A Game of Flipping Triangles

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1. Introduction and Rules

Coin flipping games have become more popular ever since the introduction of the commercially available game “Lights Out” (which has been studied by Anderson and Feil [1]). Coin flipping games, also known as s-games, are games in which coins adjacent to the one being flipped are also flipped (in certain versions only the only the neighboring coins are flipped). s-games are also popularly known as light-switching games; the objective of these games being to turn all the lights off (or coins over) given an initial configuration. s-games have been studied by Sutner [4, 5, 6] and by Barua and Ramakrishnan [2].

In these games the boards are commonly $m \times m$ grids. We present here a game on a triangulated graph, where the coin flipping is related to the triangles that they form. The more general game in n-dimensions with an arbitrary labeling set (as opposed to a heads/tails or black/white labeling) has been studied in [7].

The game is played on a board of coins (or disks with each side a different color), which are tightly packed. That is, coins laid out so that the board is made up of little triangles of coins (i.e. each triangle consisting of three coins). A general board can be formed by first taking three coins, each touching the other two, then adding new coins, one at a time, so that each new coin touches two other coins. Throughout this paper, all references to boards will be to boards of tightly packed coins formed in this way. Two classic board shapes are the triangle and the hexagon, Fig. 1. A move in the game, which we call a ‘*push*’, consists of turning over any triangle that is made up of three coins, Fig. 2. The object, classically, is to turn a board made up only of heads to a board made up only of tails. For small boards trial and error would be sufficient to determine whether and what solutions exist. And given enough time, for boards of specific shapes, certain patterns might arise with regards to the solutions.

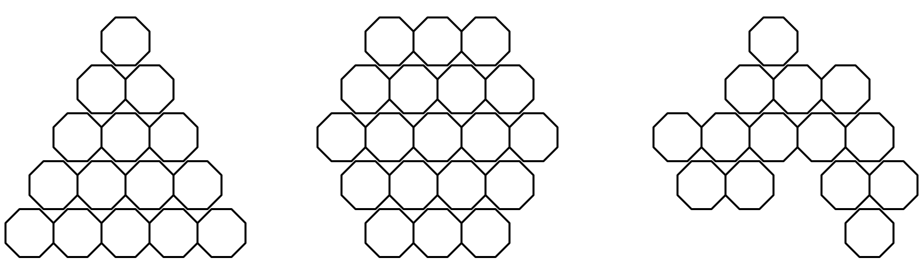

Figure 1. A triangular, hexagonal, and arbitrarily shaped board.

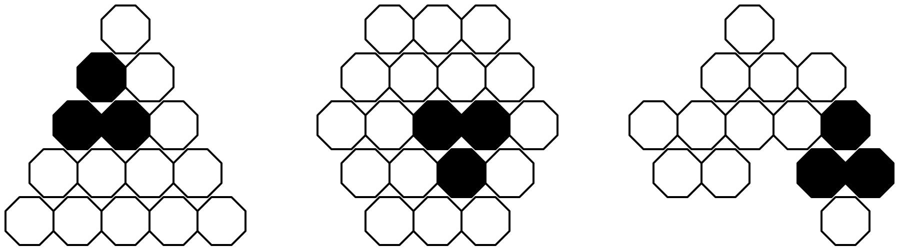

Figure 2. A few different pushes.

While the game is classically played with the objective of turning a board made up exclusively of disks (coins) of one color (heads) to disks of a different color (tails), the game can be generalized to any two colorings of a board. That is, given two colorings of a board, where a coloring is simply the assignment of different colors to different disks, we call one board the initial board and the other board the terminal board. The object of the game then, is to find a collection of pushes (a 'solution set') that turn the coloring of the initial board into the coloring of the terminal board. We mention 'collection,' as opposed to an ordered set, because, as is easily seen, pushes are commutative, and hence the order of the pushes in the solution set is unimportant. In this paper we solve the following more general problems:

1. Given any two colorings of a board, is there a collection of pushes with which we can turn the coloring of the initial board into that of the terminal board?
2. Given that a solution to Problem 1 exists, how many different solutions are there?

To find a solution to Problem 1, we introduce in Section 2 an invariant of the game. We use this in Section 3 to solve the puzzle, while in Section 4 we determine the number of different solutions that exist. In Section 5 a few specific examples are worked out.

2. Underlying Graphs and an Invariant

If we replace each disk with a vertex, and for every two disks that touch assign an edge, then we can associate with each board an underlying graph, *G* (Fig. 3). The underlying graph of a board can be thought of as little triangles pasted together by their edges. We now associate with each color of a disk a labeling of its respective vertex. Choosing one side of a disk as 'face down' and the other side as 'face up' we might label a vertex 0 if it was face down and 1 if it was face up. Therefore, based on the coloring of the board we attain a labeling of the graph. Due to this association we will often refer to the board as the graph and to the graph as the board.

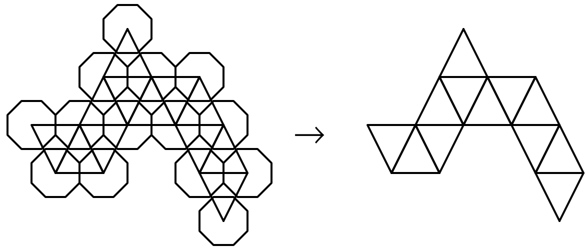

Figure 3. The underlying graph of a board

Let $L(G): V(G) \rightarrow Z_2$ be a labeling of the vertices of the underlying graph, G, from the set {0, 1}. A *push* is a function $f_{T_i}: L_1(G) \rightarrow L_2(G)$, acting on a triangle $T_i = \{v_{i_0}, v_{i_1}, v_{i_2}\}$, such that $f_{T_i}[L(v_j)] = L(v_j) + 1$ (mod 2). (The abelian nature of the pushes can be seen from this definition.)

What we would like to find is a series of pushes, $f_j: L_j(G) \rightarrow L_{j+1}(G)$, that induce a series of labelings, $L_1(G), L_2(G), ..., L_n(G)$, in which $L_1(G)$ is our initial board and $L_n(G)$ is our terminal board.

For the given boards, it can be easily shown that $c(G) = 3$ (see Appendix). We can therefore color the underlying graph with the symbols i, j, and k, such that i and j generate the Klein 4-group. I.e. $i^2 = j^2 = k^2 = ijk = e$ and $ij = k, ik = j, jk = i$. (Note: we are using the term 'color' here in the sense of graph coloring. This coloring of the underlying graph is unrelated to the original colorings of the board.)

For each vertex v_i, we now have both a color and a label. Let $\tilde{v}_i$ be the value of the color raised to the power of the label. So, for example, if a given vertex v_j had color k and label 1, we would assign $\tilde{v}_j$ a value of $k^1 = k$, if its label, however, was 0, we would assign it a value of $k^0 = e$. What we have accomplished is that the only colorings showing (i.e. which are not the identity) are those of vertices labeled 1. I.e. of those disks that are face up.

Let $P[L(G)] = \prod_{v_i \in V(G)} \tilde{v}_i$, then

Lemma 1 *$P[L(G)]$ is invariant under pushes.*

Proof Let i^a, j^b, k^g be the set of values assigned to the vertices of an arbitrary triangle in G. A push on this triangle would send $i^a \mapsto i^{a+1}, j^b \mapsto j^{b+1}, k^g \mapsto k^{g+1}$. Thus we have: $i^a j^b k^g \rightarrow i^{a+1} j^{b+1} k^{g+1} = i^a j^b k^g (ijk) = i^a j^b k^g$ (since $ijk = e$). Therefore $P[L(G)]$ is invariant under pushes.

3. A Solution to the Puzzle

The fact of whether a solution exists is dependent upon this invariant. We show that as long as two labelings have the same value for this invariant then it is possible to construct a solution.

Theorem 2 *Given two labelings, L_1 and L_2, of a board, there exists a set, F, of pushes, such that $F[L_1(G)] = L_2(G)$ iff $P[L_1(G)] = P[L_2(G)]$, where G is the underlying graph of the board.*

Proof By Lemma 1, $P[L_1(G)]$ is invariant under pushes; the necessity of the equality therefore follows.

To show the sufficiency of the equality we construct a solution. By the construction of our boards it is easy to see that we can form a path from any triangle to any other triangle via adjoining triangles (that is, triangles which share an edge with another triangle).

We can therefore find three paths *I, J, and K*, such that:

(i) $I = I_1, I_2, \cdots, I_{t_i}$ is a path of triangles such that $\left|I_{(2k-1)} \cap I_{(2k)}\right| = 2$, and the coloring of the vertices in $\overline{I_{(2k-1)} \cap I_{(2k)}}$ are both *i*. (Similarly for J and K replacing *i* with *j* or *k*, respectively.)

(ii) Given any $v \in V(G)$ with coloring *i*, $v \in t_r$ for some *r*.

(iii) Paths *I, J,* and *K* all end with the same triangle.

Remark: No other conditions are placed upon these paths, therefore it is possible for one of these paths to cross itself or even retrace part of itself.

Let us assume that the two labelings of G differ in only one triangle. Let f be a push on this triangle. Since $P[L_1(G)] = P[L_2(G)]$ (by assumption), by the identities of the Klein 4-group it is clear that all the vertices must now be labeled the same. This can be seen to be true by letting, without loss of generality, the vertex colored *i* be a vertex that is labeled differently. The push, then, will result in them being labeled the same. Due to the invariant, however, the product of the two remaining vertices must have the same value, but this is only possible if they are either both labeled 0, or both labeled 1. Thus all the labelings will be the same.

Let us now assume that the two labelings differ arbitrarily. We form three sequences of pushes along the paths *I*, *J*, and *K*, as follows: Where the sequence for the path *I* is

$f_{1,i}^{p_{1,i}} f_{2,i}^{p_{1,i}+1} f_{3,i}^{p_{3,i}} f_{4,i}^{p_{3,i}+1} \cdots$, where the $(2k-1)^{st}$ term is of the form $f_{(2k-1),i}^{p_{(2k-1),i}}$ and the $(2k)^{th}$ term is of the form $f_{2k,i}^{p_{1,i}+1}$, for $k = \{1,2,\ldots\}$, and where $p_{(2k-1),i}$ is the power (either 0 or 1) necessary so that if v is the vertex in the triangle colored *i*, then $l\left\lfloor f_{(2k-1),i}^{p_{(2k-1),i}}(v)\right\rfloor = l_2(v)$. I.e. the labeling of that vertex is (now) the same as in $L_2(G)$. The sequences for the paths *J* and *K* are formed in a similar manner.

The last term in each of the sequences will either be of the form $f_{(2k-1),i}^{p_{(2k-1),i}}$ or of the form $f_{2k,i}^{p_{(2k-1),i}+1}$, depending upon whether there is an odd or an even number of elements in the path.

Note that since $\left|t_{(2k-1),i} \cap t_{(2k),i}\right| = 2$, where the *t's* are elements in one of the paths, $f_{(2k-1),i}^{p_{(2k-1),i}} f_{2k,i}^{p_{(2k-1),i}+1}$ raises the exponent of the values assigned to each vertex, except those vertices that are colored *i*, by 2. Therefore, since the order of all the elements in the Klein 4-group is 2, all vertices, except those colored *i*, remain unchanged. (And similarly with *j* and *k*.)

The sequence on the path *I* will therefore change the labelings of all those vertices of *G* colored *i*, except for possibly that of the last triangle of the path, as will the sequences on the path *J* and *K* change the labelings of all those vertices colored *j* and *k* (respectively), except for possibly that of the last triangle of the paths. We need only concern ourselves, therefore, with this last triangle, which, by construction, is the same for each of the paths. However, having already proven the theorem for that case our proof is complete.

4. The Number of Different Solution

Now that we know when a solution exists, we wish to determine how many different solutions there are. To accomplish this, we first show that the set of labelings of G form equivalence classes of the same size. We start by defining an equivalence relation on the labelings of G.

Definition 3 *Let $L_1(G)$ & $L_2(G)$ be two labelings of a board. Then we say that $L_1(G)$ is* label-equivalent *to $L_2(G)$, written as $L_1(G) \sim L_2(G)$, if $P[L_1(G)] = P[L_2(G)]$.*

The relation $L_1(G) \sim L_2(G)$ can be shown, using Theorem 2, to be an equivalence relation. Since an equivalence relation creates a decomposition of the set into mutually disjoint subsets (see [3] for a simple proof), we have the following:

Lemma 4 *The equivalence relation ~, provides a decomposition of the set of all labelings of G into distinct (mutually disjoint) equivalence classes.*

Corollary 5 *There are 4 distinct equivalence classes.*

Proof The proof follows easily from Theorem 2 and the fact that $P[L(G)])$ can be anyone of 4 different values.

Let us add to our collection of pushes acting on G, 2 additional elements which also act upon a triangle (the specific triangle it acts upon is irrelevant). Whereas a push acts upon a triangle by increasing the labeling of each vertex in a triangle by 1, our new moves only increase the labeling of one vertex in the triangle by 1. Let h be the move that adds 1 to the labeling of the vertex colored i, and let l be the move that adds 1 to the labeling of the vertex colored j. $P[L(G)]$ can now take on any value, since given any labeling of G, multiplying it by h, l, or hl will each change the value of $P[L(G)]$ differently. Therefore it is now possible for us to change from any labeling of G to any other labeling of G.

We now form classes of labelings of G such that two labelings, K and L, are in the same class if $P[K(G)] = P[L(G)]$.

Let h (or l or hl) be a mapping from one class of labelings to another class, then if K and L are two labelings in some class, hK and hL will be two labelings in this other class. However, it is easily shown that $hK = hL$ if and only if $K = L$. Therefore there is a one to one relation between the different equivalence classes, and we have

Corollary 6 *The equivalence relation ~ divides the set of labeled graphs into equivalence classes of equal size.*

By Corollary 4 we know that there are 4 different values that P can take on. Setting $|V(G)| =$, we see that there are 2^n different labelings of G. Since, by Corollary 6, the sizes of the equivalence classes are the same, there are 2^{n-2} different labelings for each class. Therefore we have that

Corollary 7 *There are exactly 2^{n-2} elements in each equivalence class formed by the equivalence relation ~.*

We now form classes of pushes that all act in a similar manner on a labeling. Let $R(G)$ be the set of all triangles in G.

Definition 8 *Let f and g be two words in* $R(G)$*, and L some labeling of G, then we say that f is* congruent *to g, written as f º g, if* $f[L(G)] = g[L(G)]$.

The relation f º g can be shown to be an equivalence relation. As before, since an equivalence relation creates a decomposition of the set into mutually disjoint subsets (see [3]), we have that the equivalence relation $\equiv$ provides a decomposition of the set of all words in $R(G)$ into distinct (mutually disjoint) equivalence classes.

Corollary 9 *The equivalence relation* º *divides* $R(G)$ *into equivalence classes of equal size.*

Proof We form a mapping $j_{K,L} : K \to L$ from one equivalence class, K, into another equivalence class, L, by $f \mapsto mf$, where m is such that $mf \in L$. Then it can be shown that $mf \equiv mg$ iff $f \equiv g$, and $mf = mg$ iff $f = g$. Therefore, since K and L were arbitrary, there is a one to one correspondence between equivalence classes.

Let $|R(G)| = r$, then there are 2^r different sets of moves possible on G. Since we know by Corollary 9 that the classes of moves are each the same size, and by Corollary 7 that there are 2^{n-2} different classes of labelings upon which these moves act, we have that there are $\frac{2^r}{2^{n-2}} = 2^{r-n+2}$ different sets of pushes for each class of labelings. Thus we have proved

Theorem 10 *Given a graph G, as in Theorem 2, the number of solutions that exist with which one labeling can be changed into another labeling, so long as a solution exists, is* 2^{r-n+2}.

5. Examples

Let us take, as an example, a game on a triangular board of four rows, where we wish to turn a board made up exclusively of white (face down) disks (the initial board) to a board made up exclusively of black (face up) disks (the terminal board) (Fig. 4).

Coloring the vertices of the underlying graph with i, j, and k, we quickly notice that the P value of the face down board (white disks) is not equal to the P value of the face up (black disks) board. (Recall that since we label the vertices of a face down disk 0 the initial board automatically has a P value of e. And if we color the uppermost disk i, the terminal board will have a P value of i.) Therefore there is no solution to the game. I.e. there is no set of pushes that can change the face down board into the face up board. However, if instead of an all face down board we use as our initial board a board with the uppermost disk already turned over, then we find that since the P values are equal a solution does exist. (Using the same coloring as before, i.e. with the uppermost disk colored i, the P values of both boards are i.) And since $2^{r-n+2} = 2^{9-10+2} = 2^1$, there are two different solutions. See Figures 4 and 5.

Figure 4. No solution exists for the first game; however, since the P values are equal in the second game, a solution exists.

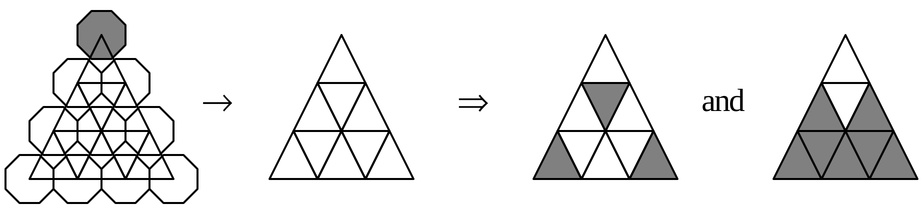

Figure 5. A four row triangular board with its two solutions (where the vertices of the colored triangles represent the coins that are flipped. That is, the pushes act upon the colored triangles).

If we now increase our board size to five rows, and have as our initial and terminal boards all face down and all face up disks (respectively), then not only are the P values equal, and hence a solution exists, but we find that since $2^{r-n+2} = 2^{16-15+2} = 2^3$, there are eight different solutions. See Fig. 6.

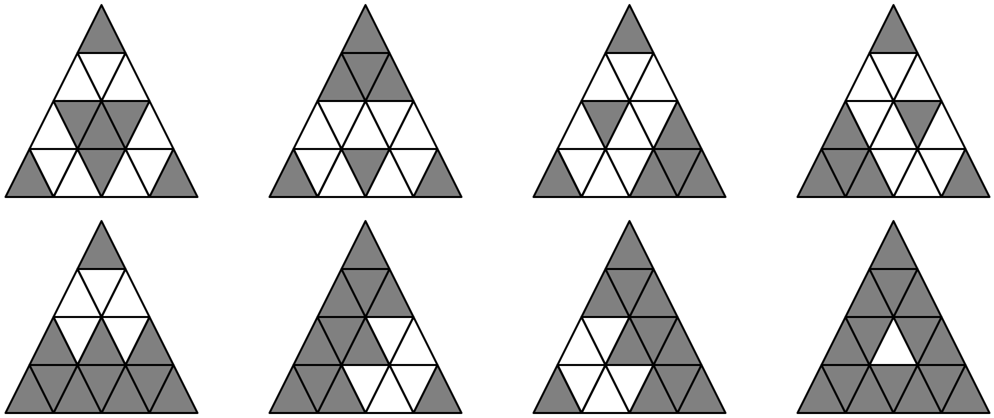

Figure 6. The eight different solutions for the triangular board of five rows.

6. Appendix: The 3-colorability of the graphs

The 3-colorability of these graphs is easily seen from the fact that since six disks fit perfectly around a seventh disk they form a hexagon, and hexagons tile the plane. Let us then tile the plane with enough of these hexagons so as to have our graph (i.e. the underlying graph of any given board) lie completely within this tiling. Since each of the triangles that make up these hexagons is equilateral, we can label all the edges (i.e. sides) 1. The six vertices (underlying the six coins) that surround the center vertex are each, therefore, at a distance 1 from the center. Let us now draw red circles of radius 1 around the center of each of these hexagons. The centers of each of these circles are then at a distance of $2\sqrt{3}$ from each other (their neighboring circles). Due to the symmetry of the equilateral triangles and the hexagons that they make up, each vertex can be viewed as the center of a hexagon. Therefore we can imagine the plane being tiled with some other hexagon; for instance one whose center is a vertex (angle) on the boundary of the first hexagon (i.e. it is one of the six that surround the center). Now let us draw blue circles of radius 1 around the center of each of these hexagons. The centers of these new hexagons, by necessity, each lie on a red circle, thus the centers of the red circles will each be at a distance of one from the centers of the blue circles, and therefore will lie on the blue circles. Since at every vertex (angle) along the boundary of a hexagon three hexagons meet, then at the center of the red circles, which is on the boundary of a blue circle, as well, three hexagons must meet. Thus there are three out of the six vertices along each boundary of the original hexagons (the ones with the red circles) that are centers of blue circles. (The three vertices are symmetrically placed, since between every two vertices is a vertex at a distance of 1 from each of these, and thus on the boundary of one of the hexagon with a blue center.) Finally, we can view the plane as being tiled with hexagons using each of the three vertices along the original hexagon (colored red) that has not been used as centers of our new hexagons. Now draw green circles of radius 1 around the center of each of these hexagons. The centers of both the red and blue circles are the boundary vertices of these new hexagons. As well, the centers of both the red and the green circles are the vertices along the boundary of those hexagons that have a blue circle around their centers. Every vertex in the plane is now the center of some circle.

If we color each vertex in the plane with the same color as that of the circle of which it is the center, then since for each vertex all of its neighbors lie on the circle surrounding it, and by symmetry we have colored all of the vertices that lie on the circle by the other two colors (for they are the centers of circles of other colors), then there is no edge of which both ends will have the same color. Therefore it is 3-colorable.

7. Acknowledgements

I would like to give special thanks to Michael Anshel for his guidance and comments and to Joseph Malkevitch for his helpful conversations and suggestions. I would also like to thank Steven Wat for all of his helpful suggestions and conversations as well as the faculty of the Graduate Center of The City University of New York.

8. References

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