Electromagnetics Theory

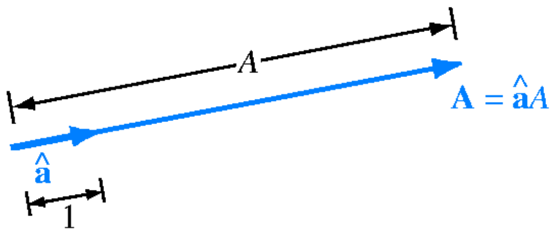
Vector Algebra

Outline:

1. **Basic Laws of Algebra**
2. **Orthogonal Coordinate System**
3. **Transformations between Coordinate System**

1.0 Basic Laws of Algebra

Magnitude and Unit Vector



Magnitude: A = |**A**|

Unit vector : $\hat{\mathbf{a}}$

$$|\hat{\mathbf{a}}| = 1 \quad \text{Magnitude}$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} \quad \text{Direction}$$

Basic Laws of Algebra

Coordinate systems:

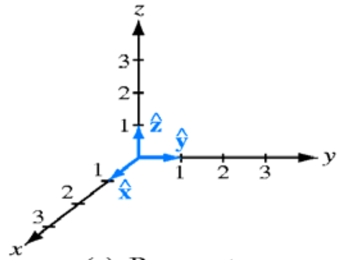
- Cartesian (rectangular)
- Cylindrical
- Spherical

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$$

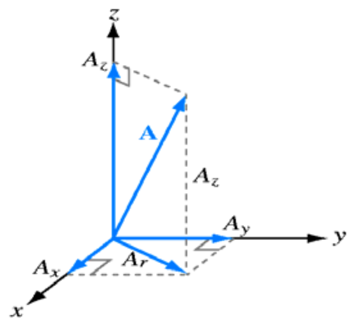
Base vectors: $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$

Components of vector A

$$A_x, A_y, A_z$$



(a) Base vectors



(b) Components of **A**

Exercise:

By using Pythagorean theorem,

a) write the expression for the magnitude of **A**.

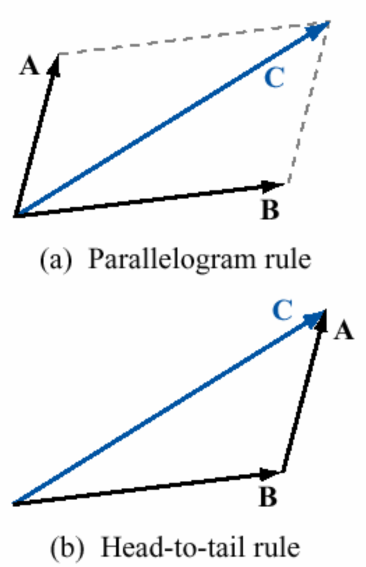
b) write the expression for the unit vector $\hat{\mathbf{a}}$.

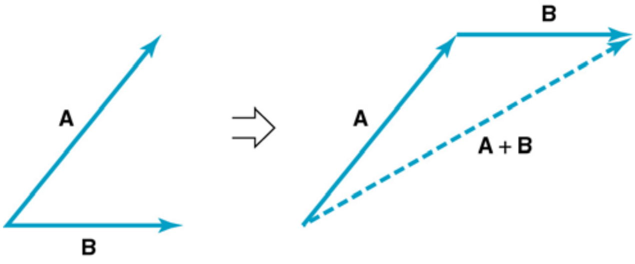
$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$$

$$A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

Vector Addition and Subtraction



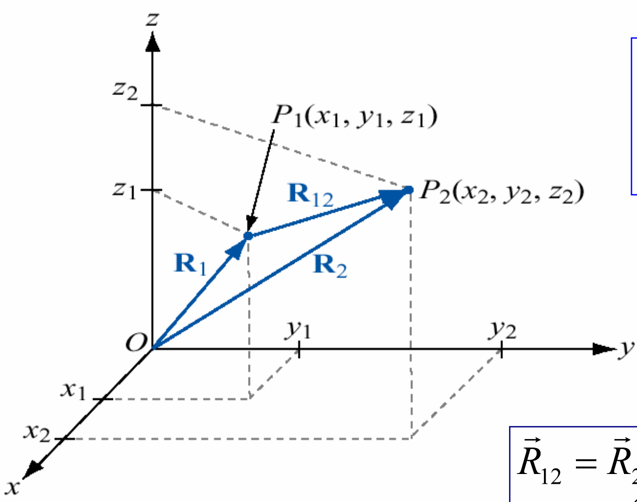


Note: A + B = B + A

Subtraction of vector **B** from vector **A** is equivalent to the addition of **A** to negative **B**.

i.e. **D** = **A** – **B** = **A** + (-**B**)

Position and Distance Vectors



Position vector

$$\overrightarrow{OP_1} = \vec{R}_1 = \hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$$
$$\overrightarrow{OP_2} = \vec{R}_2 = \hat{\mathbf{x}}x_2 + \hat{\mathbf{y}}y_2 + \hat{\mathbf{z}}z_2$$

the vector R_{12} is the vector from P_1 to P_2 and its distance (length or magnitude) is d:

Distance vector

$$\vec{R}_{12} = \vec{R}_2 - \vec{R}_1 = \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1)$$

$$d = \left|\vec{R}_{12}\right| = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2}$$

Exercise

Having P_1 (1,2,3) and P_2 (–1, –2, 3) in Cartesian coordinates. Find

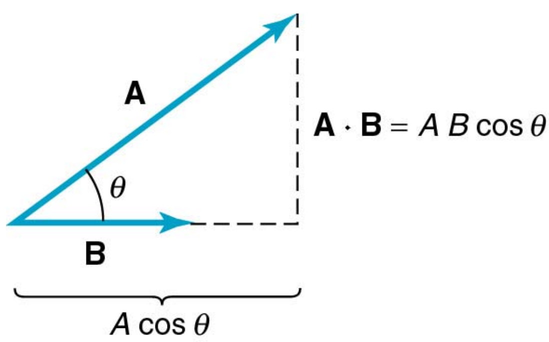
(a)The distance vector between P_1 and P_2

(b)The length of the distance vector

$$\overrightarrow{P_1P_2} = -\hat{\mathbf{x}}2 - \hat{\mathbf{y}}4$$
$$d = \sqrt{20}$$

Dot Product (scalar product)

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta_{AB}$$



- Always yields a scalar!
- $\underline{A \cos\theta}$ is the component of **A** along **B**.
- This is the projection of A on B.
- If two vectors are orthogonal their dot product is zero.
- **A**·**A**=|**A**|^2=A^2

Exercise

1. Given $\mathbf{A} = (A_x, A_y, A_z)$

$\mathbf{B} = (B_x, B_y, B_z)$

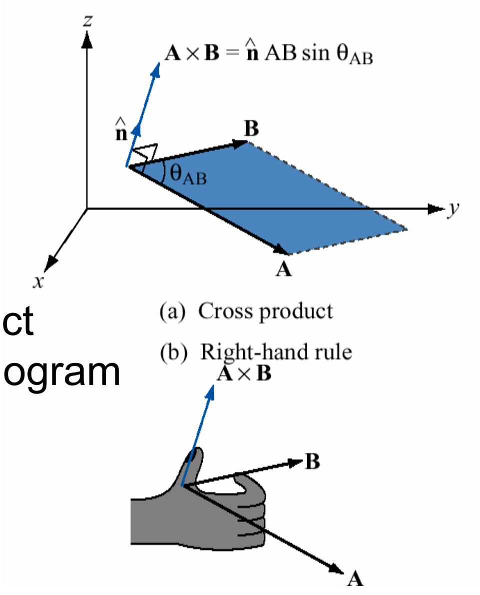
Show that:

(a) **A·B** = $A_xB_x + A_yB_y + A_zB_z$

(b) **A·A** = |**A**|^2 = A^2

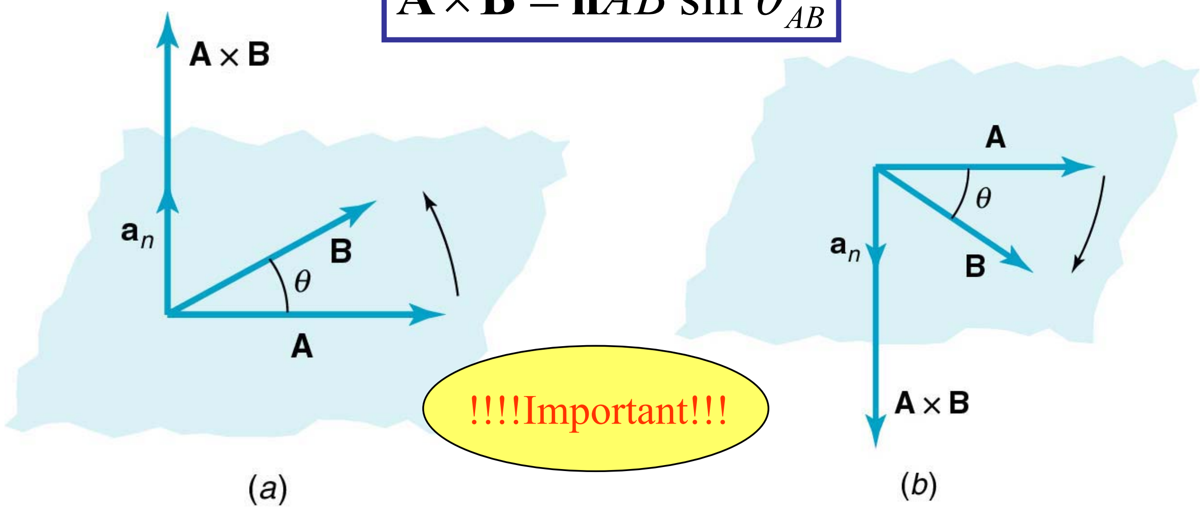
Cross Product (Vector Product)

- Always yields a vector!
  $\hat{\mathbf{n}}$ is the unit vector, which perpendicular to the plane that contains the two vectors.
- The direction of $\hat{\mathbf{n}}$ is determined by the *right-hand rule.*
- The magnitude of the cross product is equal to the **area** of the parallelogram defined by two vectors.
- If two vectors are parallel, their cross product is zero.



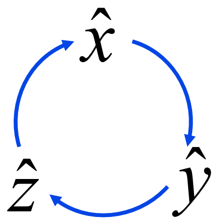
Cross Product (Cont'd)

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin \theta_{AB}$$



!!!!Important!!!

Cross Product (Cont'd)



- Move clockwise in the direction of the arrow the cross product is positive.
- Move in the counter-clockwise direction and the cross product is negative.

$$\hat{x} \times \hat{y} = \hat{z}$$
$$\hat{z} \times \hat{x} = \hat{y}$$
$$\hat{x} \times \hat{z} = -\hat{y}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_y B_z - A_z B_y)\hat{x} - (A_x B_z - A_z B_x)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

Exercise

In Cartesian coordinates, vector **A** is directed from the origin to point P_1 (2,3,3), and vector **B** is directed from P_1 to point P_2 (1,–2,2). Find

(a) Vector **A**, its magnitude A, and unit vector $\hat{\mathbf{a}}$

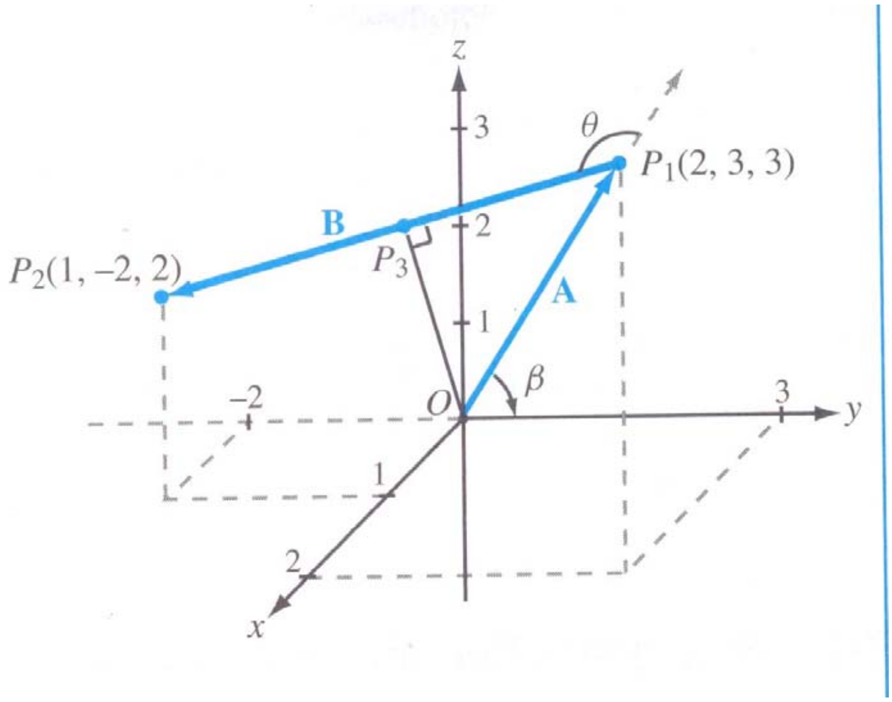
(b) The angle that **A** makes with the y-axis

(c) Vector **B**

(d) The angle between **A** and **B**, and

(e) The perpendicular distance from the origin to vector **B**.

Exercise …



2.0 Orthogonal Coordinate System

Orthogonal coordinate system:

The coordinates that are mutually perpendicular.

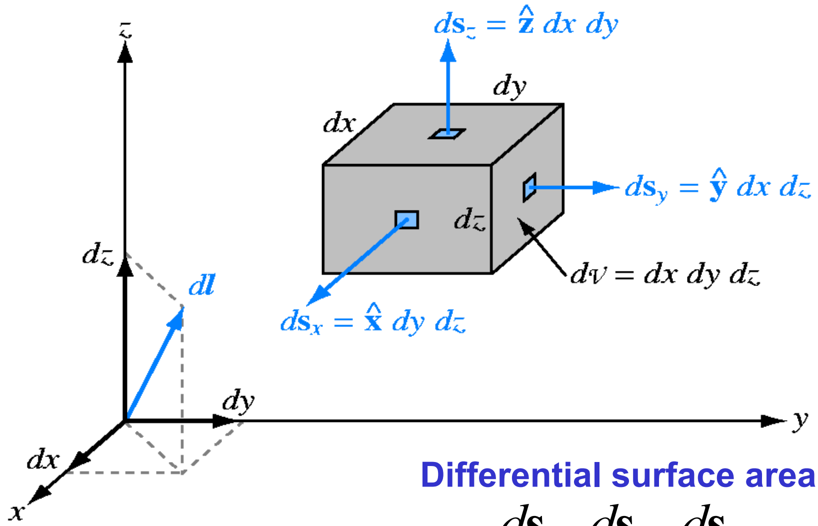
The most commonly used orthogonal coordinate system are:

(a) Cartesian coordinate system (rectangular)

(b) Cylindrical coordinate system

(c) Spherical coordinate system

(a) Cartesian Coordinates (x,y,z)



Differential length

Differential surface area

$$ds_x, ds_y, ds_z$$

Differential volume dV

$$dV = dxdydz$$

(b) Cylindrical Coordinates (r, ϕ, z)

Ranges

$$0 \le r < \infty$$
$$0 \le \phi < 2\pi$$
$$-\infty < z < \infty$$

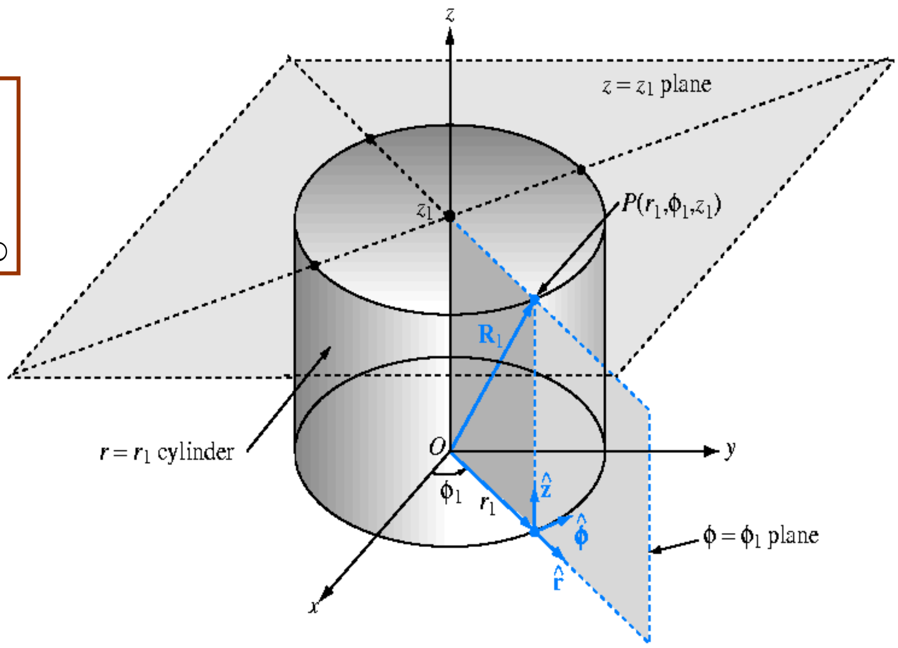


Figure3-9

Cylindrical Coordinates (r, ϕ, z)

The differential length along $\hat{\mathbf{r}}, \hat{\boldsymbol{\varphi}}, \hat{\mathbf{z}}$ **are:**

$$dl_r = dr, \qquad dl_\phi = r d\phi, \qquad dl_z = dz$$

The differential length:

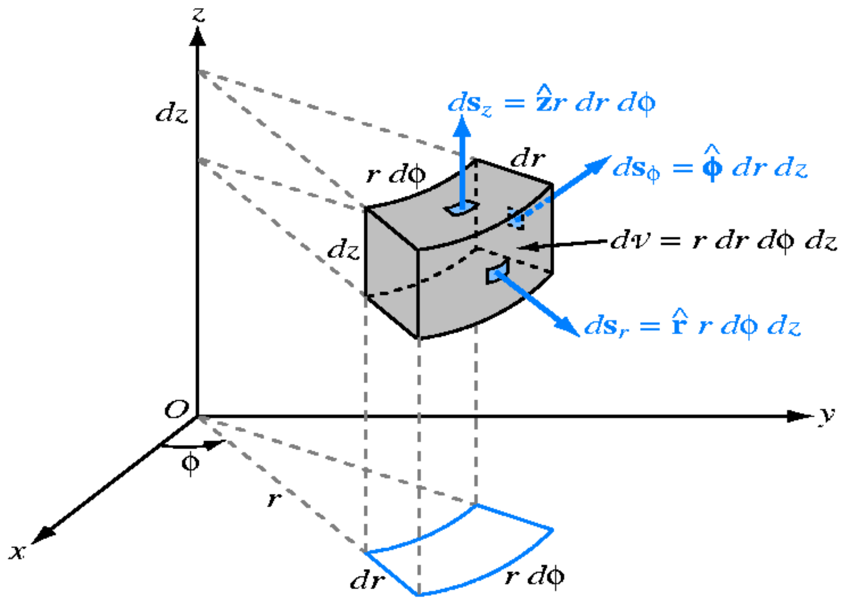
$$d\mathbf{l} = \hat{\mathbf{r}} dr + \hat{\boldsymbol{\varphi}} r d\phi + \hat{\mathbf{z}} dz$$

Try to write the expression for:

$$d\mathbf{s}_r = \qquad d\mathbf{s}_\phi = \qquad d\mathbf{s}_z =$$

$$dV =$$

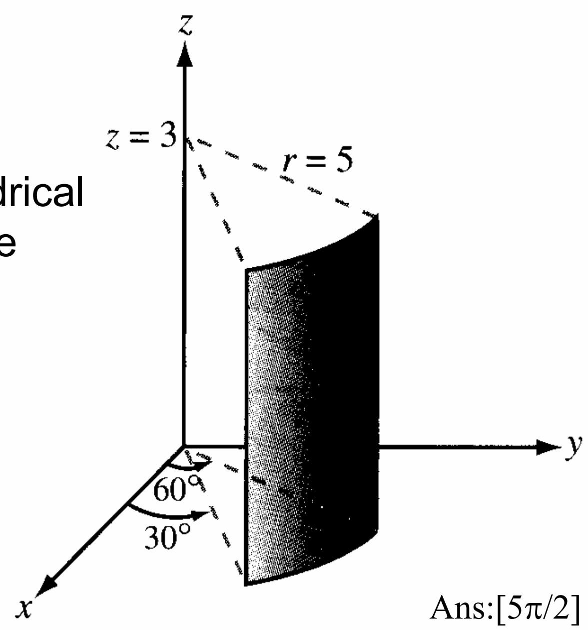
Differential areas and volume in cylindrical coordinates



e 3-10

Exercise

Find the area of a cylindrical surface described by the Figure.



Ans:[5π/2]

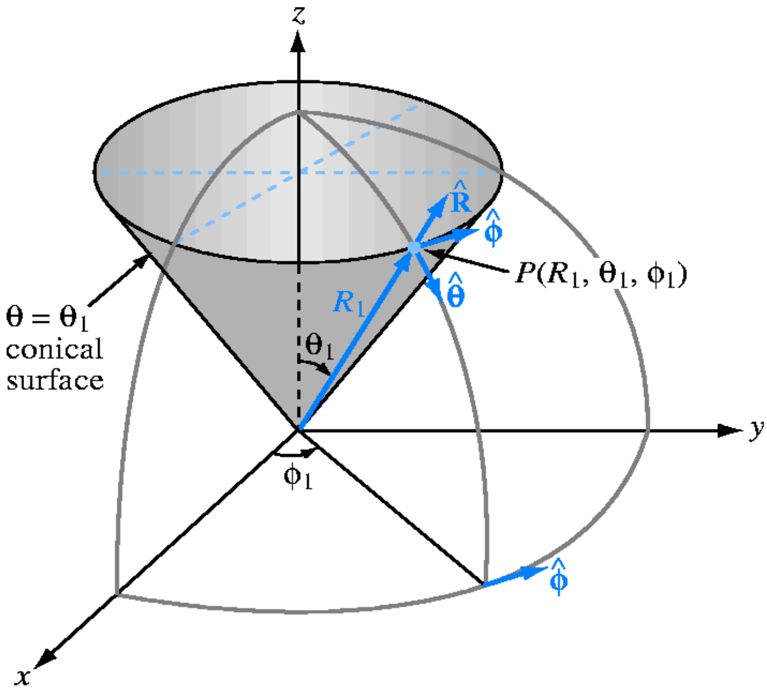
(c) Spherical Coordinates (R,θ,ϕ)

Ranges

$0 \le R < \infty$

$0 \le \theta \le \pi$

$0 \le \phi < 2\pi$



Spherical Coordinates (R,θ,ϕ)

The differential length along $\hat{\mathbf{R}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$ are:

$$dl_R = dR, \qquad dl_\theta = R d\theta, \qquad dl_\phi = R \sin\theta d\phi$$

The differential length:

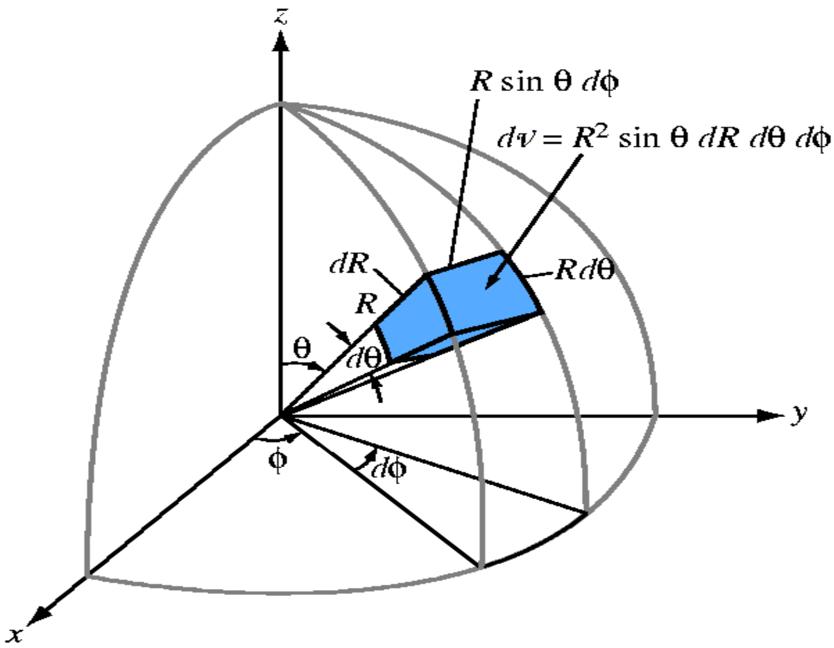
$$d\mathbf{l} = \hat{\mathbf{R}} dR + \hat{\boldsymbol{\theta}} R d\theta + \hat{\boldsymbol{\varphi}} R \sin\theta d\phi$$

Try to write the expression for:

$$d\mathbf{s}_R = \qquad d\mathbf{s}_\theta = \qquad d\mathbf{s}_\phi =$$

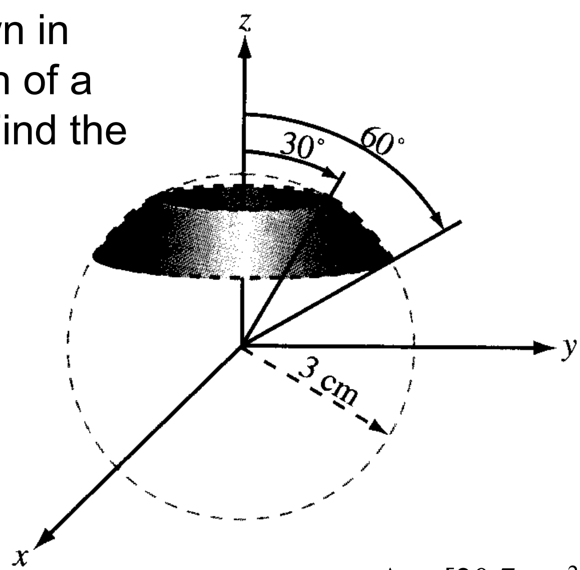
$$dV =$$

Differential volume in spherical coordinates



Exercise

The spherical strip shown in Figure below is a section of a sphere of radius 3 cm. Find the area of the strip.



Ans:[20.7 cm^2]

Exercise

A sphere of radius 2 cm contains a volume charge density ρ_V *given by*

$$\rho_V = 2\ \mathrm{c0s}^2\theta.$$

Find the total charge *Q* contained in the sphere.

Ans:[22.34 μ C]

Summary of vector relations

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, A =	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A, $\lvert A\rvert$ =	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1}$ =	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product, A · B =	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, A × B =	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l}$ =	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$

3.0 Transformation between Coordinate System

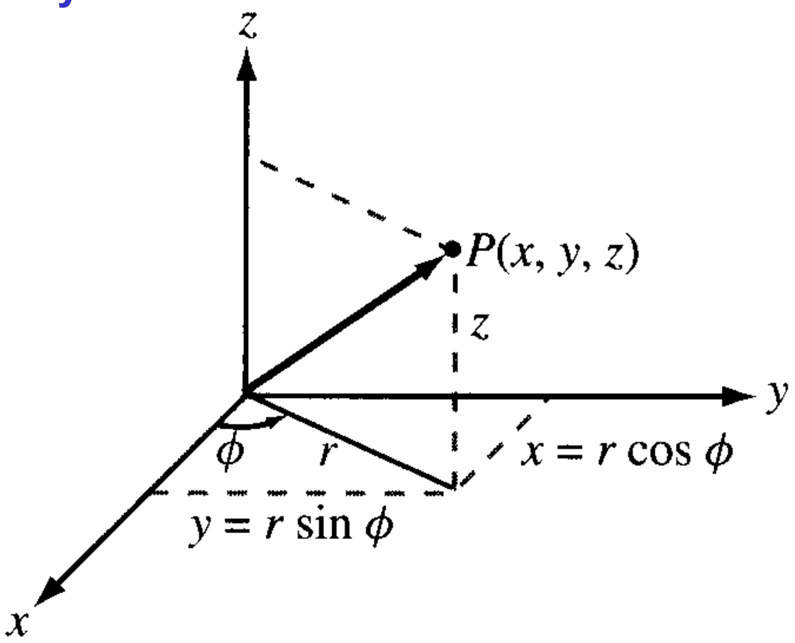
3.1 Cartesian to Cylindrical Transformations

$$r = \sqrt[+]{x^2 + y^2}$$

$$\phi = \tan^{-1}(\frac{y}{x})$$

$$x = r\cos\phi$$

$$y = r\sin\phi$$



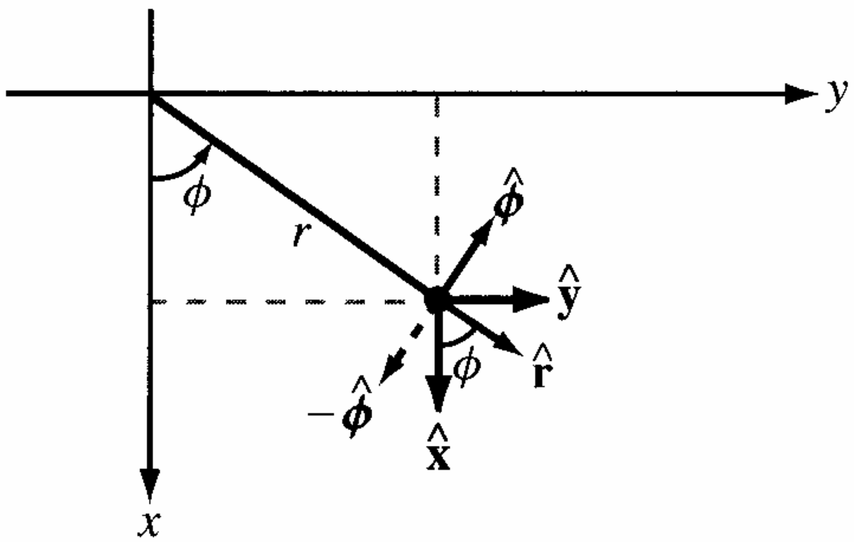
Interrelationships between base vectors Cartesian and Cylindrical

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{y}} = \sin\phi$$

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = \cos\phi$$

$$\hat{\phi} \cdot \hat{\mathbf{x}} = -\sin\phi$$

$$\hat{\phi} \cdot \hat{\mathbf{y}} = \cos\phi$$



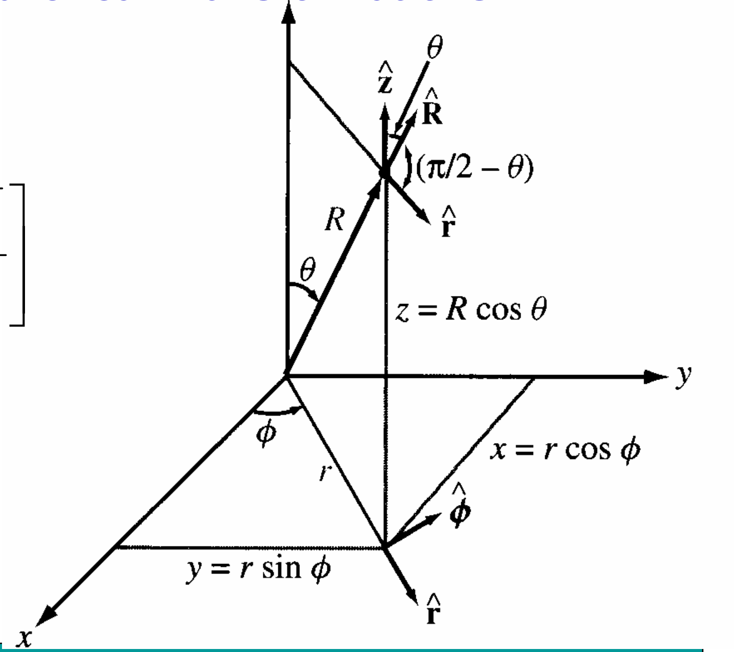
Transformation between Coordinate System

3.2 Cartesian to Spherical Transformations

$$R = \sqrt[+]{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left[\frac{\sqrt{x^2 + y^2}}{z}\right]$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$



Coordinate transformation relations

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2+y^2+z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2+y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2+z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

Exercise

Point $P(2\sqrt{3}, \pi/3, -2)$ is given in cylindrical coordinates. Express P in spherical coordinates.

Ans: $P(4, 2\pi/3, \pi/3)$

Thank You