ code.

Theorem 5.17. *There exists no* $[468, 5, 373]_5$ *code.*

Theorem 5.18. *There exists no* $[372, 5, 296]_5$ *code.*

Proof. Let $\mathcal{C}$ be a $[372, 5, 296]_5$ code. Then the spectrum of a γ_3-solid Δ is $(\tau_6, \tau_{11}, \tau_{16}) = (8, 12, 136)$ by Table 1, so $a_i = 0$ for all $i < 22$ by Lemma 2.9. Hence

$$a_i = 0 \text{ for all } i \notin \{22–26, 47, 48, 72–76\},$$

Suppose $a_{26} > 0$. For $i = 26$, $t = 1$, the equality (2.3) has no solution with $c_{76} = 0$ since a 76-solid does not contain a 1-plane while a 26-solid does. Hence $a_{26} = 0$. Similarly we can prove $a_{25} = a_{24} = a_{23} = a_{22} = 0$ using the spectra in Table 1.
For $i = 76$, $t = 6$, the equality (2.3) is

$$29c_{47} + 28c_{48} + 4c_{72} + 3c_{73} + 2c_{74} + c_{75} = 54,$$

which has no solution. This completes the proof. □

Lemma 5.19. *There exists no* $[370, 5, 295]_5$ *code.*

Proof. Let $\mathcal{C}$ be a $[370, 5, 295]_5$ code. Then a γ_3-solid Δ has no j-planes for $j \notin \{5, 6, 10, 11, 15, 16\}$ by Lemma 4.1(3). Hence it follows from Theorem 2.4 and Lemma 2.9 that

$$a_i = 0 \text{ for all } i \notin \{0, 25, 70, 75\}.$$

Since Δ has no 0-planes, $a_0 = 0$. Then the equalities of Lemma 2.8 yield $9a_{25} = 84$, a contradiction. This completes the proof. □

For a putative $[369, 5, 294]_5$ code $\mathcal{C}$, it can be proved that $a_{26} = 0$ since the equality (2.3) has no solution for $i = 26, t = 1$ with $c_{70} = c_{74} = c_{75} = 0$. Applying Theorem 2.2, $\mathcal{C}$ is extendable, contradicting the above lemma. Hence There exists no $[369, 5, 294]_5$ code. Applying Theorem 2.3 similarly after proving $a_{26} = 0$ for a putative $[368, 5, 293]_5$ code, we get the following.

Theorem 5.20. *There exists no* $[368, 5, 293]_5$ *code.*

Lemma 5.21. *There exists no* $[364, 5, 290]_5$ *code.*

Proof. Let $\mathcal{C}$ be a $[364, 5, 290]_5$ code. Then a γ_3-solid Δ has no j-planes for $j \notin \{$4-6, 9-11, 14-16$\}$ by Lemma 4.1(4), so $a_i = 0$ for all $i < 14$ by Lemma 2.9. Hence it follows from Theorem 2.4 and Lemma 2.9 that

$$a_i = 0 \text{ for all } i \notin \{24, 39, 64, 69, 74\}.$$

Estimating the LHS of (2.2) for $i = 24, 39$ using the spectra in Table 1, we can get contradictions so that $a_{24} = a_{39} = 0$. Solving the equalities of Lemma 2.8, we get $(a_{64}, a_{69}, a_{74}) = (411, -620, 990)$, a contradiction. This completes the proof. □

Lemma 5.22. *There exists no* $[363, 5, 289]_5$ *code.*

Proof. We only give a sketch of the proof. Let $\mathcal{C}$ be a putative $[363, 5, 289]_5$ code. Then, by Lemmas 4.1(4), 2.9, we have

$$a_i = 0 \text{ for all } i \notin \{23{-}26, 38, 39, 48, 63, 64, 68, 69, 70, 73, 74\}.$$

Estimating the LHS of (2.2) with the spectra in Table 1 for $i = 25, 26, 23, 24, 70$, we can get contradictions so that $a_{23} = a_{24} = a_{25} = a_{26} = a_{70} = 0$. Thus we have $a_i = 0$ for all $i \not\equiv 3, 4 \pmod 5$. Applying Theorem 2.2, $\mathcal{C}$ is extendable, which contradicts Lemma 5.21. This completes the proof. □

For a putative $[362, 5, 288]_5$ code $\mathcal{C}$, it can be proved that $a_{22} = a_{23} = a_{24} = a_{25} = a_{26} = a_{70} = 0$ by estimating the LHS of (2.2) with the spectra in Table 1 for $i = 22, 23, 24, 25, 26, 70$. Hence, applying Theorem 2.3, we get the following.

Theorem 5.23. *There exists no* $[362, 5, 288]_5$ *code.*

Theorem 5.24. *There exists no* $[341, 5, 271]_5$ *code.*

Proof. Let $\mathcal{C}$ be a $[341, 5, 271]_5$ code. Then a γ_3-solid Δ has the spectrum $(\tau_5, \tau_{10}, \tau_{15}) = (6, 22, 128)$ by Table 1. By Lemma 2.9, we have

$$a_i = 0 \quad \text{for all} \quad i \notin \{16, 26, 41, 42, 66{-}70\}.$$

Let Π be an i-solid. If $i = 26$, then Π has only 1-planes and 6-planes by Table 1, which are not contained in Δ, a contradiction. Hence $a_{26} = 0$.
For $i = 16$, $t = 0$, the equality (2.3) has no solution while a 16-solid has a 0-plane by Table 1. Hence $a_{16} = 0$.
For $i = 70$, $t = 5$, the equality (2.3) has no solution, contradicting the spectrum of a γ_3-solid. This completes the proof. □

Lemma 5.25. *There exists no* $[339, 5, 270]_5$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[339, 5, 270]_5$ code. Applying Theorem 2.4 we have $i \equiv 4 \pmod 5$ for all $a_i > 0$. A γ_3-solid Δ has no j-planes for $j \notin \{0, 1, 4\text{-}6, 9\text{-}11, 14{,}15\}$ by Lemma 4.1(2). Hence it follows from Theorem 2.4 and Lemma 2.9 that

$$a_i = 0 \text{ for all } i \notin \{24, 39, 64, 69\}.$$

Hence the equalities of Lemma 2.8 yield

$$12a_{24} + 5a_{39} = 135. \tag{5.3}$$

Estimating the LHS of (5.3) for $i = 24$, we get a contradiction so that $a_{24} = 0$. Then $a_{39} = 27$ by (5.3). On the other hand, setting $a = 39, c = 4$ in (2.1), we have $b \geq 44$. So $a_{39} = 1$, a contradiction. This completes the proof. □

Applying Theorems 2.2, 2.3, we get the following. See [26] for the detail of the proof.

Theorem 5.26. *There exists no* $[337, 5, 268]_5$ *code.*

Theorem 5.27. *There exists no* $[185, 5, 146]_5$ *code.*

Proof. Let $\mathcal{C}$ be a $[185, 5, 146]_5$ code. Then a γ_3-solid Δ has the spectrum $(\tau_4, \tau_9) = (39, 117)$ by Table 1. By Lemma 2.9, we have

$$a_i = 0 \quad \text{for all} \quad i \notin \{10, 11, 12, 25, 26, 35{-}39\}.$$

Let Π be an i-solid. If $i = 26$, then Π has only 1-planes and 6-planes by Table 1, which are not contained in Δ, a contradiction. Hence $a_{26} = 0$. We have $a_{25} = 0$ similarly.
For $i = 39$, the solutions of (2.3) are
$(c_{10}, c_{39}) = (1, 4)$, $(c_{11}, c_{38}, c_{39}) = (1, 1, 3)$, $(c_{12}, c_{37}, c_{39}) = (1, 1, 3)$
and $(c_{12}, c_{38}, c_{39}) = (1, 2, 2)$ for $t = 4$;
$(c_{35}, c_{39}) = (1, 4)$, $(c_{36}, c_{38}, c_{39}) = (1, 1, 3)$, $(c_{37}, c_{39}) = (2, 3)$
and $(c_{37}, c_{38}, c_{39}) = (1, 2, 2)$ for $t = 9$.
Since $c_j > 0$ with $j \in \{10, 11, 12\}$ can happen only for $t = 4$, we get

$$a_{10} + a_{11} + a_{12} = \tau_4 = 39. \tag{5.4}$$

Suppose $a_{10} > 0$. Setting $a = 10$ in (2.1), we have $b \geq 24$. So $a_{10} = 1$, $a_{11} = a_{12} = 0$, which contradicts (5.4). Hence $a_{10} = 0$. Similarly we get $a_{11} = a_{12} = 0$. Then (5.4) does not hold, a contradiction. This completes the proof. □

Lemma 5.28. *There exists no* $[183, 5, 145]_5$ *code.*

Proof. Let $\mathcal{C}$ be a $[183, 5, 145]_5$ code. Then a γ_3-solid Δ has no 2-planes and 7-planes by Lemma 4.1(1). By Theorem 2.4 and Lemma 2.9, we have

$$a_i = 0 \quad \text{for all} \quad i \notin \{23, 28\}.$$

The equalities of Lemma 2.8 yield $3a_{23} = 226$, a contradiction. □

Theorem 5.29. *There exists no* $[182, 5, 144]_5$ *code.*

Proof. Let $\mathcal{C}$ be a $[182, 5, 144]_5$ code. By Lemmas 4.1(1), 2.9, we have

$$a_i = 0 \text{ for all } i \notin \{0, 1, 12, 22{-}26, 32, 37, 38\}.$$

Suppose $a_0 > 0$. Setting $a = 0$ in (2.1), we have $b \geq 30$. So the equality (2.2) gives $15a_{32} = 8437$, a contradiction. Hence $a_0 = 0$. Similarly, we also get $a_1 = 1$.
Suppose $a_{26} > 0$. Recall that the spectrum of a 26-solid is $(\tau_1, \tau_6) = (26, 130)$ by Table 1. Setting $i = 26$, the maximum possible contributions of c_j's in (2.3) to the LHS of (2.2) are $(c_{12}, c_{37}, c_{38}) = (1, 3, 1)$ for $t = 1$ and $(c_{37}, c_{38}) = (4, 1)$ for $t = 6$. Estimating the LHS of (2.2) we get

$$9140 \leq 325 \cdot 26 + 0 \cdot 130 + 66 = 8516,$$

a contradiction. Hence $a_{26} = 0$. Similarly one can prove that $a_{25} = a_{12} = a_{24} = 0$. Applying Theorem 2.2, $\mathcal{C}$ is extendable, which contradicts Lemma 5.28. This completes the proof. □

Table 2: Values and bounds for $n_5(5,d)$, $d \le 5^4$,
except for some d with $n_5(5,d) = g_5(5,d)$.

d	$g_5(5,d)$	$n_5(5,d)$	d	$g_5(5,d)$	$n_5(5,d)$	d	$g_5(5,d)$	$n_5(5,d)$
1	5	5	51	67	67-69	101	129	130-133
2	6	6	52	68	68-70	102	130	131-134
3	7	8	53	69	69-71	103	131	132-135
4	8	9	54	70	70-72	104	132	133-137
5	9	10	55	71	71-73	105	133	134-138
6	11	11	56	73	74	106	135	136-139
7	12	13	57	74	75	107	136	137-140
8	13	14	58	75	76	108	137	138-142
9	14	15	59	76	77	109	138	139-143
10	15	16-17	60	77	78	110	139	140-144
11	17	18	61	79	80-82	111	141	142-145
12	18	19	62	80	81-83	112	142	143-147
13	19	20	63	81	82-84	113	143	144-148
14	20	21-22	64	82	83-85	114	144	145-149
15	21	22-23	65	83	84-86	115	145	146-150
16	23	24	66	85	86-88	116	147	148-151
17	24	25	67	86	87-89	117	148	149-152
18	25	26-27	68	87	88-90	118	149	150-153
19	26	27-28	69	88	89-91	119	150	151-154
20	27	28-29	70	89	90-92	120	151	152-155
21	29	30-31	71	91	92-94	121	153	155-158
22	30	31-32	72	92	93-95	122	154	156-159
23	31	32-33	73	93	94-97	123	155	157-161
24	32	33-34	74	94	95-98	124	156	158-162
25	33	35-36	75	95	96-99	125	157	159-163
26	36	36-37	76	98	99-100	126	161	161-164
27	37	37-38	77	99	100-101	127	162	162-166
28	38	38-39	78	100	101-102	128	163	163-167
29	39	39-40	79	101	102-103	129	164	164-168
30	40	40-41	80	102	103-105	130	165	165-169
31	42	42-43	81	104	105-106	131	167	167-171
32	43	43-45	82	105	106-107	132	168	168-172
33	44	45-46	83	106	107-108	133	169	169-173
34	45	46-47	84	107	108-109	134	170	170-174
35	46	47-48	85	108	109-110	135	171	171-176
36	48	48-49	86	110	111-113	136	173	173-177
37	49	49-50	87	111	112-114	137	174	174-178
38	50	51	88	112	113-115	138	175	175-179
39	51	52	89	113	114-117	139	176	176-181
40	52	53	90	114	115-118	140	177	177-182
41	54	55-56	91	116	117-119	141	179	179-183
42	55	56-57	92	117	118-120	142	180	180-185
43	56	57-59	93	118	119-121	143	181	181-186
44	57	58-60	94	119	120-122	144	182	183-187
45	58	59-61	95	120	121-123	145	183	184-188
46	60	61-62	96	122	123-125	146	185	186-190
47	61	62-63	97	123	124-126	147	186	187-191
48	62	63-64	98	124	125-127	148	187	188-192
49	63	64-65	99	125	126-128	149	188	189-193
50	64	65-66	100	126	127-129	150	189	190-195

Table 2: Continued.

d	$g_5(5,d)$	$n_5(5,d)$	d	$g_5(5,d)$	$n_5(5,d)$	d	$g_5(5,d)$	$n_5(5,d)$
151	192	192-196	201	254	255-259	251	317	317-322
152	193	193-197	202	255	256-260	252	318	318-323
153	194	194-199	203	256	257-261	253	319	319-324
154	195	195-200	204	257	258-263	254	320	320-325
155	196	196-201	205	258	259-264	255	321	321-326
156	198	198-202	206	260	261-265	256	323	323-328
157	199	199-204	207	261	262-266	257	324	324-329
158	200	200-205	208	262	263-268	258	325	325-330
159	201	201-206	209	263	264-269	259	326	326-331
160	202	202-207	210	264	265-270	260	327	327-333
161	204	205-209	211	266	267-272	261	329	329-334
162	205	206-210	212	267	268-273	262	330	330-335
163	206	207-211	213	268	269-274	263	331	331-337
164	207	208-212	214	269	270-275	264	332	332-338
165	208	209-214	215	270	271-276	265	333	333-339
166	210	211-215	216	272	273-278	266	335	335-340
167	211	212-216	217	273	274-279	267	336	336-341
168	212	213-217	218	274	275-280	268	337	338-343
169	213	214-219	219	275	276-281	269	338	339-344
170	214	215-220	220	276	277-283	270	339	340-345
171	216	217-221	221	278	279-284	271	341	342-346
172	217	218-223	222	279	280-285	272	342	343-347
173	218	219-224	223	280	281-286	273	343	344-348
174	219	220-225	224	281	282-288	274	344	345-349
175	220	221-226	225	282	283-289	275	345	346-350
176	223	223-227	226	285	286-290	276	348	348-353
177	224	224-229	227	286	287-292	277	349	349-354
178	225	225-230	228	287	288-293	278	350	350-355
179	226	226-231	229	288	289-294	279	351	351-356
180	227	227-232	230	289	290-295	280	352	352-358
181	229	229-234	231	291	292-297	281	354	354-359
182	230	230-235	232	292	293-298	282	355	355-360
183	231	231-236	233	293	294-299	283	356	356-361
184	232	232-237	234	294	295-300	284	357	357-363
185	233	233-239	235	295	296-302	285	358	358-364
186	235	236-240	236	297	298-303	286	360	360-365
187	236	237-241	237	298	299-304	287	361	361-366
188	237	238-243	238	299	300-305	288	362	363-368
189	238	239-244	239	300	301-306	289	363	364-369
190	239	240-245	240	301	302-308	290	364	365-370
191	241	242-246	241	303	304-309	291	366	366-371
192	242	243-248	242	304	305-310	292	367	367-372
193	243	244-249	243	305	306-312	293	368	369-373
194	244	245-250	244	306	307-313	294	369	370-374
195	245	246-251	245	307	308-314	295	370	371-375
196	247	248-253	246	309	310-315	296	372	373-376
197	248	249-254	247	310	311-317	297	373	374-377
198	249	250-255	248	311	312-318	298	374	375-378
199	250	251-256	249	312	313-319	299	375	376-379
200	251	252-258	250	313	314-320	300	376	377-380

Table 2: Continued.

d	$g_5(5,d)$	$n_5(5,d)$	d	$g_5(5,d)$	$n_5(5,d)$	d	$g_5(5,d)$	$n_5(5,d)$
301	379	380-384	351	441	442-446	401	504	504-508
302	380	381-385	352	442	443-447	402	505	505-509
303	381	382-386	353	443	444-448	403	506	506-510
304	382	383-388	354	444	445-449	404	507	507-511
305	383	384-389	355	445	446-451	405	508	508-512
306	385	386-390	356	447	448-452	406	510	510-514
307	386	387-391	357	448	449-453	407	511	511-515
308	387	388-393	358	449	450-454	408	512	512-516
309	388	389-394	359	450	451-455	409	513	513-517
310	389	390-395	360	451	452-457	410	514	514-519
311	391	392-396	361	453	454-458	411	516	517-520
312	392	393-398	362	454	455-459	412	517	518-521
313	393	394-399	363	455	456-460	413	518	519-522
314	394	395-400	364	456	457-462	414	519	520-523
315	395	396-401	365	457	458-463	415	520	521-525
316	397	398-403	366	459	460-464	416	522	523-526
317	398	399-404	367	460	461-465	417	523	524-527
318	399	400-405	368	461	462-467	418	524	525-528
319	400	401-406	369	462	463-468	419	525	526-530
320	401	402-408	370	463	464-469	420	526	527-531
321	403	404-409	371	465	466-470	421	528	529-532
322	404	405-410	372	466	467-471	422	529	530-533
323	405	406-411	373	467	469-473	423	530	531-535
324	406	407-412	374	468	470-474	424	531	532-536
325	407	408-414	375	469	471-475	425	532	533-537
326	410	411-415	376	473	473-476	426	535	536-538
327	411	412-416	377	474	474-478	427	536	537-539
328	412	413-417	378	475	475-479	428	537	538-541
329	413	414-419	379	476	476-480	429	538	539-542
330	414	415-420	380	477	477-481	430	539	540-543
331	416	417-421	381	479	479-482	431	541	542-544
332	417	418-422	382	480	480-484	432	542	543-545
333	418	419-424	383	481	481-485	433	543	544-547
334	419	420-425	384	482	482-486	434	544	545-548
335	420	421-426	385	483	483-487	435	545	546-549
336	422	423-427	386	485	485-489	436	547	548-550
337	423	424-428	387	486	486-490	437	548	549-551
338	424	425-430	388	487	487-491	438	549	550-553
339	425	426-431	389	488	488-492	439	550	551-554
340	426	427-432	390	489	489-494	440	551	552-555
341	428	429-433	391	491	491-495	441	553	554-556
342	429	430-435	392	492	492-496	442	554	555-557
343	430	431-436	393	493	493-497	443	555	556-558
344	431	432-437	394	494	495-499	444	556	557-559
345	432	433-438	395	495	496-500	445	557	558-560
346	434	435-439	396	497	497-501	446	559	560-562
347	435	436-441	397	498	498-502	447	560	561-563
348	436	437-442	398	499	500-503	448	561	562-564
349	437	438-443	399	500	501-504	449	562	563-565
350	438	439-444	400	501	502-505	450	563	564-567

Table 2: Continued.

d	$g_5(5,d)$	$n_5(5,d)$	d	$g_5(5,d)$	$n_5(5,d)$	d	$g_5(5,d)$	$n_5(5,d)$
451	566	567-568	481	603	604-605	536	672	672-673
452	567	568-569	482	604	605-606	537	673	673-674
453	568	569-570	483	605	606-607	538	674	675
454	569	570-571	484	606	607-608	539	675	676
455	570	571-573	485	607	608-610	540	676	677
456	572	573-574	486	609	610-611	541	678	678-679
457	573	574-575	487	610	611-612	542	679	680
458	574	575-576	488	611	612-613	543	680	681
459	575	576-577	489	612	613-614	544	681	682
460	576	577-578	490	613	614-615	545	682	683
461	578	579-580	491	615	616	546	684	685
462	579	580-581	492	616	617	547	685	686
463	580	581-583	493	617	618	548	686	687
464	581	582-584	494	618	619	549	687	688
465	582	583-585	495	619	620	550	688	689
466	584	585-586	521	653	654	561	703	703-704
467	585	586-587	522	654	655	562	704	704-705
468	586	587-588	523	655	656	563	705	706
469	587	588-590	524	656	657	564	706	707
470	588	589-591	525	657	658	565	707	708
471	590	591-592	526	660	660	566	709	710
472	591	592-593	527	661	661	567	710	711
473	592	593-594	528	662	662	568	711	712
474	593	594-595	529	663	663	569	712	713
475	594	595-597	530	664	664	570	713	714
476	597	598	531	666	666-667	571	715	716
477	598	599	532	667	667-668	572	716	717
478	599	600	533	668	668-669	573	717	718
479	600	601	534	669	669-670	574	718	719
480	601	602-603	535	670	670-671	575	719	720

Acknowledgment

The authors would like to thank the anonymous referees for their valuable comments and suggestions.

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