Solutions for exercise set 3

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1 Exercise 3.I.5

There are several ways to tackle this exercise, one that is not the quickest but that give an occasion to take a look at some interesting properties is as follows:
Given a Q-L utility function w.r.t. good one and assuming w.l.o.g. that there are only two goods, the expenditure function is found by solving the problem

$$\min_{x_1, x2} p_1^0 x_1 + p_2^0 x_2 \tag{1}$$

$$\text{s.t.} \qquad x_1 + \varphi(x_2) \geq U^0 \tag{2}$$

Normalising $p_1 = 1$ the solution to this system yields

$$1 = \lambda \tag{3}$$

$$\varphi'(x_2) = p_2 \tag{4}$$

leading to the Hicksian demands

$$h_2 = \varphi'^{-1}(p_2) \tag{5}$$

$$h_1 = U^0 - \varphi(\varphi'^{-1}(p_2)) \tag{6}$$

Now, it is easy to see that demand for good 2 does not depend on the initial utility level, so that it is easy to conclude that

$$\int_{p_2^1}^{p_2^0} h_2(p_2, \overline{p_1}, U^1) dp_2 = \int_{p_2^1}^{p_2^0} h_2(p_2, \overline{p_1}, U^0) dp_2 \tag{7}$$

which was to be expected as we knew that, when the utility function is Q-L w.r.t. good 1, there are no wealth effects for goods 2...L (i.e. $\frac{\partial x_2}{\partial w} = 0$). This leads to the conclusion that $x_2(p, w) = h_2(p, U_0) = h_2(p, U_1)$ implying the result above. For the case of good one, the calculation above leads us to write

$$CV(p_1^1, p_1^0, w) = \int_{p_1^1}^{p_1^0} (U^1 - \varphi(\varphi'^{-1}(p_2)))dp_1 \quad (8)$$

$$EV(p_1^1, p_1^0, w) = \int_{p_1^1}^{p_1^0} (U^0 - \varphi(\varphi'^{-1}(p_2)))dp_1 \quad (9)$$

Which does not need to be equal unless $U^1 = U^0$.
Remembering that these two utility levels are the two indirect utility levels associated with price levels p_1^1 and p_1^0 we can verify if they are equal by computing the Walrasian demands and substituting them in the original utility function.
The Consumer's problem in this case is

$$\max_{x_1, x2} x_1 + \varphi(x_2) \quad (10)$$

$$\text{s.t.} \quad p_1^0 x_1 + p_2^0 x_2 \leq w \quad (11)$$

which has solution

$$x_2 = \varphi'^{-1}(p_2) \quad (12)$$

$$x_1 = w - \varphi(\varphi'^{-1}(p_2)) \quad (13)$$

From this last set of equations the indirect utility can be computed as

$$v(p_1, p_2, w) = w - p_2\varphi'^{-1}(p_2) + \varphi(\varphi'^{-1}(p_2)) \quad (14)$$

now it easy to see that the maximum level off utility attainable does not depend on p_1, so it is possible to conclude that $v(p_1^0, p_2, w) = v(p_1^1, p_2, w)$ or, in other words, that $U^1 = U^0$ leading us to conclude that $CV(p_1^1, p_1^0, w) = EV(p_1^1, p_1^0, w)$.
As this kind of conclusion is valid for good 1 and for goods 2...L we can affirm that in general it is true that $CV(p^1, p^0, w) = EV(p^1, p^0, w)$ when the utility function is Q-L w.r.t. a good.

A much more elegant solution is as follows:
start again with the exp. min. problem and get the Hicksian demand functions

$$h_2 = \varphi'^{-1}(p_2) \quad (15)$$

$$h_1 = U^0 - \varphi(\varphi'^{-1}(p_2)) \quad (16)$$

from which it is possible to write the expenditure function:

$$e(p, U) = U - \varphi(\varphi'^{-1}(p_2)) + p_2\varphi'^{-1}(p_2) \tag{17}$$

From the definition of CV we have

$$CV(p^0, p^1, w) = e(p^1, U^1) - e(p^1, U^0) =$$
$$= U^1 - \varphi(\varphi'^{-1}(p_2)) + p_2\varphi'^{-1}(p_2) - (U^0 - \varphi(\varphi'^{-1}(p_2)) + p_2\varphi'^{-1}(p_2)) =$$
$$= U^1 - U^0$$

From the definition of EV we have

$$EV(p^0, p^1, w) = e(p^0, U^1) - e(p^0, U^0) =$$
$$= U^1 - \varphi(\varphi'^{-1}(p_2)) + p_2\varphi'^{-1}(p_2) - (U^0 - \varphi(\varphi'^{-1}(p_2)) + p_2\varphi'^{-1}(p_2)) =$$
$$= U^1 - U^0$$

Q.E.D.

Exercise 3.I.6

Suppose that, after a price change has been enacted, each individual is assigned a (positive or negative) monetary transfer equal to his/her CV. After all transfers have been assigned each individual will be at a new wealth level. By definition of CV, the individual will be exactly as well off with the new wealth level and the new prices as he/she was before the price change. Hence we can write

$$CV_i + w_i = w_i' \qquad \forall i \tag{18}$$

and

$$v_i(p^0, w_i) = v_i(p^1, w_i') \qquad \forall i \tag{19}$$

It can be easily seen that, as long as $\sum_i CV_i > 0$ it will be the case that $\sum_i w_i > \sum_i w_i'$.
Which leads to conclude that when $\sum_i CV_i > 0$ it is possible to give enough money to everybody to make them exactly as well off as before the price change, and there will still be room to make some additional transfers to make some at least one person strictly better off.