Pebbles and Pair Groups

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Abstract

We introduce an intuitive method to compute the cycle indices of pair groups and hence enumerate different types of graphs with Pólya's theorem. We demonstrate the method on simple graphs and bipartite graphs and present a theorem for the enumeration of generic k-partite graphs.

KEYWORDS. Graph enumeration, Pólya's theorem, k-partite graphs, ordinary generating functions.

1 Introduction

Pair groups play a fundamental role in graphical enumeration. The problem of enumerating various types of graphs reduces to the computation of the cycle index of the appropriate pair group. A pair group is obtained from a permutation group, as follows. Let the permutation group act on the vertices. The associated pair group is the result of the action on the edges, these being either unordered or ordered pairs (directed graph). The permutation α maps the edge (u, v) to $(\alpha(u), \alpha(v))$.

This note presents a highly inituitive model that facilitates the computation of pair groups. The idea is simple: visualize the vertex permutation as a set of directed cycles. Now place two pebbles anywhere on the vertex permutation: these pebbles define an edge. The goal is to compute the length of the cycle that this edge belongs to. Start moving the two pebbles in parallel along the respective cycles. The length of the cycle that the edge belongs to is the number of steps taken until both pebbles have returned to their original position. Note that every such cycle will be overcounted, the factor is the number of edges on the cycle.

This paper illustrates the pebble method to enumerate simple graphs, a known result. We then apply the method to the problem of enumerating bipartite graphs. Finally we try to solve the problem of enumerating generic k-partite graphs, which appears to be new.

2 Simple graphs

The pair group $S_p^{(2)}$ for the enumeration of simple graphs results from the action of the symmetric group S_p on the vertices. We wish to compute the cycle index of this group. It includes all permutations of p elements.

Let $J = \{\mathbf{j} \mid \sum_{k=1}^{p} k j_k = p\}$. Note that the disjoint cycle decomposition $\prod_{k=1}^{p} a_k^{j_k}$ occurs

$$\frac{p!}{\prod_{k=1}^{p} (k!)^{j_k}} \prod_{k=1}^{p} \left(\frac{k!}{k}\right)^{j_k} \prod_{k=1}^{p} \frac{1}{j_k!}$$

times. (Partition the p elements into subsets, one for each cycle. A subset of size k generates $k!/k$ cycles. The same size j_k set of k-cycles may be chosen in $j_k!$ different ways.)

Hence

$$Z(S_p) = \sum_{\mathbf{j} \in J} \frac{1}{\prod_{k=1}^{p} k^{j_k} j_k!} \prod_{k=1}^{p} a_k^{j_k}.$$

We classify pebble constellations in order to obtain the cycle index of $S_p^{(2)}$. There is exactly one constellation for each edge.

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- **Two pebbles on the same cycle.** We distinguish even and odd cycles, say of length $2k$ and $2k+1$. We must pay special attention to pairs that are placed exactly across from one another if the length is even: these k pairs generate a single cycle of length k. The remaining pairs generate cycles of length $2k$. There are $\binom{2k}{2}-k = k(2k-2)$ of these. The net contribution is

$$\prod_{k=1}^{\lfloor p/2 \rfloor} \left(a_k a_{2k}^{k-1}\right)^{j_{2k}}.$$

All $\binom{2k+1}{2}$ pairs generate cycles of length $2k+1$ when the length is odd and the contribution is

$$\prod_{k=0}^{\lfloor (p-1)/2 \rfloor} \left(a_{2k+1}^{k}\right)^{j_{2k+1}}.$$

- **Two pebbles on different cycles of the same length.** There are $\binom{j_k}{2}$ ways to choose the two cycles. The pebbles return to their initial position after k steps. There are k^2 different pairs of pebbles. The contribution is

$$\prod_{k=1}^{p} a_k^{k\binom{j_k}{2}}.$$

- **Two pebbles on different cycles of different lengths, say r and s.** The pebbles return to the starting position after $\mathrm{LCM}(r,s)$ steps. There are rsj_rj_s different pairs of pebbles, and hence $\mathrm{GCD}(r,s)j_rj_s$ cycles. The contribution is

$$\prod_{1\le r<s\le p-1} a_{\mathrm{LCM}(r,s)}^{\mathrm{GCD}(r,s)j_rj_s}.$$

It follows that

$$Z(S_p^{(2)}) = \sum_{\mathbf{j}\in \mathrm{J}} \frac{1}{\prod_{k=1}^{p} k^{j_k} j_k!} \prod_{k=1}^{\lfloor p/2 \rfloor} \left(a_k a_{2k}^{k-1}\right)^{j_{2k}} \prod_{k=0}^{\lfloor (p-1)/2 \rfloor} \left(a_{2k+1}^{k}\right)^{j_{2k+1}} \prod_{k=1}^{p} a_k^{k\binom{j_k}{2}} \prod_{1\le r<s\le p-1} a_{\mathrm{LCM}(r,s)}^{\mathrm{GCD}(r,s)j_rj_s}.$$

The first few cycle indices are

$$\begin{aligned}
Z(S_1^{(2)}) &= 1\\
Z(S_2^{(2)}) &= a_1\\
Z(S_3^{(2)}) &= \frac{1}{3}a_3 + \frac{1}{2}a_1a_2 + \frac{1}{6}a_1^3\\
Z(S_4^{(2)}) &= \frac{1}{4}a_2a_4 + \frac{1}{3}a_3^2 + \frac{3}{8}a_1^2a_2^2 + \frac{1}{24}a_1^6\\
Z(S_5^{(2)}) &= \frac{1}{6}a_1a_3a_6 + \frac{1}{5}a_5^2 + \frac{1}{4}a_2a_4^2 + \frac{1}{6}a_1a_3^3 + \frac{1}{8}a_1^2a_2^4 + \frac{1}{12}a_1^4a_2^3 + \frac{1}{120}a_1^{10}
\end{aligned}$$

3 Bipartite graphs

Consider $B = ((S,T),E)$ where $|S| = |T| = p$ and $E \subseteq S\times T$. There are two types of permutations in the permutation group B_p that acts on the $2p$ vertices: permutations that take S to S and T to T (type I) and permutations that take S to T and T to S (type II).

3.1 Type I

Consider $\mathbf{i},\mathbf{j} \in \mathrm{J}$. Place the first pebble anywhere on the cycles of $\mathbf{i}$ and the second one on the cycles of $\mathbf{j}$, say these cycles have length r and s. As before, the pebbles return to the starting position after $\mathrm{LCM}(r,s)$ steps. There are rsi_rj_s different pairs of pebbles, and hence $\mathrm{GCD}(r,s)i_rj_s$ cycles. The contribution is

$$\sum_{\mathbf{i}\in \mathrm{J}} \sum_{\mathbf{j}\in \mathrm{J}} \frac{p!}{\prod_{k=1}^p k^{i_k} i_k!} \frac{p!}{\prod_{k=1}^p k^{j_k} j_k!} \prod_{1\le r,s\le p} a_{\mathrm{LCM}(r,s)}^{\mathrm{GCD}(r,s)i_rj_s}.$$

3.2 Type II

The permutations of this type alternate between vertices from S and T and contain only cycles of even length. We obtain such a permutation by merging two permutations of the same shape $\mathbf{j}$. They can be merged in $\prod_{k=1}^p j_k! k^{j_k}$ different ways, so that $\mathbf{j}$ yields

$$\left(\frac{p!}{\prod_{k=1}^p k^{j_k} j_k!}\right)^2 \prod_{k=1}^p j_k! k^{j_k} = \frac{(p!)^2}{\prod_{k=1}^p k^{j_k} j_k!}$$

permutations. Every such permutation contains j_k cycles of length $2k$. It may help to think of the vertices as being colored, say, red and blue.

Start placing pebbles. Note that the two pebbles must always be placed on different colors.

- **Two pebbles on the same cycle.** First treat the case $2k+1$. There are $(2k+1)^2$ possible pairs, $2k+1$ of these directly across from one another. The former ($2k(2k+1)$ pairs) generate cycles of length $4k+2$, the latter cycles of length $2k+1$. The contribution is

$$\prod_{k=0}^{\lfloor (p-1)/2 \rfloor} \left(a_{2k+1} a_{4k+2}^k\right)^{j_{2k+1}}.$$

All $(2k)^2$ pairs generate cycles of length $4k$ when the length is even and the contribution is

$$\prod_{k=1}^{\lfloor p/2 \rfloor} \left(a_{4k}^k\right)^{j_{2k}}.$$

- **Two pebbles on different cycles of the same length.** There are $\binom{j_k}{2}$ ways to choose the two cycles of length $2k$ and $2k^2$ possible pairs of pebbles (connect red vertices of the first cycle to blue ones of the second cycle and vice versa), each of which generates a cycle of length $2k$. The contribution is

$$\prod_{k=1}^p a_{2k}^{k\binom{j_k}{2}}.$$

- **Two pebbles on different cycles of different lengths, say $2r$ and $2s$.** The pebbles return to the starting position after $2\mathrm{LCM}(r,s)$ steps. There are $2rsj_rj_s$ different pairs of pebbles, and hence $\mathrm{GCD}(r,s)j_rj_s$ cycles. The contribution is

$$\prod_{1\le r<s\le p-1} a_{2\mathrm{LCM}(r,s)}^{\mathrm{GCD}(r,s)j_rj_s}.$$

The cycle index of $B_p^{(2)}$ is thus given by

$$Z(B_p^{(2)}) \quad = \quad \frac{1}{2} \sum_{\mathbf{i}\in \mathrm{J}} \sum_{\mathbf{j}\in \mathrm{J}} \frac{1}{\prod_{k=1}^p k^{i_k} i_k!} \frac{1}{\prod_{k=1}^p k^{j_k} j_k!} \prod_{1\le r,s\le p} a_{\mathrm{LCM}(r,s)}^{\mathrm{GCD}(r,s)i_rj_s}$$

$$+\quad \frac{1}{2}\sum_{\mathbf{j}\in \mathrm{J}} \frac{1}{\prod_{k=1}^{p} k^{j_k} j_k!} \prod_{k=0}^{\lfloor (p-1)/2\rfloor} \left(a_{2k+1} a_{4k+2}^{k}\right)^{j_{2k+1}} \prod_{k=1}^{\lfloor p/2\rfloor} \left(a_{4k}^{k}\right)^{j_{2k}} \prod_{k=1}^{p} a_{2k}^{k\binom{j_k}{2}} \prod_{1\le r<s\le p-1} a_{2\mathrm{LCM}(r,s)}^{\mathrm{GCD}(r,s) j_r j_s}.$$

The first few cycle indices are

$$\begin{aligned}
Z(B_1^{(2)}) &= a_1 \\
Z(B_2^{(2)}) &= \frac{a_4}{4} + \frac{3\,a_2^2}{8} + \frac{a_1^2\,a_2}{4} + \frac{a_1^4}{8} \\
Z(B_3^{(2)}) &= \frac{a_3\,a_6}{3} + \frac{a_1\,a_4^2}{4} + \frac{a_3^3}{9} + \frac{a_1\,a_2^4}{8} + \frac{a_1^3\,a_2^3}{6} + \frac{a_1^9}{72} \\
Z(B_4^{(2)}) &= \frac{a_4\,a_{12}}{12} + \frac{a_8^2}{8} + \frac{a_1\,a_3\,a_6^2}{6} + \frac{a_2^2\,a_6^2}{24} + \frac{a_1^2\,a_2\,a_3^2\,a_6}{12} + \frac{19\,a_4^4}{96} + \frac{a_1^2\,a_2\,a_4^3}{8} \\
&+ \frac{a_1\,a_3^5}{18} + \frac{a_1^4\,a_3^4}{72} + \frac{17\,a_2^8}{384} + \frac{5\,a_1^4\,a_2^6}{96} + \frac{a_1^8\,a_2^4}{96} + \frac{a_1^{16}}{1152} \\
Z(B_5^{(2)}) &= \frac{a_5\,a_{20}}{20} + \frac{a_{10}\,a_{15}}{30} + \frac{a_5^2\,a_{15}}{30} + \frac{a_3\,a_4\,a_6\,a_{12}}{12} + \frac{a_2\,a_3\,a_4^2\,a_{12}}{24} + \frac{a_1^2\,a_3\,a_4^2\,a_{12}}{24} + \frac{a_5\,a_{10}^2}{8} \\
&+ \frac{a_5^3\,a_{10}}{60} + \frac{a_1\,a_8^3}{8} + \frac{a_1^2\,a_2\,a_3\,a_6^3}{12} + \frac{a_2^2\,a_3^3\,a_6^2}{72} + \frac{a_2^5\,a_3\,a_6^2}{48} + \frac{a_1^2\,a_2^4\,a_3\,a_6^2}{48} + \frac{a_2^2\,a_3^5\,a_6}{36} \\
&+ \frac{a_2^5\,a_3^3\,a_6}{72} + \frac{a_1^6\,a_2^2\,a_3^3\,a_6}{72} + \frac{13\,a_5^5}{600} + \frac{3\,a_1\,a_4^6}{32} + \frac{a_1\,a_2^2\,a_4^5}{32} + \frac{a_1^3\,a_2\,a_4^5}{48} + \frac{a_1^5\,a_4^5}{480} \\
&+ \frac{a_1^3\,a_2^3\,a_4^4}{24} + \frac{a_1^4\,a_3^7}{72} + \frac{a_2^5\,a_3^5}{720} + \frac{a_1^{10}\,a_3^5}{720} + \frac{a_1\,a_2^{12}}{128} + \frac{a_1^3\,a_2^{11}}{96} + \frac{a_1^5\,a_2^{10}}{192} + \frac{a_1^9\,a_2^8}{288} + \frac{a_1^{15}\,a_2^5}{1440} + \frac{a_1^{25}}{28800}.
\end{aligned}$$

The generating functions are

$$\begin{aligned}
Z(B_1^{(2)}; 1+z) &= z+1 \\
Z(B_2^{(2)}; 1+z) &= z^4 + z^3 + 2\,z^2 + z + 1 \\
Z(B_3^{(2)}; 1+z) &= z^9 + z^8 + 2\,z^7 + 4\,z^6 + 5\,z^5 + 5\,z^4 + 4\,z^3 + 2\,z^2 + z + 1 \\
Z(B_4^{(2)}; 1+z) &= z^{16} + z^{15} + 2\,z^{14} + 4\,z^{13} + 10\,z^{12} + 13\,z^{11} + 23\,z^{10} + 26\,z^9 + 32\,z^8 \\
&+ 26\,z^7 + 23\,z^6 + 13\,z^5 + 10\,z^4 + 4\,z^3 + 2\,z^2 + z + 1 \\
Z(B_5^{(2)}; 1+z) &= z^{25} + z^{24} + 2\,z^{23} + 4\,z^{22} + 10\,z^{21} + 20\,z^{20} + 39\,z^{19} + 72\,z^{18} + 128\,z^{17} + 198\,z^{16} \\
&+ 280\,z^{15} + 353\,z^{14} + 399\,z^{13} + 399\,z^{12} + 353\,z^{11} + 280\,z^{10} + 198\,z^9 + 128\,z^8 \\
&+ 72\,z^7 + 39\,z^6 + 20\,z^5 + 10\,z^4 + 4\,z^3 + 2\,z^2 + z + 1.
\end{aligned}$$

4 Generic k-partite graphs

We try to solve the enumeration problem for generic k-partite graphs. Consider an undirected graph ("source") $G = (V, E)$ on k vertices. The corresponding k-partite graph on kp vertices contains p vertices for every vertex of G. These vertices may be connected by an edge only if there is an edge that connects the two clusters in G. We write $B = (\{V_m\}_{m=1}^{k}, F)$ where $F \subseteq \sum_{(u_1,u_2)\in E} V_{u_1} \times V_{u_2}$. We obtain bipartite graphs on $2p$ vertices when G consists of 2 vertices that are connected by an edge.

Let H be the automorphism group of G, and let B_p be the group that acts on the kp vertices. Start by computing the cycle structure of the permutations that fit a given $\alpha \in H$, indices in g_k. There are k colors. Every cycle of length l (index g_l) yields permutations that visit vertices from $V_{m_1}, V_{m_2} \ldots V_{m_l}$ in some order. These permutations contain l colors, have length divisible by l and are obtained by merging l permutations of the same shape $\mathbf{j}$ in $\prod_{k=1}^{p} (j_k!)^{l-1} (k^{l-1})^{j_k}$ different ways, so that $\mathbf{j}$ yields

$$\left(\frac{p!}{\prod_{k=1}^{p} k^{j_k} j_k!}\right)^{l} \prod_{k=1}^{p} (j_k!)^{l-1} (k^{l-1})^{j_k} = \frac{(p!)^l}{\prod_{k=1}^{p} k^{j_k} j_k!}$$

permutations. Every such permutation contains j_k cycles of length lk. This formula may also be derived by selecting a shape $\mathbf{j}$ and repeatedly placing additional nodes on $p!$ edges.

Let $x(\alpha)$ be the set of cycles of α, let $q = |x(\alpha)|$. Let l_c where $c \in x$ be the length of the cycle c and let $l_d = l_{\alpha_d}$. Finally let $\mathbf{s} \in \mathrm{J}^q$ be a vector of shapes, one for each cycle of α. Start placing pebbles.

4.1 Two pebbles on cycles obtained from different cycles of α

Say the two cycles are c_1 and c_2. Iterate over all possible shapes $\mathbf{i}$,$\mathbf{j}$ from which the cycles of B_p are obtained. The cycle lengths in $\mathbf{i}$ and $\mathbf{j}$, say r and s are scaled by l_{c_1} and l_{c_2}, respectively. The pebbles return to the starting position after $\mathrm{LCM}(l_{c_1}r, l_{c_2}s)$ steps. Define

$$e(c) = \{\, (u,v) \mid u \in c,\ v \in c,\ \{u,v\} \in E\} \quad \text{and} \quad e(c_1,c_2) = \{\, (u,v) \mid u \in c_1,\ v \in c_2,\ \{u,v\} \in E\}.$$

There are $|e(c_1,c_2)|\, rsi_rj_s$ different pairs of pebbles, and hence $|e(c_1,c_2)|\, rsi_rj_s/\mathrm{LCM}(l_{c_1}r, l_{c_2}s)$ cycles. The contribution $A(\alpha)$ is

$$A(\alpha, \mathbf{s}) = \prod_{\substack{1 \le d_1 < d_2 \le q \\ |e(\alpha_{d_1}, \alpha_{d_2})| > 0}} \ \prod_{\substack{1 \le r,s \le p \\ \mathbf{i}=s_{d_1},\, \mathbf{j}=s_{d_2}}} a_{\mathrm{LCM}(l_{d_1}r, l_{d_2}s)}^{|e(\alpha_{d_1},\alpha_{d_2})|\, rsi_rj_s/\mathrm{LCM}(l_{d_1}r, l_{d_2}s)}.$$

4.2 Two pebbles on cycles obtained from the same cycle of α

Such a cycle c must contain at least two vertices and we must have $|e(c)| > 0$.

- **Two pebbles on the same cycle, say of length l_cr.** We must check for the possibility of the two pebbles being placed exactly across from one another. This requires that

  $$l_cr \equiv 0(2) \quad \text{and} \quad l_cr/2 \not\equiv 0(l_c).$$

  If l_c is odd, this forces r to be even and hence $l_cr/2 \equiv 0(l_c)$. The two conditions can only be fulfilled if l_c is even and r is odd, say $r = 2r_1 + 1$. The two pebbles that lie across from one another are separated by a distance of $l_c(2r_1+1)/2 \equiv 1/2l_c(l_c)$, so this case only occurs if there is an edge between c_t and $c_{t+1/2l_c(l_c)}$. There are $l_cr/2$ such pairs. The total number of pairs is $|e(c)|\,(2\binom{r}{2} + r) = |e(c)|\, r^2$.

  Let

  $$y(\alpha) = \left\{ c \mid c \in x(\alpha),\ l_c \equiv 0(2),\ \{c_0, c_{1/2l_c}\} \in E \right\} \quad \text{and} \quad z(\alpha) = \{\, c \mid c \in x(\alpha),\ |e(c)| > 0,\ c \notin y(\alpha)\}.$$

  The contribution for $y(\alpha)$ consists of $l_cr/2$ opposite pairs which generate cycles of length $l_cr/2$ and $|e(c)|\, r^2 - l_cr/2$ pairs which generate cycles of length l_cr when r is odd, i.e.

  $$\prod_{k=0}^{\lfloor (p-1)/2 \rfloor} \left(a_{l_c(2k+1)/2}\, a_{l_c(2k+1)}^{|e(c)|(2k+1)/l_c - 1/2} \right)^{j_{2k+1}}$$

  and $|e(c)|\, r^2$ pairs in cycles of length l_cr when r is even, i.e.

  $$\prod_{k=1}^{\lfloor p/2 \rfloor} \left(a_{2l_ck}^{2|e(c)|k/l_c} \right)^{j_{2k}}.$$

  The contribution for $z(\alpha)$ consists of $|e(c)|\, r^2$ pairs in cycles of length l_cr, i.e.

  $$\prod_{k=1}^{p} \left(a_{l_ck}^{|e(c)|k/l_c} \right)^{j_k}.$$

- **Two pebbles on different cycles of the same length.** There are $\binom{j_k}{2}$ ways to choose the two cycles of length $l_c k$ and $2\,|e(c)|\,k^2$ possible pairs of pebbles, each of which generates a cycle of length $l_c k$. The contribution is

$$\prod_{k=1}^{p} a_{l_c k}^{2|e(c)|/l_c k \binom{j_k}{2}}.$$

- **Two pebbles on different cycles of different lengths, say $l_c r$ and $l_c s$.** The pebbles return to the starting position after $\mathrm{LCM}(l_c r, l_c s) = l_c \mathrm{LCM}(r,s,)$ steps. There are $2\,|e(c)|\,rsj_rj_s$ different pairs of pebbles, and hence $2\,|e(c)|\,rs/l_c/\mathrm{LCM}(r,s)j_rj_s$ cycles. The contribution is

$$\prod_{1\le r<s\le p-1} a_{l_c \mathrm{LCM}(r,s)}^{2|e(c)|rs/l_c/\mathrm{LCM}(r,s)j_rj_s}.$$

Let

$$\begin{aligned}
B(\alpha,\mathbf{s}) &= \prod_{\substack{1\le d\le q\\ \mathbf{j}=\mathbf{s}_d,\ \alpha_d\in y(\alpha)}} \prod_{k=0}^{\lfloor (p-1)/2\rfloor} \left(a_{l_d(2k+1)/2}\, a_{l_d(2k+1)}^{|e(\alpha_d)|(2k+1)/l_d-1/2}\right)^{j_{2k+1}} \prod_{k=1}^{\lfloor p/2\rfloor} \left(a_{2l_dk}^{2|e(\alpha_d)|k/l_d}\right)^{j_{2k}} \\
&\times \prod_{k=1}^{p} a_{l_dk}^{2|e(\alpha_d)|/l_dk\binom{j_k}{2}} \prod_{1\le r<s\le p-1} a_{l_d\mathrm{LCM}(r,s)}^{2|e(\alpha_d)|rs/l_d/\mathrm{LCM}(r,s)j_rj_s} \quad \text{and} \\
C(\alpha,\mathbf{s}) &= \prod_{\substack{1\le d\le q\\ \mathbf{j}=\mathbf{s}_d,\ \alpha_d\in z(\alpha)}} \prod_{k=1}^{p} \left(a_{l_dk}^{|e(\alpha_d)|k/l_d}\right)^{j_k} \prod_{k=1}^{p} a_{l_dk}^{2|e(\alpha_d)|/l_dk\binom{j_k}{2}} \prod_{1\le r<s\le p-1} a_{l_d\mathrm{LCM}(r,s)}^{2|e(\alpha_d)|rs/l_d/\mathrm{LCM}(r,s)j_rj_s}
\end{aligned}$$

Then

$$Z(B) = \frac{1}{|H|\,(p!)^k} \sum_{\alpha\in H} \sum_{\mathbf{s}\in \mathrm{J}^q} \prod_{d=1}^{q} \frac{(p!)^{l_{\alpha_d}}}{\prod_{k=1}^{p} k^{\mathbf{s}_{d,k}} \mathbf{s}_{d,k}!} A(\alpha,\mathbf{s})B(\alpha,\mathbf{s})C(\alpha,\mathbf{s}).$$

4.3 Method by Harary and Palmer, after Robinson

Compare this result with the theorem by Harary and Palmer, found in section 4.5 of [2]. The two results apply to two sets of graphs G that are neither equal nor disjoint.

- Harary and Palmer applies when G is complete and the clusters have different sizes, a case that is not covered by our theorem.
- Harary and Palmer and our result apply when G is complete and all clusters have the same size.
- Harary and Palmer does not apply when G is a graph other than the complete graph and the clusters have the same size, a case that is covered by our theorem.

View the contribution of $\prod_{k=1}^{p} a_k^{j_k}$ from S_p to the pair group $S_p^{(2)}$ as a map on monomials that maps

$$\prod_{k=1}^{p} a_k^{j_k} \quad \text{to} \quad \left(\prod_{k=1}^{p} a_k^{j_k}\right)^{(2)}$$

where

$$\left(\prod_{k=1}^{p} a_k^{j_k}\right)^{(2)} = \prod_{k=1}^{\lfloor p/2\rfloor} \left(a_k a_{2k}^{k-1}\right)^{j_{2k}} \prod_{k=0}^{\lfloor (p-1)/2\rfloor} \left(a_{2k+1}^{k}\right)^{j_{2k+1}} \prod_{k=1}^{p} a_k^{k\binom{j_k}{2}} \prod_{1\le r<s\le p-1} a_{\mathrm{LCM}(r,s)}^{\mathrm{GCD}(r,s)j_rj_s}.$$

Harary and Palmer show that supergraphs of a graph M without additional vertices are counted by the cycle index

$$Z(\Gamma_1(\bar{M})) = \rho Z(\Gamma_{0,1}(M))$$

where Γ_1 is the line cycle index and $\Gamma_{0,1}$ is the bivariate point-line cycle index. (The latter two indices include the edges that are present in the graph among the objects being permuted and introduce the variables $b_1, b_2, \ldots$ to refer to line cycles.) Robinson's operator ρ is a map on monomials where

$$\rho\left(\prod_{k=1}^{p} a_k^{j_k} \prod_{k=1}^{p} b_k^{i_k}\right) = \left(\prod_{k=1}^{p} a_k^{j_k}\right)^{(2)} \prod_{k=1}^{p} a_k^{-i_k}.$$

Now observe that k-partite graphs (with or without equal-sized clusters; we consider the latter) can be counted by considering supergraphs of the union U of k copies of K_p. (The interior edges of each copy of K_p are precisely the edges that are forbidden in a k-partite graph.) The point-line cycle index $Z(\Gamma_{0,1}(U))$ can be computed using the wreath product and has the form

$$Z(\Gamma_{0,1}(U)) = Z(S_k)[Z(\Gamma_{0,1}(K_p))].$$

Two important questions arise: how does the complexity of this result compare to ours (consider the number of monomials being computed) and can it be adapted to our result, where there are pairs of clusters so that two vertices from these clusters may not be connected by an edge.

Let ϕ be the number of partitions of p. Our result iterates over the permutations α of H, computing ϕ^q monomials, where q is the number of cycles of the permutation.

However, the term $Z(\Gamma_{0,1}(K_p))$ of the formula by Harary and Palmer contains precisely ϕ monomials which are substituted into the terms of the cycle index $Z(S_k)$, and every such term again contributes ϕ^q monomials where q is the number of cycles in the term. In this special case, a formula for the terms of $Z(S_k)$ that contains $\phi(k)$ terms instead of $|S_k| = k!$ terms is known, so Harary and Palmer includes a lesser number of terms.

Clearly the goal for future research should be a formula that is applicable to graphs U where there are forbiddem edges between clusters and has the complexity of the result by Harary and Palmer. The question is how to evaluate $Z(\Gamma_{0,1}(U))$ when U consists of k copies of K_p and sets of p^2 edges between certain copies of K_p. Note that the appropriate composition of cycle index must somehow take into account the two types of lines (at the interior of K_p and between copies of K_p), so an ordinary wreath product will apparently not suffice.

4.4 Application I: cycle of length 4

The permutations in H are

$$\{[0,1,2,3],[1,2,3,0],[2,3,0,1],[3,0,1,2],[0,3,2,1],[2,1,0,3],[3,2,1,0],[1,0,3,2]\}.$$

The first few cycle indices are

$$\begin{aligned}
Z(B_1) &= \frac{a_4}{4} + \frac{3\,a_2^2}{8} + \frac{a_1^2\,a_2}{4} + \frac{a_1^4}{8}\\
Z(B_2) &= \frac{a_8^2}{8} + \frac{13\,a_4^4}{32} + \frac{a_1^2\,a_2\,a_4^3}{8} + \frac{27\,a_2^8}{128} + \frac{3\,a_1^4\,a_2^6}{32} + \frac{a_1^8\,a_2^4}{32} + \frac{a_1^{16}}{128}\\
Z(B_3) &= \frac{a_{12}^3}{12} + \frac{a_6^2\,a_{12}^2}{18} + \frac{a_1\,a_3\,a^24\,a_6^2\,a_{12}}{12} + \frac{a_2^3\,a_4^3\,a_6\,a_{12}}{18} + \frac{a_4\,a_8^4}{8} + \frac{25\,a_6^6}{216} + \frac{a_3^2\,a_6^5}{36}\\
&+ \frac{a_3^4\,a_6^4}{144} + \frac{a_1^3\,a_2^3\,a_3\,a_6^4}{36} + \frac{a_2^9\,a_6^3}{54} + \frac{a_3^8\,a_6^2}{72} + \frac{a_1\,a_2^4\,a_3^5\,a_6^2}{72} + \frac{a_1^2\,a_2^8\,a_3^2\,a_6^2}{48} + \frac{a_1^6\,a_2^6\,a_3^2\,a_6^2}{144} + \frac{a_1^3\,a_2^3\,a_3^7\,a_6}{108}\\
&+ \frac{a_1^4\,a_2^7\,a_3^4\,a_6}{72} + \frac{a_1^{12}\,a_2^3\,a_3^4\,a_6}{216} + \frac{a_4^9}{24} + \frac{a_2^2\,a_4^8}{32} + \frac{a_1^2\,a_2\,a_4^8}{16} + \frac{11\,a_2^6\,a_4^6}{144} + \frac{a_1^4\,a_2^6\,a_4^5}{24} + \frac{7\,a_3^{12}}{1296} + \frac{a_1^9\,a_3^9}{648}\\
&+ \frac{a_1^6\,a_2^6\,a_3^6}{432} + \frac{a_1^{18}\,a_3^6}{1296} + \frac{19\,a_2^{18}}{864} + \frac{a_1^4\,a_2^{16}}{128} + \frac{a_1^6\,a_2^{15}}{144} + \frac{a_1^8\,a_2^{14}}{96} + \frac{a_1^{12}\,a_2^{12}}{576} + \frac{a_1^{16}\,a_2^{10}}{288} + \frac{a_1^{24}\,a_2^6}{864} + \frac{a_1^{36}}{10368}
\end{aligned}$$

The generating functions are

$$\begin{aligned}
Z(B_1; 1+z) &= z^4 + z^3 + 2\,z^2 + z + 1\\
Z(B_2; 1+z) &= z^{16} + z^{15} + 5\,z^{14} + 10\,z^{13} + 33\,z^{12} + 53\,z^{11} + 101\,z^{10}\\
&+ 122\,z^9 + 153\,z^8 + 122\,z^7 + 101\,z^6 + 53\,z^5 + 33\,z^4 + 10\,z^3 + 5\,z^2 + z + 1
\end{aligned}$$

$$
\begin{aligned}
Z(B_3; 1+z) &= z^{36} + z^{35} + 5\,z^{34} + 15\,z^{33} + 63\,z^{32} + 201\,z^{31} + 704\,z^{30} \\
&+ 2124\,z^{29} + 6220\,z^{28} + 16196\,z^{27} + 39014\,z^{26} + 84119\,z^{25} + 165072\,z^{24} + 290483\,z^{23} \\
&+ 462801\,z^{22} + 662604\,z^{21} + 857916\,z^{20} + 999685\,z^{19} + 1053498\,z^{18} + 999685\,z^{17} \\
&+ 857916\,z^{16} + 662604\,z^{15} + 462801\,z^{14} + 290483\,z^{13} + 165072\,z^{12} + 84119\,z^{11} \\
&+ 39014\,z^{10} + 16196\,z^{9} + 6220\,z^{8} + 2124\,z^{7} + 704\,z^{6} + 201\,z^{5} \\
&+ 63\,z^{4} + 15\,z^{3} + 5\,z^{2} + z + 1
\end{aligned}
$$

4.5 Application II: bowtie / hourglass on five vertices

The permutations in H are

$$\{[0,1,2,3,4],[1,0,2,3,4],[0,1,2,4,3],[1,0,2,4,3],[3,4,2,0,1],[4,3,2,0,1],[3,4,2,1,0],[4,3,2,1,0]\}.$$

The first few cycle indices are

$$
\begin{aligned}
Z(B_1; 1+z) &= \frac{a_2\,a_4}{4} + \frac{a_2^3}{4} + \frac{a_1^2\,a_2^2}{8} + \frac{a_1^4\,a_2}{4} + \frac{a_1^6}{8} \\
Z(B_2; 1+z) &= \frac{a_8^3}{8} + \frac{3\,a_4^6}{32} + \frac{a_2^2\,a_4^5}{8} + \frac{a_2^4\,a_4^4}{8} + \frac{a_2^6\,a_4^3}{16} \\
&+ \frac{a_1^2\,a_2^5\,a_4^3}{16} + \frac{3\,a_1^4\,a_2^4\,a_4^3}{64} + \frac{a_1^{12}\,a_4^3}{64} + \frac{13\,a_2^{12}}{128} + \frac{a_1^2\,a_2^{11}}{16} + \frac{9\,a_1^4\,a_2^{10}}{128} \\
&+ \frac{3\,a_1^6\,a_2^9}{64} + \frac{5\,a_1^8\,a_2^8}{256} + \frac{a_1^{12}\,a_2^6}{128} + \frac{a_1^{14}\,a_2^5}{64} + \frac{a_1^{16}\,a_2^4}{64} + \frac{a_1^{24}}{256} \\
Z(B_3; 1+z) &= \frac{a_2\,a_8^2\,a_{12}\,a_{24}}{24} + \frac{a_6\,a_{12}^4}{12} + \frac{a_2^3\,a_4^3\,a_{12}^3}{72} + \frac{a_6^5\,a_{12}^2}{36} \\
&+ \frac{a_2\,a_4^4\,a_6^2\,a_{12}^2}{48} + \frac{a_1^2\,a_4^4\,a^26\,a_{12}^2}{96} + \frac{a_1\,a_3\,a_4^2\,a_6^5\,a_{12}}{72} + \frac{5\,a_2^3\,a_4^3\,a_6^4\,a_{12}}{72} \\
&+ \frac{a_1^4\,a_2^3\,a_4^2\,a_6^4\,a_{12}}{144} + \frac{a_1\,a_3^5\,a_4^2\,a_6^3\,a_{12}}{72} + \frac{a_1^2\,a_2^4\,a_3^2\,a_4^2\,a_6^3\,a_{12}}{96} + \frac{a_1^4\,a_2^3\,a_3^4\,a_4^2\,a_6^2\,a_{12}}{144} + \frac{a_1\,a_3^9\,a_4^2\,a_6\,a_{12}}{108} \\
&+ \frac{a_1^{10}\,a_3^6\,a_4^2\,a_6\,a_{12}}{864} + \frac{a_2\,a_4^3\,a_8^5}{12} + \frac{a_6^9}{27} + \frac{a_3^2\,a_6^8}{72} + \frac{a_1^3\,a_2^3\,a_3\,a_6^7}{216} \\
&+ \frac{a_3^6\,a_6^6}{72} + \frac{a_1\,a_2^4\,a_3^3\,a_6^6}{48} + \frac{a_2^9\,a_6^6}{48} + \frac{a_1^6\,a_2^6\,a_6^6}{864} + \frac{a_1^3\,a_2^3\,a_3^5\,a_6^5}{54} + \frac{a_1^4\,a_2^7\,a_3^2\,a_6^5}{288} \\
&+ \frac{a_1\,a_2^3\,a_3\,a_4^5\,a_6^4}{36} + \frac{17\,a_3^{10}\,a_6^4}{1296} + \frac{5\,a_1\,a_2^4\,a_3^7\,a_6^4}{432} + \frac{a_1^9\,a_3^7\,a_6^4}{432} + \frac{11\,a_1^2\,a_2^8\,a_3^4\,a_6^4}{1152} \\
&+ \frac{a_1^6\,a_2^6\,a_3^4\,a_6^4}{432} + \frac{17\,a_1^3\,a_2^{12}\,a_3\,a_6^4}{864} + \frac{a_1^7\,a_2^{10}\,a_3\,a_6^4}{96} \\
&+ \frac{a_1^{15}\,a_2^6\,a_3\,a_6^4}{288} + \frac{a_1^{27}\,a_3\,a_6^4}{2592} + \frac{a_1^3\,a_2^3\,a_3^9\,a_6^3}{108} + \frac{7\,a_1^4\,a_2^7\,a_3^6\,a_6^3}{864} \\
&+ \frac{a_1^{12}\,a_2^3\,a_3^6\,a_6^3}{2592} + \frac{a_1\,a_2^3\,a_3^5\,a_4^5\,a_6^2}{144} + \frac{a_1^2\,a_2^7\,a_3^2\,a_4^5\,a_6^2}{48} + \frac{a_3^{14}\,a_6^2}{162} \\
&+ \frac{a_1\,a_2^4\,a_3^{11}\,a_6^2}{216} + \frac{a_1^9\,a_3^{11}\,a_6^2}{1296} + \frac{11\,a_1^6\,a_2^6\,a_3^8\,a_6^2}{2592} + \frac{a_1^{10}\,a_2^4\,a_3^8\,a_6^2}{1728} \\
&+ \frac{5\,a_1^3\,a_2^{12}\,a_3^5\,a_6^2}{864} + \frac{a_1^7\,a_2^{10}\,a_3^5\,a_6^2}{432} + \frac{a_1^{15}\,a_2^6\,a_3^5\,a_6^2}{2592} + \frac{5\,a_1^4\,a_2^{16}\,a_3^2\,a_6^2}{288} \\
&+ \frac{a_1^8\,a_2^{14}\,a_3^2\,a_6^2}{144} + \frac{a_1^{16}\,a_2^{10}\,a_3^2\,a_6^2}{864} + \frac{a_1^4\,a_2^6\,a_3^4\,a_4^5\,a_6}{72} + \frac{a_1^3\,a_2^3\,a_3^{13}\,a_6}{216} \\
&+ \frac{5\,a_1^{12}\,a_2^3\,a_3^{10}\,a_6}{2592} + \frac{7\,a_1^6\,a_2^{15}\,a^43\,a_6}{864} + \frac{5\,a_1^{10}\,a_2^{13}\,a_3^4\,a_6}{864} + \frac{7\,a_1^{18}\,a_2^9\,a_3^4\,a_6}{2592} \\
&+ \frac{a_1^{30}\,a_2^3\,a_3^4\,a_6}{2592} + \frac{a_2^3\,a_4^{12}}{36} + \frac{a_2^7\,a_4^{10}}{24} + \frac{a_1^2\,a_2^6\,a_4^{10}}{48} + \frac{a_2^{15}\,a_4^6}{36}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{a_1\, a_2^3\, a_3^9\, a_4^5}{432} + \frac{a_1^{10}\, a_2^3\, a_3^6\, a_4^5}{432} + \frac{17\, a_1^4\, a_2^{15}\, a_4^5}{576} + \frac{a_1^8\, a_2^{13}\, a_4^5}{64} \\
&+ \frac{a_1^{16}\, a_2^9\, a_4^5}{192} + \frac{a_1^{28}\, a_2^3\, a_4^5}{1728} + \frac{a_3^{18}}{432} \\
&+ \frac{a_1^9\, a_3^{15}}{972} + \frac{a_1^{18}\, a_3^{12}}{3456} + \frac{a_1^3\, a_2^{12}\, a_3^9}{1296} \\
&+ \frac{a_1^7\, a_2^{10}\, a_3^9}{864} + \frac{a_1^{15}\, a_2^6\, a_3^9}{1296} + \frac{a_1^{27}\, a_3^9}{7776} + \frac{a_1^{12}\, a_2^{12}\, a_3^6}{1296} + \frac{a_1^{16}\, a_2^{10}\, a_3^6}{864} \\
&+ \frac{a_1^{24}\, a_2^6\, a_3^6}{1296} + \frac{a_1^{36}\, a_3^6}{7776} + \frac{a_2^{27}}{216} + \frac{79\, a_1^6\, a_2^{24}}{6912} + \frac{a_1^{10}\, a_2^{22}}{96} \\
&+ \frac{7\, a_1^{14}\, a_2^{20}}{2304} + \frac{a_1^{18}\, a_2^{18}}{384} + \frac{a_1^{22}\, a_2^{16}}{432} \\
&+ \frac{17\, a_1^{30}\, a_2^{12}}{20736} + \frac{a_1^{34}\, a_2^{10}}{3456} + \frac{a_1^{42}\, a_2^6}{5184} \\
&+ \frac{a_1^{54}}{62208}
\end{aligned}
$$

The generating functions are

$$
\begin{aligned}
Z(B_1) &= z^6 + 2\,z^5 + 5\,z^4 + 5\,z^3 + 5\,z^2 + 2\,z + 1 \\
Z(B_2) &= z^{24} + 2\,z^{23} + 13\,z^{22} + 42\,z^{21} + 155\,z^{20} + 423\,z^{19} \\
&+ 1111\,z^{18} + 2403\,z^{17} + 4688\,z^{16} + 7699\,z^{15} + 11162\,z^{14} \\
&+ 13759\,z^{13} + 14899\,z^{12} + 13759\,z^{11} + 11162\,z^{10} + 7699\,z^9 \\
&+ 4688\,z^8 + 2403\,z^7 + 1111\,z^6 + 423\,z^5 + 155\,z^4 + 42\,z^3 + 13\,z^2 + 2\,z + 1 \\
Z(B_3) &= z^{54} + 2\,z^{53} + 13\,z^{52} + 59\,z^{51} + 296\,z^{50} + 1336\,z^{49} + 5930\,z^{48} + 24517\,z^{47} + 96504\,z^{46} + 356557\,z^{45} \\
&+ 1239926\,z^{44} + 4034500\,z^{43} + 12269851\,z^{42} + 34771282\,z^{41} + 91729422\,z^{40} \\
&+ 224986965\,z^{39} + 512914199\,z^{38} + 1086583577\,z^{37} + 2139459095\,z^{36} \\
&+ 3916236659\,z^{35} + 6667229612\,z^{34} + 10561091956\,z^{33} + 15572574899\,z^{32} + 21383135142\,z^{31} \\
&+ 27353118148\,z^{30} + 32605463886\,z^{29} + 36226243954\,z^{28} + 37519703908\,z^{27} \\
&+ 36226243954\,z^{26} + 32605463886\,z^{25} + 27353118148\,z^{24} \\
&+ 21383135142\,z^{23} + 15572574899\,z^{22} + 10561091956\,z^{21} \\
&+ 6667229612\,z^{20} + 3916236659\,z^{19} + 2139459095\,z^{18} \\
&+ 1086583577\,z^{17} + 512914199\,z^{16} + 224986965\,z^{15} + 91729422\,z^{14} \\
&+ 34771282\,z^{13} + 12269851\,z^{12} + 4034500\,z^{11} \\
&+ 1239926\,z^{10} + 356557\,z^9 + 96504\,z^8 + 24517\,z^7 + 5930\,z^6 + 1336\,z^5 + 296\,z^4 + 59\,z^3 + 13\,z^2 + 2\,z + 1
\end{aligned}
$$

References

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