Electromagnetics Theory

Vector Calculus

Outline:

1. **Gradient of a Scalar Field**
2. **Divergence of a Vector Field**
3. **Curl of a Vector Field**
4. **Laplacian Operator**

Vector Operators

To describe the differential spatial variations of field value, we use three operators:

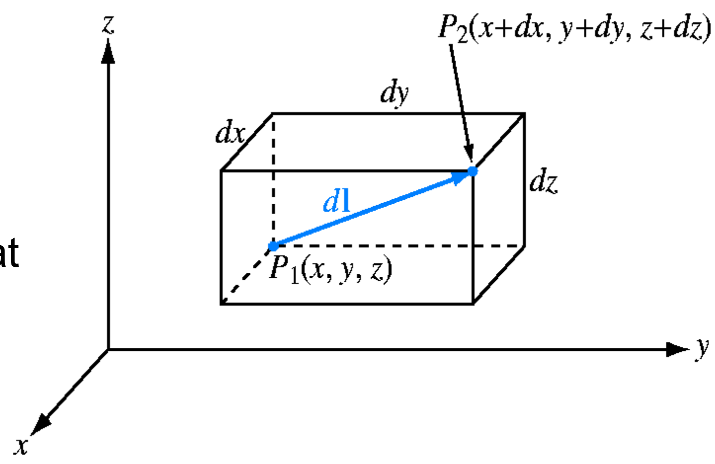
- Gradient (for scalar field)
- Divergence (for vector field)
- Curl (for vector field)

1.0 Gradient of a Scalar Field

Assume:

T_1 (x,y,z) at $P_1(x,y,z)$

T_2 $(x+dx,y+dy,z+dz)$ at $P_2(x+dx,y+dy,z+dz)$



The *gradient* defines the change in temperature dT corresponding to a vector change in position dl

Gradient of a Scalar Field (cont'd)

$$dT = \left[\hat{\mathbf{x}} \frac{\partial T}{\partial x} + \hat{\mathbf{y}} \frac{\partial T}{\partial y} + \hat{\mathbf{z}} \frac{\partial T}{\partial z} \right] \cdot d\mathbf{l}$$

Cartesian Coordinates

⇓

Gradient / grad / del* of *T

⇓

$$\nabla T$$

⇓

$$\nabla T = \hat{\mathbf{x}} \frac{\partial T}{\partial x} + \hat{\mathbf{y}} \frac{\partial T}{\partial y} + \hat{\mathbf{z}} \frac{\partial T}{\partial z}$$

? Cylindrical, Spherical.

Exercise

Find the directional derivative of $T = x^2 + y^2 z$ along the direction $\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2$ and evaluate it at (1, −1, 3).

Useful information:

$$dT = \nabla T \cdot d\mathbf{l}$$

$$\hat{\mathbf{a}}_l = \frac{d\mathbf{l}}{dl}$$

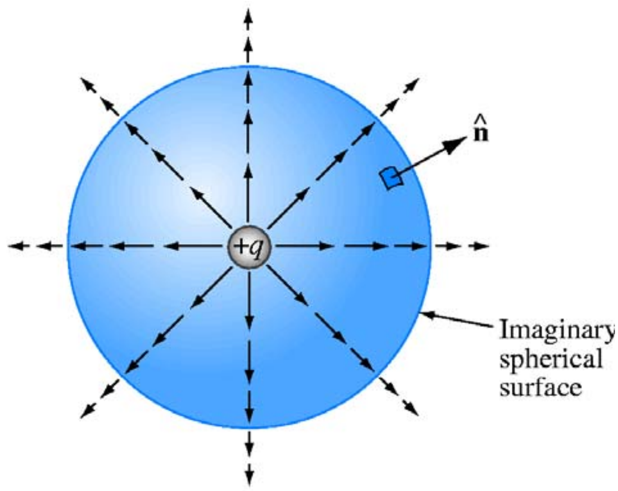
$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l$$

← *directional derivative of T*

2.0 Divergence of a Vector Field

Divergence of a vector field:

- The divergence of a vector **E** at a point is defined as the *net outward flux* of **E** per unit volume as the volume about the point tends to zero.
- It is a measure of the strength of the flow source of **E**.



Flux lines of the electric field **E** due to a positive charge q

Divergence of a Vector Field (Cont'd)

$$\text{div}\mathbf{E} \equiv \nabla \cdot \mathbf{E} = \lim_{\Delta v \to 0} \frac{\oint_s \mathbf{E} \cdot d\mathbf{s}}{\Delta v}$$

$$\nabla \cdot \mathbf{E} = \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right]$$

Cartesian Coordinates

?

Cylindrical coordinates,
Spherical coordinates.

Divergence of a Vector Field (Cont'd)

- The field **E** has positive divergence if the net flux out of surface S is positive, which may be viewed as if the volume Δv contains a *source* of flux.
- If the divergence is negative, Δv might be viewed as *sink.*
- For the uniform field, the divergence is zero, and the field is said to be *divergenceless.*
- Divergence operator is used for vector, and the result is scalar (while gradient operator is for scalar, resulting vector)

Divergence Theorem

Divergence Theorem:

The volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

$$\int_V \nabla \cdot \mathbf{E}\, dv = \oint_S \mathbf{E} \cdot d\mathbf{s}$$

Exercise

Determine the divergence of the following vector fields and then evaluate it at the indicated point:

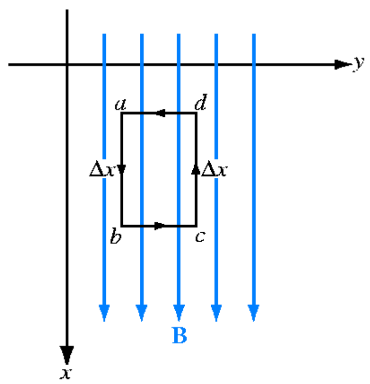
(a) $\mathbf{E} = \hat{\mathbf{x}}3x^2 + \hat{\mathbf{y}}2z + \hat{\mathbf{z}}x^2 z$ at $(3, -2, 0)$

3.0 Curl of a Vector Field

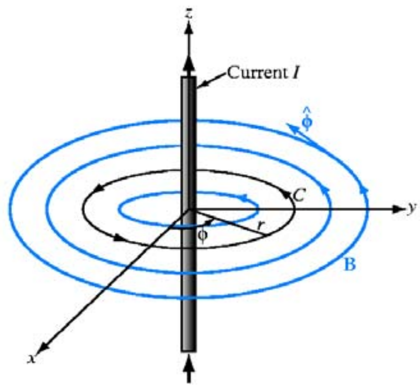
Curl of a vector field:

- The curl of a vector field **B** is a vector whose magnitude is the maximum net **circulation** of **B** per unit area as the area tends to zero.
- The direction is the normal direction of the area when the area is oriented to make the net circulation maximum.
- It is a measure of the strength of the vortex source of **B.**

Curl of a Vector Field (Cont'd)



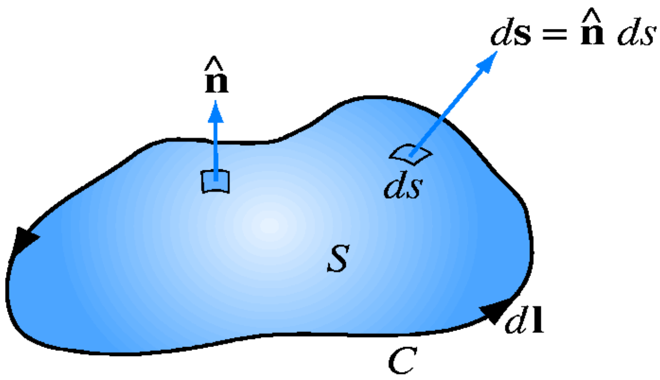
(a) Uniform field



(b) Azimuthal field

Circulation is zero for the uniform field in (a), but it is not zero for the azimuthal field in (b)

Curl of a Vector Field (Cont'd)



The **direction** of the unit vector $\hat{\mathbf{n}}$ is along the thumb when the other four fingers of the right hand follow *dl*

Curl of a Vector Field (Cont'd)

$$\operatorname{curl} \mathbf{B} \equiv \nabla \times \mathbf{B} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\hat{\mathbf{n}} \oint_c \mathbf{B} \cdot d\mathbf{l} \right]_{\max}$$

$$\nabla \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

Cartesian Coordinates

?

Cylindrical coordinates,
Spherical coordinates.

Stoke's Theorem

Stokes's Theorem:

The surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

What if the field **B** is said to be *conservative or irrotational?*

Exercise

Find $\nabla \times$ **A** at (2, 0 , 3) in cylindrical coordinates for the vector field

$$\mathbf{A} = \hat{\mathbf{r}} 10 e^{-2r} \cos\phi + \hat{\mathbf{z}} 10 \sin\phi$$

4.0 Laplacian Operator

- The combined operator which takes the gradient followed by taking the divergence is called the **Laplacian operator.**

$$\nabla \cdot (\nabla V) = \nabla^2 V$$

- Laplacian operator is called as **"del square" ($\nabla^2 V$)**

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cartesian Coordinates

Thank You