CSLI Publications
Center for the Study of Language and Information
Leland Stanford Junior University
Printed in the United States
05 04 03 02 01 1 2 3 4 5

Library of Congress Cataloging-in-Publication Data

Foundations of real-world intelligence / edited by Yoshinori Uesaka, Pentti Kanerva, Hideki Asoh.
p. cm. -- (CSLI lecture notes ; no. 125)
Includes bibliographical references and index.
ISBN 1-57586-339-1 (acid-free paper) -- ISBN 1-57586-338-3 (pbk. : acid-free paper)
1. Artificial intelligence. 2. Neural networks (Computer science) 3. Evolutionary programming (Computer science) I. Uesaka, Yoshinori, 1936– . II. Kanerva, Pentti, 1937– . III. Asoh, Hideki, 1958– . IV. Series.
Q335 .F685 2001
006.3--dc21

2001043252

∞ The acid-free paper used in this book meets the minimum requirements of the American National Standard for Information Sciences—Permanence of Paper for Printed Library Materials, ANSI Z39.48-1984.

Please visit our web site at
http://cslipublications.stanford.edu/
for comments on this and other titles, as well as for changes and corrections by the author and publisher.

Contents

III

Evolutionary Computation and Beyond

RWC Theoretical Foundation GMD Laboratory

Heinz Mühlenbein and Thilo Mahnig

Simulating evolution as seen in nature has been identified as one of the key computing paradigms for the new decade. Today evolutionary algorithms have been used successfully in a number of applications. These include discrete and continuous optimization problems, synthesis of neural networks, synthesis of computer programs from examples (also called genetic programming), and even evolvable hardware. But all application areas have encountered problems where evolutionary algorithms performed badly. Therefore a mathematical theory of evolutionary algorithms is urgently needed. Theoretical research so far has evolved from two opposing ends: from the theoretical approach there are theories emerging that are getting closer to practice, and from the applied side ad hoc theories have arisen that often lack theoretical justification.

In this chapter we concentrate on the analysis of evolutionary algorithms for optimization. The first section introduces the most popular algorithm, the *simple genetic algorithm*. This algorithm has many degrees of freedom, especially in the choice of recombination scheme. We show the shortcomings of the traditional *schema analysis*. We prove that all genetic algorithms behave very similarly, if recombination is done without selection a sufficient number of times before the next selection step. We approximate this conceptual algorithm by the *univariate marginal distribution algorithm* (UMDA), which is analyzed in section 12. We compute the difference equation for the univariate marginal distribu-

Foundations of Real-World Intelligence.
Yoshinori Uesaka et al. (eds.).

tions under the assumption of proportionate selection. This equation has been proposed in population genetics by Sewall Wright as early as 1937 (Wright, 1970). This is an independent confirmation of our claim that UMDA approximates any genetic algorithm. Using *Wright's equation* we show that UMDA solves a *continuous optimization problem*. The function to be optimized is given by the average fitness of the population.

Proportionate selection is far too weak for optimization. This was recognized very early in the breeding of livestock. *Artificial selection* as done by breeders is a much better model for optimization than *natural selection* modeled by proportionate selection. Unfortunately an exact mathematical analysis of efficient artificial selection schemes seems impossible. Therefore breeders have developed an approximate theory, using the concepts of regression of offspring to parent, heritability, and response to selection. This theory is discussed in section 13. At the end of the section we represent numerical results that show the strength and the weakness of UMDA as a numerical optimization method.

UMDA optimizes very efficiently some difficult optimization problems, but it fails on some simple problems. These problems require higher-order marginal distributions that capture the nonlinear dependency between variables. In section 14 UMDA is extended to the *factorized distribution algorithm* (FDA). We prove convergence of the algorithm to the global optima if *Boltzmann selection* is used. The theory of factorization connects FDA with the theory of *graphical models* and *Bayesian networks*. We derive a new adaptive Boltzmann selection schedule SDS using ideas from the science of breeding.

In section 15 we use results from the theory of Bayesian networks for the *learning factorized distribution algorithm* (LFDA), which computes a factorization from the data. We make a preliminary comparison between the efficiency of FDA and LFDA.

In section 16 we describe the *system dynamics approach to optimization*. The difference equations obtained for UMDA are iterated until convergence. Thus the continuous optimization problem is solved mathematically without using a population of points. We present numerical results for three system dynamics equations: Wright's equation, the *diversified replicator equation,* and a modified version of Wright's equation that converges more quickly.

In the final section we classify the various evolutionary computation methods presented. The classification criterion is whether the method uses a microscopic or a macroscopic model for selection and/or recombination.

11 Analysis of the Simple Genetic Algorithm

In this section we investigate the standard genetic algorithm, also called the simple genetic algorithm (SGA). The algorithm has been described by Holland (1975/1992) and Goldberg (1989). It consists of

- fitness proportionate selection
- recombination/crossover, and
- mutation.

Here we will analyze only selection and recombination. Mutation is considered to be a background operator. It can be analyzed by known techniques from stochastics (Mühlenbein and Schlierkamp-Voosen, 1994; Mühlenbein, 1997).

There have been many claims concerning the optimization power of SGA. Most of them are based on a rather qualitative application of the *schema theorem*. We will show the shortcomings of this approach. Our analysis is based on techniques used in population genetics. The analysis reveals that an exact mathematical analysis of SGA is possible only for small problems. For a binary problem of size n the exact analysis needs the computation of 2^n equations. But we propose an approximation, often used in population genetics, that assumes that the gene frequencies are in *linkage equilibrium*. The main result is that *any genetic algorithm can be approximated by an algorithm using only n parameters, the univariate marginal gene frequencies.*

11.1 Definitions

Let $\mathbf{x} = (x_1, \ldots, x_n)$ denote a binary vector. For notational simplicity we restrict the discussion to binary variables $x_i \in \{0, 1\}$. We use the following conventions. Capital letters X_i denote variables, small letters x_i assignments.

Definition 11.1. *Let a function $f : \boldsymbol{X} \to R^{\geq 0}$ be given. We consider the optimization problem*

$$\mathbf{x}_{\text{opt}} = \text{argmax}\, f(\mathbf{x})\,. \tag{11.1}$$

We will use $f(\mathbf{x})$ as the fitness function for the SGA. We will investigate two widely used recombination/crossover schemes.

Definition 11.2. *Let two strings* $\mathbf{x}$ *and* $\mathbf{y}$ *be given. In* one-point crossover *the string* $\mathbf{z}$ *is created by randomly choosing a crossover point $0 < l < n$ and setting $z_i = x_i$ for $i \leq l$ and $z_i = y_i$ for $i > l$. In* uniform crossover *z_i is randomly chosen with equal probability from $\{x_i, y_i\}$.*

Definition 11.3. *Let $p(\mathbf{x}, t)$ denote the probability of $\mathbf{x}$ in the population at generation t. Then $p_i(x_i, t) = \sum_{\mathbf{x}, X_i = x_i} p(\mathbf{x}, t)$ defines a univariate marginal distribution.*

We will often write $p_i(x_i)$ for simplicity if just one generation is discussed. In this notation the average fitness of the population and the variance is given by

$$\begin{aligned} \bar{f}(t) &= \sum_x p(\mathbf{x}, t) f(\mathbf{x}) \\ V(t) &= \sum_x p(\mathbf{x}, t) \left(f(\mathbf{x}) - \bar{f}(t) \right)^2 . \end{aligned}$$

The *response to selection* $R(t)$ is defined by

$$R(t) = \bar{f}(t+1) - \bar{f}(t) \tag{11.2}$$

11.2 Proportionate Selection

Proportionate selection changes the probabilities according to

$$p(\mathbf{x}, t+1) = p(\mathbf{x}, t) \frac{f(\mathbf{x})}{\bar{f}(t)} . \tag{11.3}$$

Lemma 11.1. *For proportionate selection the response is given by*

$$R(t) = \frac{V(t)}{\bar{f}(t)} . \tag{11.4}$$

Proof. We have

$$R(t) = \sum_x p(\mathbf{x}, t) \frac{f(\mathbf{x})^2}{\bar{f}(t)} - \bar{f}(t) = \frac{V(t)}{\bar{f}(t)} . \tag{11.5}$$

□

With proportionate selection the average fitness never decreases. This is true for every rational selection scheme.

11.3 Recombination

For the analysis of recombination we introduce a special distribution.

Definition 11.4. *Robbins's proportions are given by the distribution π:*

$$\pi(\mathbf{x}, t) := \prod_{i=1}^{n} p_i(x_i, t) . \tag{11.6}$$

A population in Robbins's proportions is also said to be in linkage equilibrium.

Geiringer (1944) has shown that all reasonable recombination schemes lead to the same limit distribution.

Theorem 11.1 (Geiringer). *Recombination does not change the univariate marginal frequencies, i.e.* $p_i(x_i, t+1) = p_i(x_i, t)$. *The limit distribution of any complete recombination scheme is Robbins's proportions* $\pi(\mathbf{x}, 0)$.

Complete recombination means that for each subset S of $\{1, \ldots, n\}$, the probability of an exchange of genes by recombination is greater than zero. Convergence to the limit distribution is very fast. We will prove this for $n = 2$ loci.

Theorem 11.2. *Let* $D(t) = p(0,0,t)p(1,1,t) - p(0,1,t)p(1,0,t)$. *If there is no selection then we have for two loci and uniform crossover:*

$$D(t) = (-1)^{|x|^2} \big(p(\mathbf{x}, t) - p_1(x_1, 0)p_2(x_2, 0)\big)\,. \tag{11.7}$$

$|x|^2$ *denotes the number of ones in* $\mathbf{x}$. $p_i(x_i, 0)$ *denotes the univariate marginal frequency at* $t = 0$. *The factor* $D(t)$ *is halved each generation:*

$$D(t+1) = \frac{1}{2} D(t)\,. \tag{11.8}$$

Proof. Without selection the univariate marginal frequencies are independent of t, because in an infinite population they are unchanged by a recombination operator. Then from

$$\begin{aligned} & p(1,1,t) - p_1(1,0)p_2(1,0) \\ &\quad = \; p(1,1,t) - \big(p(1,0,t) + p(1,1,t)\big)\big(p(0,1,t) + p(1,1,t)\big) \\ &\quad = \; p(1,1,t) - p(0,1,t)p(1,0,t) - p(1,1,t)(1 - p(0,0,t)) \\ &\quad = \; D(t) \end{aligned}$$

we obtain equation (11.7) for $\mathbf{x} = (1,1)$. The other cases are proven in the same way.

The gene frequencies after recombination are obtained as follows. We consider only $p(1,1,t)$. The probability of $p(1,1,t+1)$ can be computed from the probability that recombination generates string $(1,1)$. The probability is given by

$$\begin{aligned} p(1,1,t+1) &= p(1,1,t) \cdot \big(\tfrac{1}{2}p(0,0,t) + p(0,1,t) + p(1,0,t) + p(1,1,t)\big) \\ &\quad + \frac{1}{2}p(0,1,t)p(1,0,t) \\ &= p(1,1,t) - \tfrac{1}{2}\big(p(1,1,t)p(0,0,t) - p(0,1,t)p(1,0,t)\big) \\ &= p(1,1,t) + (-1)^{|\mathbf{x}|^2+1}\frac{1}{2}D(t)\,. \end{aligned}$$

By computing D(t+1) equation (11.8) is obtained. □

We will use as a measure for the deviation from Robbins's proportions

the mean square error DSQ(t):

$$\mathrm{DSQ}(t) = \sum_{\mathbf{x}} \big(p(\mathbf{x},t) - p_1(x_1,t)p_2(x_2,t)\big)^2. \tag{11.9}$$

From the above theorem we obtain the following corollary.

Corollary 11.1. *For two loci the mean square error is reduced each step by one fourth:*

$$\mathrm{DSQ}(t+1) = \frac{1}{4}\mathrm{DSQ}(t)\,.$$

For more than two loci the equations for uniform crossover and one-point crossover get more complicated. Uniform crossover converges faster to linkage equilibrium because it "mixes" the genes much more than does one-point crossover.

In a finite population linkage equilibrium cannot be achieved exactly. Consider as an example the uniform distribution. Here linkage equilibrium is given by $p(\mathbf{x}) = 2^{-n}$. This value can be obtained only if the size of the population is substantially larger than 2^n. In a finite population we observe first a fast decrease of linkage disequilibrium. Then DSQ slowly increases owing to stochastic fluctuations by *genetic drift*. Ultimately the population will consist of only one genotype. Genetic drift has been analyzed by Asoh and Mühlenbein (1994b). It will not be considered here.

11.4 Selection and Recombination

We have shown that the average $\bar{f}(t)$ never decreases after selection and that any complete recombination scheme rearranges the genetic population to Robbins's proportions. Now the difficult question arises: What happens if recombination is applied *after* selection? This problem is still a puzzle in population genetics (Nagylaki, 1992). Formally, the difference equations can be easily written. Let a recombination distribution R be given. $R_{x,yz}$ denotes the probability that y and z produce x after recombination. Then

$$p(\mathbf{x},t+1) = \sum_{y,z} R_{x,yz}\, p^s(\mathbf{y},t)\, p^s(\mathbf{z},t)\,, \tag{11.10}$$

where $p^s(x)$ denotes the probability of string x after selection. For n loci the recombination distribution R consists of $2^n \times 2^n$ parameters. Recently Christiansen and Feldman (1998) have written a survey about the mathematics of selection and recombination from the viewpoint of population genetics. A new technique to obtain the equations has been developed by Vose (1999). In both frameworks one needs a computer program to compute the equations for a given fitness function.

We discuss the problem only for a special case, uniform crossover for $n = 2$ loci.

Theorem 11.3. *For proportionate selection and uniform crossover the gene frequencies obey the following difference equation:*

$$p(\mathbf{x}, t+1) = p(\mathbf{x}, t)\frac{f(\mathbf{x})}{\bar{f}(t)} + (-1)^{|\mathbf{x}|^2+1}\frac{1}{2}\frac{D_s(t)}{\bar{f}(t)^2}\,. \tag{11.11}$$

$|\mathbf{x}|$ *denotes the number of ones in* $\mathbf{x}$. $\bar{f}(t) = \sum_{\mathbf{x}} p(\mathbf{x}, t) f(\mathbf{x})$ *is the average fitness of the population; and* $D_s(t)$ *is defined as*

$$\begin{aligned} D_s(t) = {} & f(0,0)f(1,1)p(0,0,t)p(1,1,t) \\ & - f(0,1)f(1,0)p(1,0,t)p(0,1,t)\,. \end{aligned} \tag{11.12}$$

Proof. For proportionate selection the gene frequencies $p^s(\mathbf{x}, t)$ after selection are given by

$$p^s(\mathbf{x}, t) = p(\mathbf{x}, t)\frac{f(\mathbf{x})}{\bar{f}(t)}\,.$$

Now we pair randomly between the selected parents and count how often genotype $\mathbf{x}$ arises after uniform crossover. Taking $\mathbf{x} = (0, 0)$ as an example, and computing the probabilities of mating, we obtain

$$\begin{aligned} & p(0,0,t+1) \\ & \quad = \quad p^s(0,0,t)\left(p^s(0,0,t) + p^s(0,1,t) + p^s(1,0,t) + \tfrac{1}{2}p^s(1,1,t)\right) \\ & \qquad + \tfrac{1}{2}p^s(0,1,t)p^s(1,0,t)\,. \end{aligned}$$

Using the fact that $p^s(0,0,t) + p^s(0,1,t) + p^s(1,0,t) + p^s(1,1,t) = 1$ we obtain the theorem for $\mathbf{x} = (0, 0)$. The remaining equations are obtained in the same manner. □

A rigorous analysis of the mathematical properties of n loci systems is difficult. For a problem of size n we have 2^n equations. Furthermore the equations depend on the recombination operator used. If the gene frequencies remain in linkage equilibrium, then only n equations are needed for the marginal frequencies. Thus the crucial question is: Does the optimization process get worse because of this simplification? The answer is no. Evidence for this statement is found in a theorem from Mühlenbein (1997). It shows that the univariate marginal frequencies are the same for all recombination schemes if applied to the same distribution $p(\mathbf{x}, t)$.

Theorem 11.4. *For* any *complete recombination/crossover scheme used after proportionate selection the univariate marginal frequencies are determined by*

$$p_i(x_i, t+1) = \sum_{\mathbf{x}|X_i=x_i} \frac{p(\mathbf{x}, t) f(\mathbf{x})}{\bar{f}(t)}\,. \tag{11.13}$$

Proof. After selection the univariate marginal frequencies are given by

$$p_i^s(x_i,t) = \sum_{\mathbf{x}|X_i=x_i} p^s(\mathbf{x},t) = \sum_{\mathbf{x}|X_i=x_i} \frac{p(\mathbf{x},t)f(\mathbf{x})}{\bar{f}(t)}\,.$$

Now the selected individuals are paired randomly. Since complete recombination does not change the allele frequencies, these operators do not change the univariate marginal frequencies. Therefore

$$p_i(x_i,t+1) = p_i^s(x_i,t)\,.$$

□

11.5 Schema Analysis Demystified

Many of the more intuitive arguments about the behavior of genetic algorithm are based on the analysis of "schemata" and their evolution in a population. The theory has been developed by Holland (1975/1992). By using probability distributions and an ideal schema equation we demonstrate by a simple example that the popular conclusions about the proliferation of schemata are wrong. Our analysis is based on an exact solution of the probability distribution for proportionate selection.

Definition 11.5. *Let $p(\mathbf{x},t)$ denote the probability of $\mathbf{x}$ in the population at generation t. Let $\mathbf{x}_s = (x_{s_1},\ldots,x_{s_i}) \subset \{x_1,\ldots,x_n\}$. Thus $\mathbf{x}_s$ denotes a sub vector of $\mathbf{x}$ defined by the indices $s_1,\ldots,s_i$. Then the probability of schema $H(\boldsymbol{s})$ is defined by*

$$p(H(\boldsymbol{s}),t) = \sum_{X|X_s=x_s} p(\mathbf{x},t)\,. \tag{11.14}$$

The summation is done by fixing the values of $\mathbf{x}_s$. Thus the probability of a schema is just the corresponding marginal distribution $p(\mathbf{x}_s)$. If $\mathbf{x}_s$ consists of only a single element, we have a univariate marginal distribution.

SGA uses fitness proportionate selection, which means that the probability of $\mathbf{x}$ being selected is given by

$$p^s(\mathbf{x},t) = p(\mathbf{x},t)\frac{f(\mathbf{x})}{\bar{f}(t)}\,. \tag{11.15}$$

$\bar{f}(t) = \sum_x p(\mathbf{x},t)f(\mathbf{x})$ is the average fitness of the population. Let us now assume that we have an algorithm that generates new points according to the distribution of selected points, or more formally:

$$p(\mathbf{x},t+1) = p(\mathbf{x},t)\frac{f(\mathbf{x})}{\bar{f}(t)}\,. \tag{11.16}$$

$p(\mathbf{x},t+1)$ can be seen as the ideal probability distribution of SGA.

Definition 11.6. *The fitness of schema $H(\boldsymbol{s})$ is defined by*

$$f(H(\boldsymbol{s}),t) = \sum_{X|X_s=x_s} \frac{p(\mathbf{x},t)}{p(H(\boldsymbol{s}),t)} f(\mathbf{x})\,. \tag{11.17}$$

Theorem 11.5 (Schema Theorem). *The probability of schema $H(\boldsymbol{s})$ is given by*

$$p(H(\boldsymbol{s}),t+1) = p(H(\boldsymbol{s},t))\frac{f(H(\boldsymbol{s}),t)}{\bar{f}(t)}\,. \tag{11.18}$$

Holland (1975/1992, Theorem 6.2.3) computed for SGA the following inequality:

$$p(H(\boldsymbol{s}),t+1) \geq (1-\delta)p\big(H(\boldsymbol{s},t)\big)\frac{f(H(\boldsymbol{s}),t)}{\bar{f}(t)}\,, \tag{11.19}$$

where δ is a small factor that captures the loss by mutation and crossover. The inequality only complicates the analysis. Equation (11.17) is obviously an ideal case for Holland's analysis. The mathematical difficulty of using the inequality (11.19) to estimate the distribution of schemata lies in the fact that the fitness of a schema depends on $p(\mathbf{x},t)$, i.e. the distribution of the genotypes of the population. This is a defining fact of Darwinian *natural selection*. The fitness is always relative to the current population. To cite a proverb: *the one-eyed is the king of the blind.*

Thus an application of the inequality (11.19) is not possible without computing $p(\mathbf{x},t)$. Goldberg (1989) circumvented this problem by assuming

$$p(H(\boldsymbol{s}),t) \geq (1+c)\bar{f}(t)\,. \tag{11.20}$$

With this assumption we estimate $p(H(\boldsymbol{s}),t) \geq (1+c)^t p(H(\boldsymbol{s}),0)$. But the assumption can never be fulfilled for all t. When approaching an optimum, the fitness of all schemata in the population will be only $1 \pm \epsilon$ away from the average fitness. Here proportionate selection meets difficulties.

The typical folklore that arose from the schema analysis is nicely summarized by Ballard (1997, page 270). He is not biased toward or against genetic algorithms. He just cites the commonly used arguments:

- *Short schemata have a high probability of surviving the genetic operations.*
- *Focusing on short schemata that compete shows that, over the short run, the fittest are increasing at an exponential rate.*
- *Ergo, if all of the assumptions hold (we cannot tell whether they do, but we suspect they do), GAs are optimal.*

We will not investigate the optimality argument, but will show that the basic conclusion of exponential increasing schemata does not hold.

It turns out that equation (11.16) for proportionate selection admits an analytical solution.

Theorem 11.6 (Convergence). *The distribution $p(\mathbf{x}, t)$ for proportionate selection is given by*

$$p(\mathbf{x}, t) = \frac{p(\mathbf{x}, 0) f(\mathbf{x})^t}{\sum_y p(\mathbf{y}, 0) f(\mathbf{y})^t} \,. \tag{11.21}$$

Let $\mathcal{M}$ be the set of global optima, then

$$\lim_{t \to \infty} p(\mathbf{x}, t) = \begin{cases} 1/|\mathcal{M}| & \mathbf{x} \in \mathcal{M} \\ 0 & \text{otherwise.} \end{cases} \tag{11.22}$$

Proof. The proof is by induction. The assumption is fulfilled for $t = 1$. Then

$$\begin{aligned} p(\mathbf{x}, t+1) &= \frac{p(\mathbf{x}, 0) f(\mathbf{x})^{t+1}}{\sum_y p(\mathbf{y}, 0) f(\mathbf{y})^{t+1}} \\ &= \frac{p(\mathbf{x}, 0) f(\mathbf{x})^t}{\bar{f}(t)} \cdot \frac{f(\mathbf{x})}{\sum_y \frac{p(\mathbf{y}, 0) f(\mathbf{y})^t \cdot f(\mathbf{y})}{\bar{f}(t)}} \\ &= \frac{p(\mathbf{x}, 0) f(\mathbf{x})^{t+1}}{\sum_y p(\mathbf{y}, 0) f(\mathbf{y})^{t+1}} \,. \end{aligned}$$

Let $\mathbf{x}_{\max} \in \mathcal{M}$ and $f(\mathbf{x}) < f(\mathbf{x}_{\max})$. Then

$$\frac{p(\mathbf{x}, t)}{p(\mathbf{x}_{\max}, t)} = \frac{p(\mathbf{x}, 0) f(\mathbf{x})^t}{p(\mathbf{x}_{\max}, 0) f(\mathbf{x}_{\max})^t} \to 0 \,.$$

□

This shows that our algorithm is ideal in the sense that it converges even to the set of global optima. Equation (11.21) was already used by Goldberg and Deb (1991).

By using equation (11.21) we can make a correct schema analysis. We compute the probabilities of all schemata. We discuss only the interesting case of a *deceptive function*. We take the 3-bit deceptive function defined by

$$\begin{aligned} \text{Decep}(\mathbf{x}) = 0.9 &- 0.1(x_1 + x_2 + x_3) \\ &- 0.7(x_1 x_2 + x_2 x_3 + x_1 x_3) + 2.5 x_1 x_2 x_3 \,. \end{aligned}$$

The function is called *deceptive* because the global optimum $(1, 1, 1)$ is isolated, whereas the local optimum $(0, 0, 0)$ is surrounded by strings of high fitness. We now look at the behavior of some schemata.

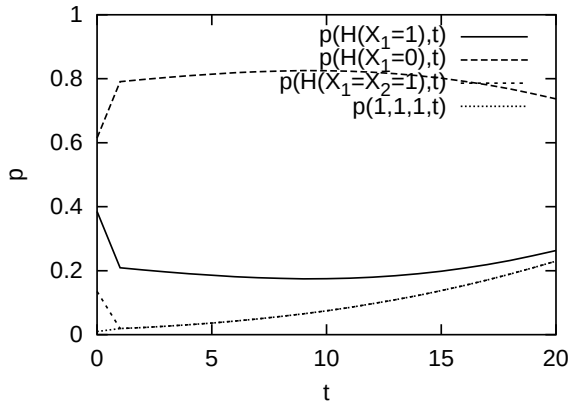

FIGURE 11.1. Evolution of some schemata.

Definition 11.7. *A schema is* optimal *if its defining string s is contained in an optimal string.*

For the deceptive function, $H(X_1=1)$ and $H(X_1=X_2=1)$ are optimal schemata. These are displayed in figure 11.1. We see that the probability of the optimal schema $p(H(X_1=1)$ decreases for about 8 generations, then it increases fairly slowly. This behavior is contrary to the folklore arising from Holland's schema analysis. Schema $H(X_1=X_2=1)$ decreases dramatically at the first generation; then its probability is almost identical to the probability of the optimum $(1,1,1)$.

To summarize the results: All complete recombination schemes lead to the same univariate marginal distributions after one step of selection and recombination. If recombination is used a number of times without selection, then the genotype frequencies converge to linkage equilibrium. This means that *all genetic algorithms are identical if after one selection step recombination is done without selection a sufficient number of times.* This fundamental algorithm keeps the population in linkage equilibrium.

12 The Univariate Marginal Distribution Algorithm (UMDA)

The univariate marginal distribution algorithm UMDA generates new points according to $p(\mathbf{x},t)=\prod_{i=1}^{n} p_i^s(x_i,t)$. Thus UMDA keeps the gene frequencies in linkage equilibrium. This makes a mathematical analysis possible. We derive a difference equation for proportionate selec-

tion. This equation has already been proposed by Sewall Wright in 1937 (Wright, 1970). Wright's equation shows that UMDA is trying to solve a continuous optimization problem. The continuous function to be optimized is the average fitness of the population $W(\mathbf{p})$. The variables are the univariate marginal distributions. In a fundamental theorem we show the relation between the attractors of the continuous problem and the local optima of the fitness function $f(\mathbf{x})$.

12.1 Definition of UMDA

Instead of performing recombination a number of times in order to converge to linkage equilibrium, one can achieve this in one step by *gene pool recombination* (Mühlenbein and Voigt, 1996). In gene pool recombination a new string is computed by randomly taking for each locus a gene from the distribution of the selected parents. This means that gene x_i occurs with probability $p_i^s(x_i)$ in the next population. $p_i^s(x_i)$ is the distribution of x_i in the selected parents. New strings $\mathbf{x}$ are generated according to the distribution

$$p(\mathbf{x}, t+1) = \prod_{i=1}^{n} p_i^s(x_i, t)\,. \tag{12.1}$$

One can simplify the algorithm by directly computing the univariate marginal frequencies from the data. Then equation (12.1) can be used to generate new strings. This method is used by UMDA.

UMDA

- STEP 0: Set $t \Leftarrow 1$. Generate $N \gg 0$ points randomly.
- STEP 1: Select $M \leq N$ points according to a selection method. Compute the marginal frequencies $p_i^s(x_i, t)$ of the selected set.
- STEP 2: Generate N new points according to the distribution $p(\mathbf{x}, t+1) = \prod_{i=1}^{n} p_i^s(x_i, t)$. Set $t \Leftarrow t+1$.
- STEP 3: If termination criteria are not met, go to STEP 1.

For proportionate selection we need the average fitness of the population $\bar{f}(t)$. We consider $\bar{f}(t)$ as a function that depends on $p(x_i)$. To emphasize this dependency we write

$$W(p_1(X_1 = 0), p_1(X_1 = 1), \ldots, p_n(X_n = 1)) := \bar{f}(t)\,. \tag{12.2}$$

W depends formally on $2n$ parameters. $p_i(X_i = 1)$ and $p_i(X_i = 0)$ are considered to be two independent parameters despite the constraint $p_i(X_i = 0) = 1 - p_i(X_i = 1)$. We abbreviate $p_i := p_i(X_i = 1)$. If we insert $1 - p_i$ for $p_i(X_i = 0)$ into W, we obtain $\tilde{W}$. $\tilde{W}$ depends on n parameters. Now we can formulate the main theorem.

Theorem 12.1. *For infinite populations and proportionate selection the difference equations for the gene frequencies used by UMDA are given by*

$$p_i(x_i, t+1) = p_i(x_i, t)\frac{\bar{f}_i(x_i, t)}{W(t)} = p_i(x_i, t)\frac{\frac{\partial W}{\partial p_i(x_i)}}{W(t)} \tag{12.3}$$

with

$$\bar{f}_i(x_i, t) := \sum_{\mathbf{x}, X_i = x_i} f(\mathbf{x}) \prod_{j \neq i}^{n} p(x_j, t)\,. \tag{12.4}$$

The equation can also be written as

$$p_i(t+1) = p_i(t) + p_i(t)(1 - p_i(t))\frac{\frac{\partial \tilde{W}}{\partial p_i}}{\tilde{W}(t)}\,. \tag{12.5}$$

The response $R(t)$ is given by

$$\begin{aligned} R(t) = \frac{\mathrm{VA}(t)}{\tilde{W}} &+ \frac{1}{2}\sum_{i \neq j} \frac{\alpha_i \cdot \alpha_j}{\tilde{W}^2}\frac{\partial^2 \tilde{W}}{\partial p_i \partial p_j} \\ &+ \frac{1}{3!}\sum_{i \neq j, j \neq k, i \neq k} \frac{\alpha_i \cdot \alpha_j \cdot \alpha_k}{\tilde{W}^3}\frac{\partial^3 \tilde{W}}{\partial p_i \partial p_j \partial p_k} + \cdots\,, \end{aligned} \tag{12.6}$$

$$\mathrm{VA}(t) = \sum_i p_i(1,t)\big(f_i(1,t) - W\big)^2 + p_i(0,t)\big(f_i(0,t) - W\big)^2 \tag{12.7}$$

$$\alpha_i = p_i(t)\big(1 - p_i(t)\big)\frac{\partial \tilde{W}}{\partial p_i}\,.$$

$\mathrm{VA}(t)$ *is called the* additive genetic variance. *The average fitness never decreases:*

$$W(t+1) \geq W(t)\,. \tag{12.8}$$

Proof. Equation (12.3) has been proven in Mühlenbein (1997). We have to prove equation (12.5). Note that

$$p_i(t+1) - p_i(t) = p_i(t)\frac{\bar{f}_i(x_i = 1, t) - \tilde{W}(t)}{\tilde{W}(t)}\,.$$

Obviously we have

$$\frac{\partial \tilde{W}}{\partial p_i} = \bar{f}(x_i = 1, t) - \bar{f}(x_i = 0, t)\,.$$

From $p_i(t)\bar{f}_i(x_i = 1, t) + (1 - p_i(t))\bar{f}_i(x_i = 0, t) = \tilde{W}(t)$, we obtain

$$\bar{f}(x_i{=}1, t) - \tilde{W}(t) - \big(1 - p_i(t)\big)\bar{f}(x_i{=}1, t) + \big(1 - p_i(t)\big)\bar{f}(x_i{=}0, t) = 0\,.$$

This gives

$$\bar{f}_i(x_i = 1, t) - \tilde{W}(t) = \big(1 - p_i(t)\big)\frac{\partial \tilde{W}}{\partial p_i}\,.$$

Inserting this equation into the difference equation yields equation (12.5). Equation (12.6) is just the multidimensional Taylor expansion. The first term follows from

$$\begin{aligned}
&\sum_i (p_i(t+1) - p_i(t))\frac{\partial \tilde{W}}{\partial p_i} \\
&\qquad = \sum_i p_i(t)(1 - p_i(t))\left(\frac{\partial \tilde{W}}{\partial p_i}\right)^2 \\
&\qquad = \sum_i p_i(t)(f_i(1,t) - \tilde{W})\big(f_i(1,t) - \tilde{W} + \tilde{W} - f_i(0,t)\big) \\
&\qquad = \sum_i p_i(t)(f_i(1,t) - \tilde{W})^2 + (1 - p_i(t))(f_i(0,t) - \tilde{W}) \\
&\qquad = \mathrm{VA}(t)\,.
\end{aligned}$$

□

The above equations completely describe the dynamics of UMDA with proportionate selection. Mathematically UMDA performs gradient ascent in the landscape defined by W or $\tilde{W}$.

Equation (12.5), Wright's equation, is especially suited for the theoretical analysis. Wright's (1970, p. 8) remarks are still valid today:

> The appearance of this formula is deceptively simple. Its use in conjunction with other components is not such a gross oversimplification in principle as has sometimes been alleged. ... Obviously calculations can be made only from rather simple models, involving only a few loci or simple patterns of interaction among many similarly behaving loci. ... Apart from application to simple systems, the greatest significance of the general formula is that its form brings out properties of systems that would not be apparent otherwise.

The restricted application lies in the following fact. In general the difference equations requite the evaluation of 2^n terms. The computational complexity can be reduced drastically if the fitness function has a special form.

Example 12.1. $f(\mathbf{x}) = \sum_i a_i x_i, \quad x_i \in \{0, 1\}$.

After some tedious manipulations one obtains:

$$\begin{aligned} W(\mathbf{p}) &= \sum_i a_i p_i(1) \\ \frac{\partial W}{\partial p_i(1)} &= a_i + \sum_{j \neq i} a_j p_j(1)\,. \end{aligned}$$

This gives the difference equation:

$$\Delta p_i(1) = p_i(1,t)\big(1 - p_i(1,t)\big)\frac{a_i}{\sum_i a_i p_i(1,t)}\,. \tag{12.9}$$

Given that $\frac{\partial \tilde{W}}{\partial p_i(1)} = a_i$, we have proven nothing other than Wright's equation. This equation has been solved approximately in Mühlenbein and Mahnig (1999a).

This example shows that the expressions for W and its derivatives can be surprisingly simple. $W(\mathbf{p})$ can be obtained from $f(\mathbf{x})$ by exchanging x_i with $p_i(1)$. But the formal derivation of $W(\mathbf{p})$ cannot be obtained from the simplified $W(\mathbf{p})$ expression.

Another interesting example is a multiplicative function.

Theorem 12.2. *For a multiplicative function $f(\mathbf{x}) = \prod_{i=1}^n f_i(x_i)$ we have*

$$R(t) = \frac{V(t)}{\tilde{W}} = S(t) \tag{12.10}$$

Proof. The proof is technically somewhat complicated. We just sketch the proof for $n = 2$. We set $p_1 = p_1(x_1, t)$ and $p_2 = p_2(x_2, t)$. Using equation (12.3) we obtain

$$\begin{aligned} p(\mathbf{x}, t+1) &= \frac{p_1 \bar{f}_1(x_1,t) p_2 \bar{f}(x_2,t)}{W^2} \\ \bar{f}_1(x_1,t) &= p_2 f(x_1,x_2) + (1-p_2) f(x_1, 1-x_2) \\ \bar{f}_2(x_2,t) &= p_1 f(x_1,x_2) + (1-p_1)) f(1-x_1, x_2)\,. \end{aligned}$$

After some manipulations we obtain

$$\bar{f}_1(x_1,t)\bar{f}_2(x_2,t) = f(x_1,x_2)W,$$

and finally

$$p(\mathbf{x}, t+1) = p(\mathbf{x}, t)\frac{f(\mathbf{x})}{W}\,.$$

From lemma 11.1 we obtain $R(t) = V(t)/W$. □

12.2 Computing the Average Fitness

In the this section we investigate the computation of W and its gradient. Wright is also the originator of the landscape metaphor now popular in evolutionary computation and population genetics. Unfortunately

Wright used two quite different definitions for the landscape, apparently without realizing the fundamental distinction between them. The first landscape describes the relation between the genotypes and their fitness, whereas the second describes the relation between the allele frequencies in a population and its mean fitness.

The first definition is just the fitness function $f(\mathbf{x})$ used in evolutionary computation; the second is the average fitness $\tilde{W}(\mathbf{p})$. The second definition is much more useful, because it lends itself to a quantitative description of the evolutionary process, i.e. Wright's equation.

For notational simplicity we derive only the relation between $f(\mathbf{x})$ and $\tilde{W}$ for binary alleles. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \{0, 1\}$ be a multi-index. We define with $0^0 := 1$:

$$\mathbf{x}^\alpha := \prod_i x_i^{\alpha_i}.$$

Definition 12.1. *The representation of a binary discrete function using the ordering according to function values is given by*

$$f(\mathbf{x}) = f(0, \ldots, 0)(1 - x_1) \cdots (1 - x_n) + \cdots + f(1, \ldots, 1)x_1 \cdots x_n \,.$$

The representation using the ordering according to variables is

$$f(\mathbf{x}) = \sum_\alpha a_\alpha x^\alpha. \tag{12.11}$$

$\max\{|\alpha|_1 = \sum_i \alpha_i : \ a_\alpha \neq 0\}$ *is called the* order *of the function.*

In both representations the function is linear in each variable x_i. The following lemma is obvious.

Lemma 12.1. *The two representations are unique. There exists a unique matrix A of dimension $2^n \times 2^n$ such that*

$$a_\alpha = (Af)_\alpha \,.$$

We now use this result for $\tilde{W}$.

Lemma 12.2. *$\tilde{W}(\mathbf{p}) := \bar{f}(t)$ is an extension of $f(x)$ to S. There exist two representations for $\tilde{W}(p)$. These are given by*

$$\tilde{W}(\mathbf{p}) = f(0, \ldots, 0)(1 - p_1) \cdots (1 - p_n) + \cdots + f(1, \ldots, 1)p_1 \cdots p_n \tag{12.12}$$

$$\tilde{W}(\mathbf{p}) = \sum_\alpha a_\alpha p^\alpha. \tag{12.13}$$

The proofs in this section have been informal. The above lemma can be proven rigorously by Moebius inversion. If the function is given in analytical form (equation (12.11)) and the order of the function is bounded by a constant independent of n, $\tilde{W}(\mathbf{p})$ can be computed in polynomial

time. The equation can also be used to compute the derivative of $\tilde{W}$, which is needed for Wright's equation. It is given by

$$\frac{\partial \tilde{W}(p)}{\partial p_i(1)} = \sum_{\alpha|\alpha_i=1} a_\alpha p^{\alpha'} \tag{12.14}$$

with $\alpha'_i = 0, \alpha'_j = \alpha_j$.

We will now characterize the attractors of UMDA. Let $S_i = \{q_i | \sum_{k\in\{0,1\}} q_i(x_k) \leq 1;\ 0 \leq q_i(x_k) \leq 1\}$ and $S = \prod_i S_i$ the Cartesian product. Then $S = [0,1]^n$ is the unit cube.

Theorem 12.3. *The stable attractors of Wright's equation are at the corners of S, i.e. $p_i \in \{0,1\}$, $i = 1,\ldots,n$. In the interior there are only saddle points or local minima where* grad $W(p)) = 0$. *The attractors are local maxima of $f(x)$ according to one bit changes. Wright's equation solves the continuous optimization problem* $\text{argmax}\{\tilde{W}(\mathbf{p})\}$ *in S by gradient ascent.*

Proof. W is linear in p_i; therefore it cannot have any local maxima in the interior. Points with $\text{grad}\, W(p) = 0$ are unstable fix points of UMDA.

We next show that boundary points that are not local maxima of $f(x)$ cannot be attractors. We prove the conjecture indirectly. Without loss of generality, let the boundary point be $\hat{p} = (1,\ldots,1)$. We now consider an arbitrary neighbor, i.e. $p^* = (0,1,\ldots,1)$. The two points are connected at the boundary by

$$p(z) = (1-z, 1, \ldots, 1) \qquad z \in [0,1]\,.$$

We know that $\tilde{W}$ is *linear* in the parameters p_i. Because $\tilde{W}(p^*) = f(0,1,\ldots,1)$ and $\tilde{W}(\hat{p}) = f(1,\ldots,1)$ we have

$$\tilde{W}(p(z)) = f(1,\ldots,1) + z \cdot \big[f(0,1,\ldots,1) - f(1,\ldots,1)\big]\,. \tag{12.15}$$

If $f(0,1,\ldots,1) > f(1,\ldots,1)$ then $\hat{p}$ cannot be an attractor of UMDA. The mean fitness increases with z. □

The extension of the above lemma to multiple alleles and multivariate distributions is straightforward, but the notation becomes difficult.

13 The Science of Breeding

Fitness proportionate selection is the undisputed selection method in population genetics. It is considered a model for *natural selection*. But for proportionate selection the following problem arises. When the population approaches an optimum, selection gets weaker and weaker because the fitness values become similar. Therefore breeders of livestock use

other selection methods. These are called *artificial selection* methods. For large populations they mainly apply *truncation selection*. It works as follows. A truncation threshold τ is fixed. Then the τN best individuals are selected as parents for the next generation. These parents are then randomly mated.

The science of breeding is the domain of *quantitative genetics*. The theory is based on macroscopic variables. Because an exact mathematical analysis is impossible, many statistical techniques are used. In fact, the concepts of regression, correlation, heritability, and decomposition of variance were first developed and applied in quantitative genetics.

13.1 Single Trait Theory

For a single trait the theory is easily summarized. Starting with the fitness distribution, the *selection differential* $S(t)$ is introduced. It is the difference between the average of the selected parents and the average of the population:

$$S(t) = W(\mathbf{p}^s(t+1)) - W(\mathbf{p}(t))\,. \tag{13.1}$$

Similarly, the response $R(t)$ is defined:

$$R(t) = W(\mathbf{p}(t+1)) - W(\mathbf{p}(t))\,. \tag{13.2}$$

Next a linear regression is performed:

$$R(t) = b(t)S(t)\,. \tag{13.3}$$

$b(t)$ is called *realized heritability*. The most difficult part of applying the theory is predicting $b(t)$. The first estimate uses the *regression of offspring to parent*. Let f_i, f_j be the phenotypic values of parents i and j. Then

$$\bar{f}_{i,j} = \frac{f_i + f_j}{2}$$

is the mid-parent value. Let the stochastic variable $\bar{F}$ denote the mid-parent value.

Theorem 13.1. *Let $P(t) = (f_1, \ldots, f_N)$ be the population at generation t, where f_i denotes the phenotypic value of individual i. Assume that an offspring generation $O(t)$ is created by random mating, without selection. If the regression equation*

$$o_{ij}(t) = a(t) + b_{\bar{P}O}(t) \cdot \frac{f_i + f_j}{2} + \epsilon_{ij} \tag{13.4}$$

with

$$E(\epsilon_{ij}) = 0$$

is valid, where o_{ij} is the fitness value of an offspring of i and j, then

$$b_{\bar{P}O}(t) \approx b(t)\,. \tag{13.5}$$

Proof. From the regression equation we obtain for the expected averages:

$$E(O(t)) = a(t) + b_{\bar{P}O}(t)M(t)\,.$$

Because the offspring generation is created by random mating without selection, the expected average fitness remains constant:

$$E(O(t)) = M(t)\,.$$

Let us now select a subset as parents. The parents will be randomly mated, producing the offspring generation. If the subset is large enough, we may still use the regression equation and obtain for the averages:

$$M(t+1) = a(t) + b_{\bar{P}O}(t) \cdot M_s(t)\,.$$

Here $M(t+1)$ is the average fitness of the offspring generation produced by the selected parents. Subtracting the above equations we obtain

$$M(t+1) - M(t) = b_{\bar{P}O}(t) \cdot (M_s(t) - M(t))\,.$$

This proves $b_{\bar{P}O}(t) = b(t)$. □

The importance of regression for estimating the heritability was discovered by Galton and Pearson at the end of the nineteenth century. They computed the regression coefficient intuitively, using scatter diagrams of mid-parent and offspring (Freedman et al., 1991). The problem of computing a good regression coefficient is solved mathematically by the theorem of Gauss-Markov. Here we just cite the theorem. The proof can be found in any textbook on statistics (e.g. Rao, 1973).

Theorem 13.2. *A good estimate for the regression coefficient of mid-parent and offspring is given by*

$$b_{\bar{P}O}(t) = \frac{\mathrm{cov}(O(t), \bar{P}(t))}{\mathrm{var}(\bar{P}(t))}\,. \tag{13.6}$$

The covariance of O and $\bar{P}$ is defined by

$$\mathrm{cov}(O(t), \bar{P}(t)) = \frac{1}{N}\sum_{i,j}(o_{i,j} - \mathrm{av}(O(t))) \cdot (\bar{f}_{i,j} - \mathrm{av}(\bar{P}(t)))\,,$$

where *av* denotes the average and *var* the variance. Closely related to the regression coefficient is the correlation coefficient $\mathrm{cor}(\bar{F}, O)$. It is given by

$$\mathrm{cor}(\bar{P}(t), O(t)) = b_{\bar{P}O}(t) \cdot \left(\frac{\mathrm{var}(\bar{P}(t))}{\mathrm{var}(O(t))}\right)^{1/2}.$$

The concept of covariance is restricted to parents producing offspring. It cannot be used for UMDA. Here the *analysis of variance* helps. We will decompose the fitness value $f(\mathbf{x})$ recursively into an additive part and interaction parts. Recall the definition of conditional probability:

Definition 13.1. *Let $p(\mathbf{x})$ denote the probability of $\mathbf{x}$. Then the conditional probability $p(\mathbf{x}|y)$ of $\mathbf{x}$ given y is defined by*

$$p(\mathbf{x}|y) = \frac{p(\mathbf{x}, y)}{p(y)} \,. \tag{13.7}$$

The decomposition works as follows. First we extract the average:

$$f(\mathbf{x}) = \bar{f} + r_0(\mathbf{x}) \,. \tag{13.8}$$

Then we extract the first-order (additive) part from the residual $r_0(\mathbf{x})$:

$$r_0(\mathbf{x}) = \sum_{i=1}^{n} f_{(i)}(x_i) + r_1(\mathbf{x}) \,, \tag{13.9}$$

where $f_{(i)}(x_i)$ is given by

$$f_{(i)}(x_i) = \sum_{\mathbf{x}|x_i} p(\mathbf{x}|x_i) r_0(\mathbf{x}) = \sum_{\mathbf{x}|x_i} p(\mathbf{x}|x_i) f(\mathbf{x}) - \bar{f} \,.$$

Here $\sum_{\mathbf{x}|x_i}$ means that the ith locus is fixed to the value x_i. The $f_{(i)}(x_i)$ minimize the quadratic error $\sum_{\mathbf{x}} p(\mathbf{x}) r_1(\mathbf{x})^2$.

If $r_1(\mathbf{x}) \not\equiv 0$, we can proceed further to extract the second-order terms from $r_1(\mathbf{x})$:

$$r_1(\mathbf{x}) = \sum_{\substack{(i,j) \\ i<j}} f_{(i,j)}(x_i, x_j) + r_2(\mathbf{x}) \,, \tag{13.10}$$

where

$$\begin{aligned} f_{(i,j)}(x_i, x_j) &= \sum_{\mathbf{x}|x_i,x_j} p(\mathbf{x}|x_i, x_j)\; r_1(\mathbf{x}) \\ &= \sum_{\mathbf{x}|x_i,x_j} p(\mathbf{x}|x_i, x_j)\; f(\mathbf{x}) - f_{(i)}(x_i) - f_{(j)}(x_j) \,. \end{aligned}$$

If we have n loci, we can iterate this procedure $n-1$ times recursively and finally get the decomposition of f as

$$\begin{aligned} f(\mathbf{x}) &= \bar{f} + \sum_i f_{(i)}(x_i) + \sum_{(i,j)} f_{(i,j)}(x_i, x_j) + \cdots \\ &\quad + \sum_{\substack{(i_1,\ldots,i_{n-1}) \\ i_1<\cdots<i_{n-1}}} f_{(i_1,\ldots,i_{n-1})}(x_{i_1}, \ldots, x_{i_{n-1}}) + r_{n-1}(\mathbf{x}) \,. \end{aligned}$$

Let V_k for $k = 1$ to $n-1$ be defined as

$$V_k = \sum_{\substack{(i_i,\ldots,i_k) \\ i_1<\cdots<i_k}} \sum_{x_{i_1},\ldots,x_{i_k}} p(x_{i_1}, \ldots, x_{i_k}) f_{(i_i,\ldots,i_k)}(x_{i_i}, \ldots, x_{i_k})^2, \tag{13.11}$$

and

$$V_n = \sum_{\mathbf{x}} p(\mathbf{x}) r_{n-1}(\mathbf{x})^2. \tag{13.12}$$

If the population is in linkage equilibrium, the reader can easily verify that V_1 is the *additive genetic variance* defined by equation (12.7). $p(x_{i_1}, \ldots, x_{i_k})$ is a marginal probability distribution defined by $p(\mathbf{x})$. We are now able to formulate the theorem.

Theorem 13.3. *Let the population be in linkage equilibrium, i.e.*

$$p(\mathbf{x}) = \prod_{i=1}^{n} p_i(x_i)\,. \tag{13.13}$$

Then the variance of the population is given by

$$V = V_1 + V_2 + \cdots + V_{n-1} + V_n\,. \tag{13.14}$$

The covariance of mid-parent and offspring can be computed from

$$\mathrm{cov}(\bar{P}, O) = \frac{1}{2}V_1 + \frac{1}{4}V_2 + \cdots + \frac{1}{2^n}V_n = \sum_{k=1}^{n} \frac{1}{2^k} V_k\,. \tag{13.15}$$

The proof can be found in Asoh and Mühlenbein (1994a). We now compare the estimates for heritability. For proportionate selection, we have from theorem 12.1:

$$R_{\mathrm{UMDA}}(t) = \frac{\mathrm{VA}(t)}{V(t)} S(t) + \mathrm{error}_1(t)\,.$$

For two-parent recombination (TPR), Mühlenbein (1997) has shown for $n = 2$ loci:

$$R_{\mathrm{TPR}}(t) = 2\frac{\mathrm{cov}(\bar{P}(t), O(t))}{V(t)} S(t) + \frac{1}{2}\mathrm{error}_2(t)\,.$$

If the population is in linkage equilibrium we have $\mathrm{error}_1 = \mathrm{error}_2$. Using the covariance decomposition we can write

$$R_{\mathrm{TPR}}(t) = \frac{\mathrm{VA}(t)}{V(t)} S(t) + \frac{1}{2}\frac{V_2(t)}{V(t)} S(t) + \frac{1}{2}\mathrm{error}(t)\,.$$

Thus the first term of the expansion is identical to the UMDA term. This shows again the similarity between two-parent recombination and the UMDA method.

Breeders typically use the expression $b(t) = \mathrm{VA}(t)/V(t)$ as an estimate. This is called *heritability in the narrow sense* (Falconer, 1981). But note that the variance decomposition seems to be true only for Robbins's proportions.

The selection differential is not suitable for mathematical analysis. For truncation selection it can be approximated by

$$S(t) \approx I_\tau V^{\frac{1}{2}}(t)\,, \tag{13.16}$$

where I_τ is called the *selection intensity*. Combining the two equations we obtain the famous *equation for the response to selection*:

$$R(t) = b(t) I_\tau V^{\frac{1}{2}}(t)\,. \tag{13.17}$$

These equations are discussed in depth by Mühlenbein (1997). The theory of breeding uses macroscopic variables, the average and the variance of the population. But we have derived only one equation, the *response-to-selection equation*. We need a second equation connecting the average fitness and the variance in order to be able to compute the evolution over time of the average fitness and the variance. There have been many attempts in population genetics to find a second equation. But all equations assume that the variance of the population decreases continuously. This is not the case for arbitrary fitness functions. Prügel-Bennet and Shapiro (1997) have proposed to use moments for describing genetic algorithms. They apply methods of statistical physics to derive equations for higher moments for special fitness functions.

13.2 Tournament Selection

Besides proportionate and truncation selection, *tournament selection of size k* is another popular selection method. Here k individuals are chosen randomly. The best individual is taken as parent. Here we model binary tournament selection ($k = 2$) as a game. Two individuals with genotype $\mathbf{x}$ and $\mathbf{y}$ "play" against each other. The one with the larger fitness gets a payoff of 2. If the fitness values are equal, both will win half of the games. This gives a payoff of 1. The game is defined by a *payoff matrix* with coefficients as follows:

$$a_{xy} = \begin{cases} 2 & f(\mathbf{x}) > f(\mathbf{y}) \\ 1 & f(\mathbf{x}) = f(\mathbf{y}) \\ 0 & f(\mathbf{x}) < f(\mathbf{y})\,. \end{cases}$$

With some effort one can show that

$$\sum_{\mathbf{x}} \sum_{\mathbf{y}} p(\mathbf{x}, t) a_{xy} p(\mathbf{y}, t) = 1\,. \tag{13.18}$$

After a round of tournaments the genotype frequencies are given by

$$p^s(\mathbf{x}, t+1) = p(\mathbf{x}, t) \sum_{\mathbf{y}} a_{xy} p(\mathbf{y}, t)\,. \tag{13.19}$$

If we set

$$b(\mathbf{x},t) = \sum_{\mathbf{y}} a_{xy} p(\mathbf{y},t)\,,$$

then the above equation is similar to proportionate selection using the function $b(\mathbf{x},t)$. But b depends on the genotype frequencies. Furthermore the average $\bar{b}(t) = \sum p(\mathbf{x},t) b(\mathbf{x},t)$ remains constant, $\bar{b}(t) \equiv 1$.

The difference equations for the univariate marginal frequencies can be derived in the same manner as for proportionate selection. They are given by

$$p_i(x_i, t+1) = p_i(x_i,t) \cdot \bar{B}_i(t) \tag{13.20}$$

$$\bar{B}_i(t) = \sum_{\mathbf{x}, X_i = x_i} b(\mathbf{x},t) \prod_{\substack{j=1 \\ j\neq i}}^{n} p_j(x_j,t)\,. \tag{13.21}$$

The difference equation for binary tournament selection is more difficult than the equation for proportionate selection. $\bar{B}_i$ is quadratic in $p(x_j)$. The fitness value of $\mathbf{x}$ is given by $\sum_y a_{xy} p(\mathbf{y},t)$. This is called called *frequency dependent fitness* in population genetics.

Tournament selection uses only the order relation of the fitness values. The fitness values themselves do not change the outcome of a tournament. Therefore the evolution of the univariate marginal frequencies depends only on the order relation. The same is true for truncation selection. It can be seen that tournament selection can be approximated by truncation selection. For each k there exists a selection intensity I_k with

$$S(t) = I_k V^{\frac{1}{2}}(t)\,.$$

For $k = 2$ we have $I_2 = 1/\sqrt{\pi} \approx 0.564$ (Mühlenbein, 1997).

13.3 Analytical Results for Linear Functions

For the special case where all univariate marginal distributions are equal, i.e. $p_i := p$, it is possible to obtain an analytical solution for $p(t)$.

We cite from Mühlenbein (1997) the analytical solutions for the linear function $\text{OneMax}(n) = \sum_i x_i$. For completeness we give the difference equation and its solution.

Theorem 13.4. *If in the initial population all univariate marginal frequencies are identical to $p_0 > 0$, then we obtain for UMDA and OneMax: proportionate selection:*

$$R(t) = 1 - p(t) \tag{13.22}$$

$$p(t) = 1 - (1-p_0)\left(1 - \tfrac{1}{n}\right)^t \tag{13.23}$$

truncation selection:

$$R(t) \approx I_\tau \sqrt{np(t)(1-p(t))} \tag{13.24}$$

$$p(t) \approx 0.5 \left(1 + \sin\Big(\frac{I_\tau}{\sqrt{n}} t + \arcsin(2p_0 - 1)\Big)\right) \tag{13.25}$$

tournament selection:

$$\begin{aligned} R(t) = np(1-p)\Bigg(& 2\sum_{k=1}^{n}\sum_{j=0}^{k-1}\binom{n-1}{k-1}\binom{n}{j}p^{k+j-1}(1-p)^{2n-k-j-1} \\ & + \sum_{k=1}^{n-1}\binom{n-1}{k-1}\binom{n}{k}p^{2k-1}(1-p)^{2n-2k-1} - \sum_{j=0}^{2n-2}p^j\Bigg) \end{aligned} \tag{13.26}$$

$$R(t) \approx 0.564\sqrt{np(t)(1-p(t))}\,. \tag{13.27}$$

The formulas can be used to compute the number of generations until convergence (GEN_e). For truncation selection convergence is defined by $p(t) = 1$, for proportionate selection by $p(t) = 1 - \epsilon$.

Corollary 13.1. *The number of generations until convergence is given by:*
proportionate selection:

$$\mathrm{GEN}_e = n \cdot \ln \frac{1 - p_0}{\epsilon} \tag{13.28}$$

truncation selection:

$$\mathrm{GEN}_e = \Big(\frac{\pi}{2} - \arcsin(2p_0 - 1)\Big)\frac{\sqrt{n}}{I_\tau}\,. \tag{13.29}$$

Truncation selection converges in $O(\sqrt{n})$ and proportionate selection in $O(n \cdot \ln(1/\epsilon))$ generations. Numerical results have shown that truncation selection converges in about $O\sqrt{n}$ to $O(n)$ generations (Mühlenbein and Mahnig, 2000) for all fitness functions optimized.

The analytical solutions almost perfectly match the results obtained from actual UMDA runs (see figure 13.1). With proportionate selection the population requires a long time to approach the optimum. In contrast, truncation selection and tournament selection lead to much faster convergence. p increases almost linearly until near the optimum. Equation (13.26) for binary tournament selection has p^{2n} as the largest exponent. This complicated equation can be approximated by equation (13.27) with surprising accuracy.

We next present numerical results for some popular fitness functions.

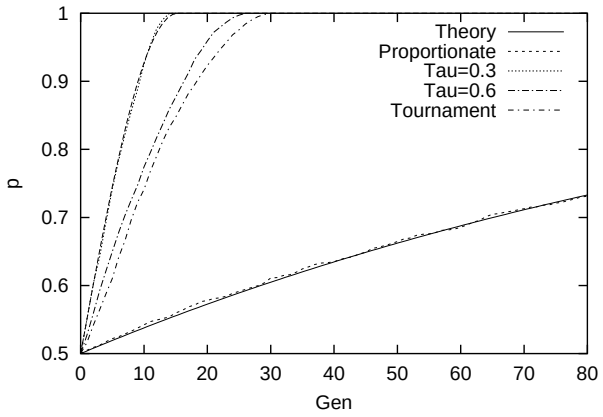

FIGURE 13.1. Comparison of selection methods for OneMax(128).

13.4 Numerical Results for UMDA

This section solves the problem put forward by Mitchell et al. (1994): to define the class of problems for which genetic algorithms are best suited, and in particular, for which they will outperform other search algorithms. We start with the *royal road* function, which was erroneously believed to lay out a "royal road" for the GA to follow to the optimal string.

13.5 Royal Road Function

The royal road function R_1 was used by Mitchell et al. (1994). It is defined as follows:

$$R_1(l, \mathbf{x}) = \sum_{i=0}^{l-1} \prod_{j=1}^{8} x_{8i+j} \,. \tag{13.30}$$

The function is of order 8. The building block hypothesis (BBH; Holland, 1975/1992) states that "the GA works well when instances of low-order, short schemas that confer high fitness can be recombined to form instances of larger schemas that confer even higher fitness." In our terminology a schema defines a marginal distribution. Thus a first-order schema defines a univariate marginal distribution. Our analysis has shown that only the first half of the BBH is correct: first-order schemata of high fitness are recombined. Larger schemata play no role.

Table 13.1 confirms and extends the results of Mitchell et al. (1994). The extremely poor performance of SGA is mainly a result of proportionate selection. UMDA with proportionate selection (U: p) requires

Table 13.1
Mean Function Evaluations for Royal Road(8)
U is UMDA, F is FDA.

$1+1$	SGA	U: p	U: $\tau = 0.3$	U: $\tau = 0.05$	F: $\tau = 0.3$
6,334	61,334	55,586	28,000	14,264	7,634

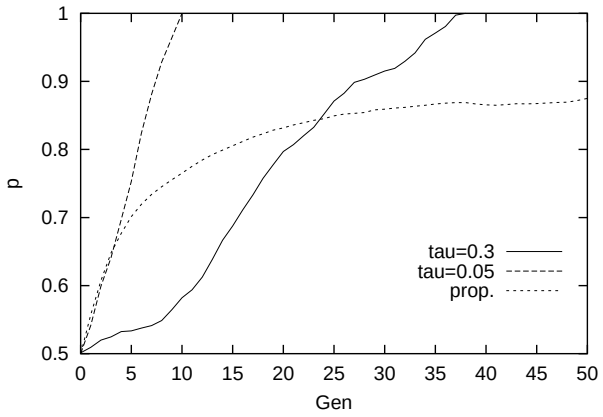

Figure 13.2. Convergence of royal road.

slightly fewer evaluations. With very strong selection, UMDA needs only about twice as many function evaluations as the $(1+1)$-algorithm. This algorithm performs a random bit flip and accepts a new configuration if its fitness is equal or better. This algorithm performs fairly well, as has already been shown by Mühlenbein (1991). But it performs well only if the fitness function never decreases with an increasing number of bits. An almost identical performance to that of the $(1+1)$-algorithm can be obtained by FDA. It uses marginal distributions of size 8 instead of univariate marginal distributions. FDA will be explained in section 14.2.

Figure 13.2 shows once again the importance of selection. Proportionate selection performs very well in the beginning, because the fitness values of all strings containing no building blocks are zero. These strings do not reproduce. But after 5 generations proportionate selection gets weaker, and truncation selection with $\tau = 0.3$ overtakes it after 23 generations. (The numerical results would be much worse for proportionate selection if we added 1 to the royal road function. In that case proportionate selection also selects many strings with no a building blocks.)

We will now explain the results by using our theory to analytically

solve the equations. We have

$$\begin{aligned}\tilde{W}(\mathbf{p}) &= \sum_{i=0}^{l-1}\prod_{j=1}^{8} p_{8i+j} \\ \frac{\partial \tilde{W}}{\partial p_k} &= \prod_{\substack{j=1 \\ 8i+j\neq k}}^{7} p_{8i+j} \qquad 8i \leq k < 8i+8\,.\end{aligned}$$

For truncation selection we apply the response-to-selection equation. Therefore we have to compute the variance $V_l(t)$. We simplify the computation by observing that the blocks of 8 variables are independent, and therefore:

$$V_l(t) = l \cdot V(t)\,.$$

Recall that all function values are 0 except $f(1,\ldots,1)$. Therefore:

$$\begin{aligned}V(t) &= \sum_x p(\mathbf{x},t) f(\mathbf{x})^2 - W^2 \\ &= \prod p_i - (\prod p_i)^2\,.\end{aligned}$$

If we assume that $p_i = p$ for all i we obtain:

$$V_8(t) = 8p(t)^8(1-p(t)^8)\,. \tag{13.31}$$

We can now formulate the theorem. The equations are exact for proportionate selection. For truncation selection the concept of heritability plays a role.

Theorem 13.5. *If all univariate marginal distributions are identical to $p(t)$ and $p(0) = p_0$, then we obtain for proportionate selection:*

$$p(t+1) - p(t) = \frac{1-p(t)}{8} \tag{13.32}$$

$$p(t) = 1 - (1-p_0)(\tfrac{7}{8})^t. \tag{13.33}$$

For truncation selection with threshold τ we get approximately

$$R(t) \approx b(t) I_\tau \sqrt{8p(t)^8(1-p(t)^8)} \tag{13.34}$$

$$p(t)^8 \approx 0.5\left(1 + \sin\Big(\frac{b(t)I_\tau}{\sqrt{8}}t + \arcsin(2p_0^8 - 1)\Big)\right). \tag{13.35}$$

Proof. The conjectures for proportionate selection follow directly from equation (12.5). From the response-to-selection equation we obtain

$$8p(t+1)^8 - 8p(t)^8 \approx b(t) I_\tau \sqrt{8p(t)^8(1-p(t)^8)}\,.$$

If we set $q(t) = p(t)^8$ the above equation is identical to the equation for OneMax(8). The approximate solution is given by equation (13.25). □

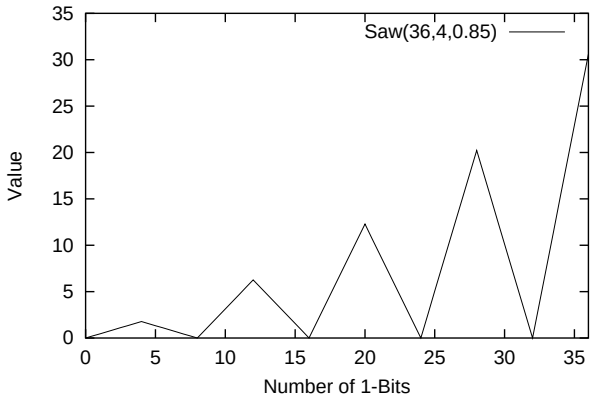

Figure 13.3. Definition of Saw$(36, 4, 0.85)$.

To apply equation (13.35) we need an estimate for the realized heritability $b(t)$. Experiments show that $b(t)$ increases approximately linearly from about 0 to 1. Thus we set $b(t) \propto t$. A numerical comparison between equation (13.35) and a simulation with a truncation threshold of 0.05 shows only 5% difference. Thus the coincidence between theory and simulation is very high.

This example shows that the response-to-selection equation can in special cases be used to compute an analytical solution for $p(t)$. The difficulty is to determine the heritability $b(t)$.

13.6 Multimodal Functions Suited for UMDA Optimization

Equation (12.5) shows that UMDA performs a gradient ascent in the landscape given by W. This helps our search for functions best suited for UMDA. We take the Saw landscape as a spectacular example. The definition of the function can be extrapolated from figure 13.3. In Saw(n, m, k), n denotes the number of bits and $2m$ the distance from one peak to the next. The highest peak is multiplied by k (with $k \leq 1$), the second highest by k^2, third by k^3, and so on. The landscape is very rugged; to get from one local optimum to another, one has to cross a deep valley.

But again the transformed landscape $W(\mathbf{p})$ is fairly smooth. An example is shown in figure 13.4. Whereas $f(\mathbf{x})$ has 5 isolated peaks, $W(\mathbf{p})$ has three plateaus, a local peak, and the global peak. We will use UMDA with truncation selection. We have not been able to derive precise analytical expressions. Figure 13.4 displays the results.

In the simulation two truncation thresholds, $\tau = 0.05$ and $\tau = 0.01$,

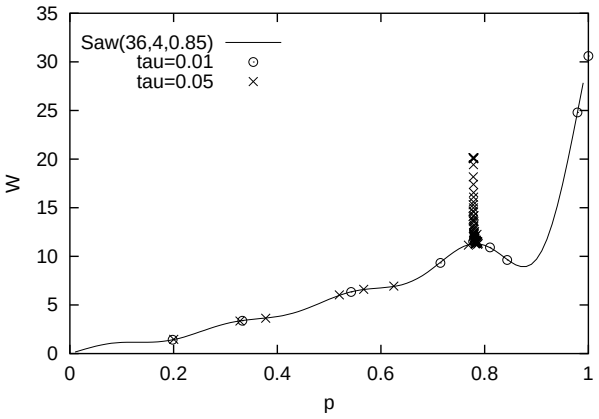

FIGURE 13.4. Results with normal and strong selection.

have been used. For $\tau = 0.05$ the probability p stops at the local maximum for $\tilde{W}(\mathbf{p})$. It is approximately $p = 0.78$. For $\tau = 0.01$ UMDA is able to converge to the optimum $p = 1$—it does so by going even downhill.

This example confirms our theory: *UMDA transforms the original fitness landscape defined by $f(\mathbf{x})$ into a fitness landscape defined by $\tilde{W}(\mathbf{p})$. This transformation smoothes the rugged fitness landscape $f(\mathbf{x})$ so that if there is a tendency toward the global optimum, UMDA may find it.*

Thus UMDA can solve difficult multimodal optimization problems. It is obvious that any search method using a single search point, like the $(1 + 1)$-algorithm, needs instead an almost exponential number of function evaluations to obtain the optimum of Saw.

13.7 Deceptive Functions

There are many optimization problems where UMDA is misleading. We demonstrate this problem with a deceptive function. We use the definition

$$\text{Decep}(\mathbf{x}, k) := \begin{cases} k - 1 - |\mathbf{x}|_1 & 0 \leq |\mathbf{x}|_1 < k \\ k & |\mathbf{x}|_1 = k\,. \end{cases} \tag{13.36}$$

The global maximum is isolated at $x = (1, \ldots, 1)$. A deceptive function of order n is a "needle in a haystack" problem. Such functions are far too difficult for any optimization method to optimize. We simplify the optimization problem by adding l distinct Decep(k) functions to give a fitness function of size $n = lk$. This function is also deceptive. The local optimum $x = (0, \ldots, 0)$ is surrounded by high fitness values, whereas

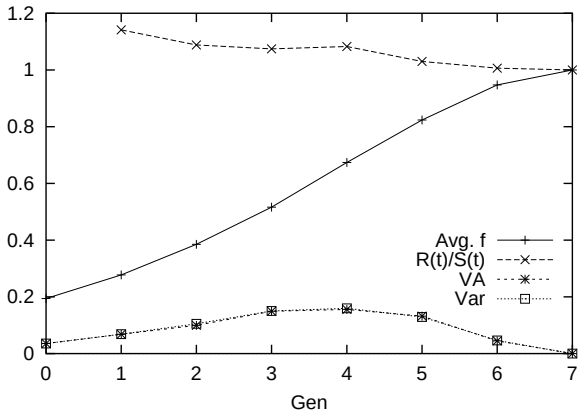

Figure 13.5. Heritability and variance for a multiplicative function (variance and VA multiplied by 10); $s = 0.1$, $n = 32$.

the global optimum is isolated.

$$\text{Decep}(n,k) = \sum_{i=1,k+1,\ldots}^{n} \text{Decep}\big((x_i, x_{i+1}, \ldots, x_{i+k-1}), k\big)\,. \tag{13.37}$$

Our theory easily shows that at $p_i = 0.5$ the gradient points to $x_i = 0$. Thus starting at $p(0) = 0.5$ UMDA converges to the local optimum $\mathbf{x} = (0, \ldots, 0)$. This problem can be solved if higher-order marginal distributions are used. This will be discussed later in the context of the factorized distribution algorithm (FDA).

We next show how the science of breeding can be used to control UMDA.

13.8 Numerical Investigations of the Science of Breeding

The application of the science of breeding requires the computation of the average fitness $\bar{f}(t)$, the variance $V(t)$, and the additive genetic variance $\text{VA}(t)$. The first two terms are standard statistical terms. The computation of VA requires $\bar{f}_i(x_i)$ and $p_i(x_i)$. The computation of the first term poses only a few difficulties. It can be approximated by

$$\bar{f}_i(X_i = 1, t) = \sum_{x, X_i = 1} \frac{p(\mathbf{x})}{p_i(X_i = 1)} f(\mathbf{x}) \approx \frac{1}{N} \sum_{k=1}^{N} \frac{f(\zeta_i^k)}{p_i(x_i)}\,, \tag{13.38}$$

where ζ_i^k are those $\mathbf{x}$ values in the population that contain $x_i = 1$.

Linear functions are the ideal case for the theory. The heritability $b(t)$ is 1 and the additive genetic variance is identical to the variance. We skip

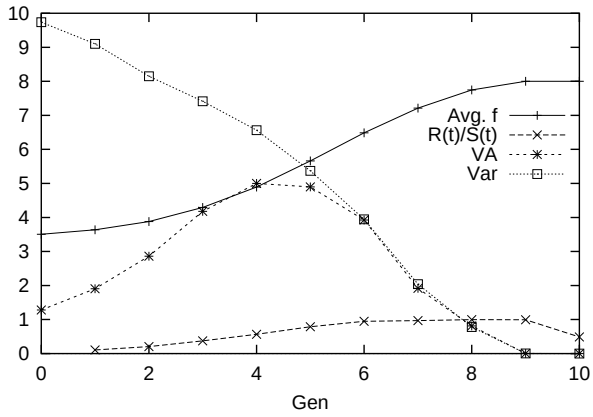

FIGURE 13.6. Heritability and variance for Decep$(32, 4)$: Average, Var, and VA divided by 3; $\tau = 0.3$, $n = 32$.

this trivial case and start with a multiplicative fitness function $f(\mathbf{x}) = \prod_i (1-s)^{1-x_i}$. For a multiplicative function we also have $R(t) = S(t)$ (theorem 12.2).

Figure 13.5 confirms the theoretical results from section 11 (VA and Var are multiplied by 10 in this figure). Additive genetic variance is almost identical to the variance and the heritability is 1. The function is highly nonlinear of order n, but nevertheless it is easy to optimize. The function has also been investigated by Rattray and Shapiro (in press), but their calculations become highly complex.

An interesting case is the function Decep$(32, 4)$. In figure 13.6 the function is optimized for 32 bits. As predicted by the theory, UMDA converges to the local optimum $\mathbf{x} = (0, \ldots, 0)$. Heritability is almost zero at the beginning, indicating that the competition between setting the genes to 0 or to 1 is undecided. UMDA decides to go to the direction of 0. If there is a high percentage of zeros in the population, then heritability increases to almost 1. In this area the fitness function is almost linear. This shows that heritability can depend strongly on the gene frequencies.

These examples demonstrate that it is worthwhile to compute the quantities used for a scientific breeding program. They indicate clearly how difficult the optimization problem is. In the breeding of livestock, heritability is normally greater than 0.2. If we optimize arbitrary fitness functions the heritability can be almost 0. But because we can easily compute 1000 generations on a computer in a few minutes, UMDA can be used for problems with very low heritability.

We have shown that UMDA can optimize difficult multimodal functions, thus explaining the success of genetic algorithms in optimization. We have also shown that UMDA can be deceived easily by simple functions called deceptive functions. These functions require more complex search distributions. This problem is investigated next.

14 Graphical Models and Optimization

The simple product distribution of UMDA cannot capture dependencies between variables. If these dependencies are necessary to find the global optimum, UMDA and simple genetic algorithms fail. We take an extreme case as example, the *"needle in a haystack" problem.* The fitness function is everywhere equal to 1, except for a single $\mathbf{x}$ where it is 10. All x_i values have to be set in the right order to obtain the optimum. Of course, there is no clever search method for this problem. But there is a continuum of increasing complexity from the simple OneMax function to the "needle in a haystack" problem. For such complex problems we need a complex search distribution. A good candidate for a search distribution for optimization is the Boltzmann distribution.

Definition 14.1. *For $\beta \geq 0$ define the* weighted Boltzmann distribution *of a function $f(\mathbf{x})$ as*

$$p_{\beta,f}(\mathbf{x}) := \frac{p_0(\mathbf{x})e^{\beta f(\mathbf{x})}}{\sum_y p_0(\mathbf{y})e^{\beta f(\mathbf{y})}} := \frac{p_0(\mathbf{x})e^{\beta f(\mathbf{x})}}{Z_f(\beta, p_0)}\,, \tag{14.1}$$

where $Z_f(\beta, p_0)$ is the partition function. To simplify the notation β and/or f can be omitted. $p_0(\mathbf{x})$ is the distribution for $\beta = 0$.

The Boltzmann distribution concentrates its search around good fitness values. Thus it is theoretically a very good candidate for a search distribution used for optimization. The problem lies in the efficient computation of the Boltzmann distribution.

In this section we will present a method for computing the Boltzmann distribution. The method is based on the factorization of the distribution. If the factorization needs only a polynomial number of parameters, then the distribution can be computed in polynomial time. The corresponding factorization theorem is important for many areas of computer science dealing with the problem of decomposition.

With a suitable annealing schedule convergence of the algorithm can easily be shown. We will derive a new annealing schedule for Boltzmann distribution algorithms. This annealing schedule makes the result of Boltzmann selection similar to the results of truncation selection.

The theory presented in this section unifies simulated annealing and population-based algorithms with the general theory of estimating distributions.

14.1 Boltzmann Selection and Convergence

The Boltzmann distribution is usually defined as $e^{-\frac{g(\mathbf{x})}{T}}/Z$. The term $g(\mathbf{x})$ is called the *energy* and $T = 1/\beta$ the *temperature*. Some properties of the weighted Boltzmann distribution are described by the following lemma.

Lemma 14.1. *Let $x_m \in \mathcal{M}$ be a global optimum of the function $f(\mathbf{x})$ and $\mathbf{x}_l$ a point with $f(\mathbf{x}_l) < f(\mathbf{x}_m)$. Then*

- *Let $g(\mathbf{x}) := f(\mathbf{x}) + c$. Then $p_{\beta,f}(\mathbf{x}) = p_{\beta,g}(\mathbf{x})$.*
- *Let $g(\mathbf{x}) := c \cdot f(\mathbf{x})$. Then $p_{\beta,g}(\mathbf{x}) = p_{c\beta,f}(\mathbf{x})$.*

The first property means that the distribution is invariant under addition of a constant. It is, however, not invariant under multiplication. We will discuss how to overcome this shortcoming in section 14.3.

Closely related to the Boltzmann distribution is Boltzmann selection. An early study about this selection method can be found in de la Maza and Tidor (1993).

Definition 14.2. *Given a distribution p and a selection parameter γ,* Boltzmann selection *calculates the distribution of the selected points according to*

$$p^s(\mathbf{x}) = p(\mathbf{x}) \frac{e^{\gamma f(\mathbf{x})}}{\sum_y p(\mathbf{y}) e^{\gamma f(\mathbf{y})}} \,. \tag{14.2}$$

Boltzmann selection can be seen as proportionate selection applied to the fitness function $e^{\beta f(\mathbf{x})}$. We now define the BEDA (Boltzmann estimated distribution algorithm).

BEDA – Boltzmann estimated distribution algorithm

- STEP 0: $t \Leftarrow 0$. Generate N points according to the $p(\mathbf{x}, 0) = p_0(\mathbf{x})$.
- STEP 1: With a given $\Delta\beta(t) > 0$, let
$$p^s(\mathbf{x}, t) = \frac{p(\mathbf{x}, t) e^{\Delta\beta(t) f(\mathbf{x})}}{\sum_y p(\mathbf{y}, t) e^{\Delta\beta(t) f(\mathbf{y})}} \,.$$
- STEP 2: Generate N new points according to the distribution $p(\mathbf{x}, t+1) = p^s(\mathbf{x}, t)$.
- STEP 3: $t \Leftarrow t + 1$.
- STEP 4: If stopping criterion not met go to STEP 1.

BEDA is a conceptional algorithm, because the calculation of the distribution requires the computation of the sum of exponentially many terms. The following convergence theorem is easily proven.

Theorem 14.1 (Convergence). *Let $\Delta\beta(t)$ be an annealing schedule, i.e. for every t increase the inverse temperature β by $\Delta\beta(t)$. Then for BEDA the distribution at time t is given by*

$$p(\mathbf{x},t) = \frac{p_0(\mathbf{x})e^{\beta(t)f(\mathbf{x})}}{Z_f(\beta(t),p_0)} \tag{14.3}$$

with the inverse temperature

$$\beta(t) = \sum_{\tau=1}^{t} \Delta\beta(\tau)\,. \tag{14.4}$$

Let $\mathcal{M}$ be the set of global optima. If $\beta(t)\to\infty$, then

$$\lim_{t\to\infty} p(\mathbf{x},t) = \begin{cases} 1/|\mathcal{M}| & x\in\mathcal{M} \\ 0 & \text{otherwise.} \end{cases} \tag{14.5}$$

Proof. Let $\mathbf{x}^m\in\mathcal{M}$ be a point with optimal fitness and $\mathbf{x}\notin\mathcal{M}$ a point with $f(\mathbf{x}) < f(\mathbf{x}^m)$. Then

$$\begin{aligned} p(\mathbf{x},t) &= \frac{p_0(\mathbf{x})e^{\beta(t)f(\mathbf{x})}}{\sum_{\mathbf{y}} p_0(\mathbf{y})e^{\beta(t)f(\mathbf{y})}} \le \frac{e^{\beta(t)f(\mathbf{x})}}{|\mathcal{M}|\cdot C\cdot e^{\beta(t)f(\mathbf{x}^m)}} \\ &\le \frac{1}{|\mathcal{M}|\cdot C\cdot e^{\beta(t)[f(\mathbf{x}^m)-f(\mathbf{x})]}}\,. \end{aligned}$$

As $\beta(t)\to\infty$, $p(\mathbf{x},t)$ converges (exponentially quickly) to 0. Because $p(\mathbf{x},t) = p(\mathbf{y},t)$ for all $\mathbf{x}^m,\mathbf{y}^m\in\mathcal{M}$, the limit distribution is the uniform distribution on the set of optima. □

Equation (14.5) shows only that the distribution converges to 0 for nonoptimal points. But we can also make an estimate for the rate of convergence.

Lemma 14.2. *Let $p_0(\mathbf{x})$ be the uniform distribution. Let there be a δ such that for any nonoptimal point $\mathbf{x}$ we have with $\mathbf{x}^m\in\mathcal{M}$:*

$$f(\mathbf{x}) \le f(\mathbf{x}^m) - \delta\,. \tag{14.6}$$

Then

$$\beta \ge \frac{n\cdot\ln 2}{\delta} \quad\Longrightarrow\quad p_\beta(\mathcal{M}) \ge 0.5. \tag{14.7}$$

Proof. Let $|\mathcal{M}|$ be the number of optima. The number of terms in the partition function is smaller than 2^n. For $\mathbf{x}^m \in \mathcal{M}$ we have with $M := f(\mathbf{x}^m)$:

$$\begin{aligned} p_\beta(\mathbf{x}^m) &= \frac{e^{\beta M}}{\sum_{\mathbf{y}} e^{\beta f(\mathbf{y})}} \\ &\geq \frac{e^{\beta M}}{2^n \cdot e^{\beta(M-\delta)} + |\mathcal{M}| \cdot e^{\beta M}} = \frac{1}{e^{n \ln 2 - \beta\delta} + |\mathcal{M}|} \\ &\overset{!}{\geq} \frac{1}{2|\mathcal{M}|}\,. \end{aligned} \tag{14.8}$$

So, to have $p_\beta(\mathcal{M}) \geq 1/2$, we need

$$e^{n \ln 2 - \beta\delta} \leq 2|\mathcal{M}| \iff \beta \geq \frac{n \cdot 2 - \ln(2|\mathcal{M}|)}{\delta} \tag{14.9}$$

or as a sufficient condition (14.7). □

Corollary 14.1. *For a binary fitness function with integer values, half of the generated points will have maximum fitness if $\beta \geq 0.7n$, independent of the fitness function.*

We next transform BEDA into a practical algorithm. This means the reduction of the parameters of the distribution and the computation of an adaptive schedule.

14.2 Factorization of the Distribution and the FDA

In this section we describe a method for computing a factorization of the probability, given an additive decomposition of the function:

Definition 14.3. *Let $s_1, \ldots, s_m$ be index sets, $s_i \subseteq \{1, \ldots, n\}$. Let f_{s_i} be functions depending only on the variables x_j with $j \in s_i$. These variables we denote as x_{s_i}. Then*

$$f(\mathbf{x}) = \sum_{i=1}^{m} f_{s_i}(\mathbf{x}) = f_i(x_{s_i}) \tag{14.10}$$

is an additive decomposition *of the fitness function f.*

We also need the following definitions.

Definition 14.4. *Given $s_1, \ldots, s_m$, we define for $i = 1, \ldots, m$ the sets d_i, b_i and c_i:*

$$d_i := \bigcup_{j=1}^{i} s_j, \qquad b_i := s_i \setminus d_{i-1}, \qquad c_i := s_i \cap d_{i-1}\,. \tag{14.11}$$

We set $d_0 = \emptyset$.

In the theory of decomposable graphs, d_i are called *histories*, b_i *residuals*, and c_i *separators* (Lauritzen, 1996). Recall the following definition.

Definition 14.5. *The* conditional probability $p(\mathbf{x}|\mathbf{y})$ *is defined as*

$$p(\mathbf{x}|\mathbf{y}) = \frac{p(\mathbf{x}, \mathbf{y})}{p(\mathbf{y})}\,. \tag{14.12}$$

In Mühlenbein et al. (1999), we have shown the following theorem.

Theorem 14.2 (Factorization Theorem). *Let $p(\mathbf{x})$ be a Boltzmann distribution with*

$$p(\mathbf{x}) = \frac{e^{\beta f(\mathbf{x})}}{Z_f(\beta)} \tag{14.13}$$

and $f(\mathbf{x}) = \sum_{i=1}^{m} f_{s_i}(\mathbf{x})$ be an additive decomposition. If

$$b_i \neq \emptyset \quad \forall i = 1, \ldots, l; \quad d_l = \tilde{X} \tag{14.14}$$

$$\forall i \geq 2\, \exists j < i \text{ such that } c_i \subseteq s_j \tag{14.15}$$

then

$$p(\mathbf{x}) = \prod_{i=1}^{m} p(x_{b_i}|x_{c_i})\,. \tag{14.16}$$

The constraint defined by equation (14.15) is called the *running intersection property* (Lauritzen, 1996).

FDA – factorized distribution algorithm

- STEP 0: Calculate b_i and c_i from the decomposition of the function.
- STEP 1: Generate an initial population with N individuals.
- STEP 2: Select N individuals using Boltzmann selection.
- STEP 3: Estimate the conditional probabilities $p(x_{b_i}|x_{c_i}, t)$ from the selected points.
- STEP 4: Generate new points according to $p(\mathbf{x}, t+1) = \prod_{i=1}^{m} p(x_{b_i}|x_{c_i}, t)$.
- STEP 5: If not stopping criterion reached: $t \Leftarrow t+1$ Go To STEP 2.

With the help of the factorization theorem, we have turned the conceptional algorithm BEDA into FDA, the factorized distribution algorithm. As the factorized distribution is identical to the Boltzmann distribution, the convergence proof of BEDA also applies to FDA. There exist a number of algorithms that compute a factorization of an additive decomposed function (Lauritzen, 1996).

Not every additive decomposed function fulfills the assumption of the factorization theorem. In these cases, more sophisticated methods have to be used. But FDA can also be used with an approximate factorization. We discuss just two simple examples.

Example 14.1. *For linear functions*

$$\text{Linear}(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i x_i \,, \tag{14.17}$$

we have $s_i = \{i\}$ *and thus all* c_i *are empty. This leads to the factorization*

$$p(\mathbf{x}) = \prod_{i=1}^{n} p_i(x_i) \,. \tag{14.18}$$

As this is the distribution used by UMDA, FDA behaves like UMDA (and thus like a simple genetic algorithm) for linear functions.

Example 14.2. *Functions with a chainlike interaction can also be factorized:*

$$\text{Chain}(\mathbf{x}) = \sum_{i=2}^{n} f_i(x_{i-1}, x_i) \,. \tag{14.19}$$

Here the factorization is

$$p(\mathbf{x}) = p(x_1) \prod_{i=2}^{n} p(x_i | x_{i-1}) \,. \tag{14.20}$$

FDA can be used with any selection scheme, but convergence has been shown only for Boltzmann selection. Therefor we think that Boltzmann selection is an essential part in using the FDA. In order to obtain a practical numerical algorithm, we still have to solve two problems: to find a good annealing schedule for Boltzmann selection and to determine a reasonable sample size (population size).

We investigate these two problems next.

14.3 A New Annealing Schedule for the Boltzmann Distribution

Boltzmann selection requires an annealing schedule. Lemma 14.2 has shown how quickly we have to anneal in order to reach convergence

within a given time frame. But if we anneal too quickly, the approximation of the Boltzmann can be very poor because of the sampling error.

14.3.1 Taylor Expansion of the Average Fitness

To determine an adaptive annealing schedule, we make a Taylor expansion of the average fitness of the Boltzmann distribution.

Definition 14.6. *The* ***average fitness*** *of a fitness function and a distribution is*

$$W_f(p) = \sum_x f(\mathbf{x})p(\mathbf{x})\,. \tag{14.21}$$

For the Boltzmann distribution, we use the abbreviation $W_f(\beta) := W_f(p_{\beta,f})$.

Theorem 14.3. *The average fitness of the Boltzmann distribution* $W_f(\beta)$ *has the following expansion in* β*:*

$$W_f(\tilde{\beta}) = W_f(\beta) + \sum_{i\geq 1} \frac{(\tilde{\beta}-\beta)^i}{i\,!} M^c_{i+1}(\beta)\,, \tag{14.22}$$

where M^c_i *are the centered moments:*

$$M^c_i(\beta) := \sum_x \big[f(\mathbf{x}) - W_f(\beta)\big]^i p(\mathbf{x}) \tag{14.23}$$

They can be calculated using the derivatives of the partition function:

$$M^c_{i+1}(\beta) = \left(\frac{Z'_f(\beta)}{Z_f(\beta)}\right)^{(i)} \qquad \text{for } i \geq 1, \quad M^c_1 = 0\,. \tag{14.24}$$

Proof. The kth derivative of the partition function obeys, for $k \geq 0$:

$$Z^{(k)}_f(\beta) = \sum_x f(\mathbf{x})^k e^{\beta f(\mathbf{x})}\,. \tag{14.25}$$

Thus the moments for $k \geq 1$ can be calculated as

$$M_k(\beta) := \sum_x f(\mathbf{x})^k p(\mathbf{x}) = \frac{Z^{(k)}_f(\beta)}{Z_f(\beta)} \tag{14.26}$$

and thus

$$W_f(\beta) = M_1(\beta) = Z'_f(\beta)/Z_f(\beta)\,. \tag{14.27}$$

Direct evaluation of the derivatives of W leads to complicated expressions. The proof, by induction, is rather technical. We omit it here. □

Corollary 14.2. *We have approximative:*

$$W_f(\tilde{\beta}) \approx W_f(\beta) + (\tilde{\beta}-\beta)\cdot\sigma^2_f(\beta)\,, \tag{14.28}$$

where $\sigma_f^2(\beta)$ *is the variance of the distribution, defined as* $\sigma_f^2(\beta) := M_2^c(\beta)$.

This approximation can also be found in Kirkpatrick et al. (1983).

Lemma 14.3. *The variance of the Boltzmann distribution obeys*

$$f(\mathbf{x}) \neq \text{const.} \implies \sigma_f^2(\beta) > 0\,. \tag{14.29}$$

Proof. We have $\forall x : p_\beta(\mathbf{x}) > 0$. In order to have

$$\sigma_f^2(\beta) = \sum_x \big[f(\mathbf{x}) - W_f(\beta)\big]^2 p_\beta(\mathbf{x}) \stackrel{!}{=} 0\,, \tag{14.30}$$

we must have for all x: $f(\mathbf{x}) = W_f$ in contradiction to the assumption. □

Corollary 14.3. *With* $f(\mathbf{x}) \neq \text{const.}$*, we have*

$$\tilde{\beta} > \beta \implies W_f(\tilde{\beta}) > W_f(\beta)\,. \tag{14.31}$$

The corollary shows that the average fitness never decreases for Boltzmann selection. A similar result was already obtained in theorem 12.1 for proportionate selection; see also Mühlenbein and Mahnig (2000).

14.3.2 The SDS Annealing Schedule

From (14.28) we can derive an adaptive annealing schedule. The variance (and the higher moments) can be estimated from the generated points. As long as the approximation is valid, one can choose a desired increase in the average fitness and set $\beta(t+1)$ accordingly. So we can set

$$\Delta\beta(t) := \beta(t+1) - \beta(t) = \frac{W_f^{\text{new}}(t) - W_f(\beta(t))}{\sigma_f^2(\beta(t))}\,. \tag{14.32}$$

From (14.28) we see that choosing $\Delta\beta$ proportional to the inverse of the variance leads in the approximation to a constant increase in the average fitness. This is much too fast, especially near the optimum. As truncation selection has proven to be a robust and efficient selection scheme, we can try to approximate the behavior of this method. For truncation selection the *response to selection* $R_f(t)$ is given approximatively by equation: (13.17)

$$R(t) := W(t+1) - W(t) \approx I_\tau b(t) \sqrt{\sigma_f^2}\,, \tag{14.33}$$

where I_τ is the selection intensity, depending on the truncation threshold τ. Because truncation selection has been shown to be an effective selection method, we will make the Boltzmann schedule proportional to the inverse of the square root of the variance:

Definition 14.7. *The* standard deviation schedule *(SDS) is defined by*

$$\Delta\beta(t) = \frac{c}{\sigma_f(\beta(t))}\,. \tag{14.34}$$

We already know that FDA with Boltzmann selection remains unchanged if we add a constant to the fitness function. For SDS we have additionally the following lemma.

Lemma 14.4. *For Boltzmann selection with SDS, BEDA is invariant under linear transformations of the fitness function with a positive factor.*

Proof. This lemma is true because the standard deviation scales linearly under multiplication. Let $f(\mathbf{x})$ be a fitness function; consider $\hat{f}(\mathbf{x}) = \hat{c} \cdot f(\mathbf{x})$. The claim is that if $\hat{\beta}(t) = \beta(t)/\tilde{c}$, then the distributions are the same for every t. With $t{=}0$, β and $\hat{\beta}$ are 0, so it is true. Now let t and $\beta{=}\beta(t)$ be given. From the previous iteration we know that $\hat{\beta} = \beta/\hat{c}$.

According to lemma 14.1, we have $p_{\beta,f}(\mathbf{x}){=}p_{\hat{\beta},\hat{f}}(\mathbf{x})$. Also, $\sigma^2_f(\beta) = \hat{c}^2 \cdot \sigma^2_{\hat{f}}(\hat{\beta})$. Hence we have $\Delta\hat{\beta}(t){=}\Delta\beta(t)/\hat{c}$. □

Corollary 14.4. *Let $\sigma(t)$ be the standard deviation. Then the response to selection for Boltzmann selection with the SDS is given by*

$$\begin{aligned} R_f(t) &= \sum_{i\geq 1} \frac{c^i}{i!\,\sigma(t)^i} M^c_{i+1} \\ &= c\cdot\sigma(t) + \frac{c^2 M^c_3}{2\,\sigma(t)^2} + \frac{c^3 M^c_4}{6\,\sigma(t)^3} + \dots \,. \end{aligned} \tag{14.35}$$

Note that this annealing schedule cannot be used for simulated annealing, as the estimation of the variance of the distribution requires samples that are drawn independently and the sequence of samples generated by simulated annealing are not independent.

14.3.3 Linear Functions

For linear functions,

$$\text{Linear}(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i x_i \,, \tag{14.36}$$

the factorization of the Boltzmann distribution was calculated in equation (14.18). We can also calculate the partition function and get

$$Z_f(\beta) = \prod_{i=1}^{n} (1 + e^{\beta\alpha_i}) \tag{14.37}$$

and

$$p_i(\beta) := p_\beta(X_i{=}1) = \frac{e^{\beta\alpha_i}}{1 + e^{\beta\alpha_i}} \,. \tag{14.38}$$

Because the variables are independent of each other, the variance is just the sum of the variance of the factors and we have

$$\sigma_f^2(\beta) = \sum_{i=1}^{n} \frac{\alpha_i^2 e^{\beta\alpha_i}}{(1+e^{\beta\alpha_i})^2} = \sum_{i=1}^{n} \alpha_i^2 p_i(\beta)\big(1-p_i(\beta)\big) \tag{14.39}$$

and thus

$$\beta(t+1) = \beta(t) + \frac{c}{\sqrt{\sum_i \alpha_i^2 p_i(\beta)\big(1-p_i(\beta)\big)}}\,. \tag{14.40}$$

By differentiating (14.38) we get

$$\begin{aligned}\frac{dp_i(\beta)}{dt} &= \frac{\alpha_i e^{\beta\alpha_i}(1+e^{\beta\alpha_i})\beta' - e^{\beta\alpha_i}\alpha_i e^{\beta\alpha_i}\beta'}{(1+e^{\beta\alpha_i})^2} \\ &= p_i(\beta)\big(1-p_i(\beta)\big)\alpha_i \tfrac{d\beta}{dt}\,. \end{aligned} \tag{14.41}$$

Therefore we obtain the differential equation

$$\frac{dp_i(\beta)}{dt} = c\cdot \frac{p_i(\beta)\big(1-p_i(\beta)\big)\alpha_i}{\sqrt{\sum_i \alpha_i^2 p_i(\beta)\big(1-p_i(\beta)\big)}}\,. \tag{14.42}$$

Note that the solution of these differential equations remains the same if we multiply all α_i by a constant factor, as predicted.

For OneMax we have $\alpha_i = 1$. In this case all marginal frequencies are equal to p_β. We obtain the differential equation

$$\frac{dp_\beta}{dt} = c\sqrt{p_\beta(1-p_\beta)/n}\,. \tag{14.43}$$

This equation is identical to the approximate equation derived for truncation selection. The solution of this equation is given by equation (13.25).

The theory presented so far has been derived under the assumption of large (infinite) populations. We turn next to the problem of finite populations.

14.4 Finite Populations

In finite populations convergence of UMDA or FDA can be only probabilistic. Since UMDA is a simplified FDA, it is sufficient to discuss FDA. This section is extracted from Mühlenbein and Mahnig (1999b).

Definition 14.8. *Let ϵ be given. Let $P_{\rm conv}(N)$ denote the probability that FDA with a population size of N converges to the optima. Then the critical population size is defined as*

$$N^*(\epsilon) = \min_N P_{\rm conv}(N) \geq 1-\epsilon\,. \tag{14.44}$$

Table 14.1
Cumulative Fixation Probability for Int(16)
Truncation selection vs. Boltzmann selection with $\Delta\beta = 0.01$
and Boltzmann SDS; N denotes size of population.

t	$\tau = 0.25$ $N = 30$	$\tau = 0.5$ $N = 30$	$\tau = 0.25$ $N = 80$	$\tau = 0.5$ $N = 60$	Boltz. $N = 700$	SDS $N = 100$
1	0.0955	0.0035	0.0	0.0	0.0885	0.0
2	0.4065	0.0255	0.0025	0.0095	0.1110	0.0
3	0.5955	0.1040	0.0165	0.0205	0.1275	0.0
4	0.6880	0.2220	0.0355	0.0325	0.1375	0.002
5	0.7210	0.3270	0.0575	0.0490	0.1455	0.002
6	0.7310	0.4030	0.0695	0.0630	0.1510	0.008
7	0.7310	0.4470	0.0740	0.0715	0.1555	0.018
8	0.7310	0.4705	0.0740	0.0780	0.1565	0.030
9	0.7310	0.4840	0.0740	0.0806	0.1575	0.036
14						0.084

If FDA with a finite population does not converge to an optimum, then at least one gene is fixed to a wrong value. The probability of fixation is reduced if the population size is increased. We have, obviously, for FDA:

$$P_{\text{conv}}(N_1) \leq P_{\text{conv}}(N_2) \quad N_1 \leq N_2 \,.$$

The critical question is: How many sample points are necessary to reasonably approximate the distribution used by FDA? A general estimate from Vapnik (1998) can be used as a guideline. One should use a sample size that is about twenty times larger than the number of free parameters.

We discuss the problem with a special function called Int. Int($\mathbf{x}$) gives the integer value of the binary representation.

$$\text{Int}(n) = \sum_{i=1}^{n} 2^{i-1} x_i \,. \tag{14.45}$$

The fitness distribution of this function is not normally distributed. The function has 2^n distinct fitness values. We show the cumulative fixation probability in table 14.1 for Int(16). The fixation probability is larger for stronger selection. For a given truncation selection the maximum fixation probability is at generation 1 for very small N. For larger values of N the fixation probability increases until a maximum is reached and then decreases again. This behavior has been observed for many fitness distributions.

For truncation selection with $\tau = 0.25$, we have for $N = 80$ a fixation probability of about 0.075. A larger τ reduces the fixation probability.

But this advantage is offset by the larger number of generations required for convergence. The problem of an optimal population size for truncation selection is investigated in Mühlenbein and Mahnig (1999b). Boltzmann selection with $\Delta\beta = 0.01$ is still very strong for the fitness distribution given by Int(16). For $N = 700$ the largest fixation probability is still at the first generation. Therefore the critical population size for Boltzmann selection for $\Delta\beta = 0.01$ is very high ($N^* > 700$). In comparison, the adaptive Boltzmann schedule SDS has a total fixation probability of 0.084 for a population size of $N = 100$. This is almost as small as truncation selection.

This example shows that Boltzmann selection in finite populations depends critically on a good annealing schedule. Normally we run FDA with truncation selection. This selection method is a good compromise, but Boltzmann selection with SDS schedule is of comparable performance.

Estimates for the necessary size of the population can also be found in Harik et al. (1999), but they use a weaker performance definition. The goal is to have a certain percentage of the bits of the optimum in the final population. Furthermore their result is valid only for fitness functions that are approximately normally distributed.

The danger of fixation can be reduced further by a technique very popular in Bayesian statistics. This is discussed in the next section.

14.5 Population Size, Mutation, and Bayesian Prior

To derive the results of this section we will use a normalized representation of the distribution. This representation is called a *Bayesian network* and can be displayed as a *directed* graph. The interested reader is referred to Jordan (1999).

Theorem 14.4 (Bayesian Factorization). *Each probability can be factored into*

$$p(\mathbf{x}) = p(x_1) \prod_{i=2}^{n} p(x_i|\mathrm{pa}_i)\,. \tag{14.46}$$

Proof. By definition of conditional probabilities we have

$$p(\mathbf{x}) = p(x_1) \prod_{i=2}^{n} p(x_i|x_1, \ldots, x_{i-1})\,. \tag{14.47}$$

Let $\mathrm{pa}_i \subset \{x_1, \ldots, x_{i-1}\}$. If x_i and $\{x_1, \ldots, x_{i-1}\} \setminus \mathrm{pa}_i$ are conditionally independent given pa_i, we can simplify $p(x_i|x_1, \ldots, x_{i-1}) = p(x_i|\mathrm{pa}_i)$. □

The PA_i are called the *parents* of variable X_i. This factorization can

be represented by a directed graph. In the context of graphical models the graph and the conditional probabilities are called a *Bayesian network* (Jordan, 1999; Frey, 1998). It is obvious that the factorization used in theorem 14.2 can be transformed easily into a Bayesian factorization.

Usually the empirical probabilities are computed by the maximum likelihood estimator. For N samples with $m \leq N$ instances of x the estimate is defined by

$$\hat{p}(\mathbf{x}) = \frac{m}{N}\,.$$

For $m = N$ we obtain $p(\mathbf{x}) = 1$ and for $m = 0$ we obtain $p(\mathbf{x}) = 0$. This leads to our gene fixation problem, because both values are attractors. The fixation problem is reduced if $\hat{p}(\mathbf{x})$ is restricted to an interval $0 < p_{\min} \leq \hat{p}(\mathbf{x}) \leq 1 - p_{\min} < 1$. This is exactly what results from the *Bayesian estimation*. The estimate $\hat{p}(\mathbf{x})$ is the expected value of the posterior distribution after applying Bayes formula to a prior distribution and the given data. For binary variables $\mathbf{x}$ the estimate

$$\hat{p}(\mathbf{x}) = \frac{m + r}{N + 2r} \tag{14.48}$$

is used with $r > 0$. r is derived from a Bayesian prior, and $r = 1$ is the result of the uniform Bayesian prior. The larger r is, the more the estimate tends toward $\hat{p}(\mathbf{x}) = 0.5$. The reader interested in a derivation of this estimate in the context of Bayesian networks is referred to Jordan (1999).

The Bayesian prior can be seen as a mutation force. Wright (1970) included mutation with a recurrent symmetric mutation rate of $0 \leq \mu < 1$ into his equation as follows:

$$\Delta p_i = p_i(t)(1 - p_i(t))\frac{\frac{\partial \tilde{W}}{\partial p_i}}{\tilde{W}} - \mu\big(p_i(t) - (1 - p_i(t)\big). \tag{14.49}$$

We will show that a similar equation is obtained if a Bayesian prior is used. We use the formula

$$p_i(t+1) = \frac{p_i^s(t)N + r}{N + 2r}$$

where $p_i^s(t)$ is given by Wright's equation (12.5). Setting $\gamma = r/N$ we obtain

$$\begin{aligned} \Delta p_i(t) &= p_i(t)(1 - p_i(t))\frac{\frac{\partial \tilde{W}}{\partial p_i}}{\tilde{W}} + \frac{\gamma}{1 + 2\gamma} \\ &\quad - \frac{2\gamma}{1 + 2\gamma}\left(p_i(t) + p_i(t)(1 - p_i(t))\frac{\partial \tilde{W}}{\tilde{W}}\right). \end{aligned} \tag{14.50}$$

Equations (14.49) and (14.50) are very similar if we set $\mu = \gamma/(1 + 2\gamma)$. Wright assumed that mutation occurs independent from selection, whereas in our model mutation occurs only during mating. The attractors of equation (14.50) are given by $\Delta p_i(t) = 0$. They are located in the interior of the unit cube. For each selection method and each fitness function the locations of the attractors are different. We give just one example.

Theorem 14.5. *For proportionate selection and OneMax the attractors are given by*

$$p_i^* = \frac{1 + \gamma(n-2)}{1 + 2\gamma(n-1)}\,. \tag{14.51}$$

If we set $N = n$ and $r = 1$ we have $\gamma = 1/(n+2)$. We obtain

$$\lim_{t\to\infty} p(x_{\text{opt}}, t) = \prod_{i=1}^{n} p_i(t) \approx (\tfrac{2}{3})^n = 0\,. \tag{14.52}$$

The theorem shows that a mutation rate of $\mu = 1/n$ is too large for proportionate selection.

The analysis is more difficult for truncation selection. In this chapter we derive only an optimistic upper bound for the mutation rate. The attractors are given by an equilibrium between selection and mutation. How can we determine an appropriate value for r for our FDA application? The location should be as far as possible in the interior under the constraint that the optima are generated with high probability.

The uniform prior gives for $m = 0$ the value $\hat{p}_{\min} = 1/(N+2)$. If N is small, then $p_{\min}$ might be so large that we generate the optima with only a very small probability. This means we perform more of a random search instead of converging to the optima. This consideration leads to a constraint: $1 - p_{\min}$ should be so large that the optima are still generated with high probability. We now heuristically derive $p_{\min}$ under the assumption that there is a unique optimum. To simplify the formulas we require that $\max \hat{p}(\mathbf{x}_{\text{opt}}) \geq e^{-1}$.

This means that the optimum string x_{opt} should be generated more than 30% at equilibrium. This is large enough to observe equilibrium and convergence. Let us first investigate the UMDA factorization $p(\mathbf{x}) = \prod p(x_i)$. For $r = 1$ the largest probability is $p_{\max} = (N+1)/(N+2)$. From this, we have

$$p_{\max} = 1 - \frac{1}{N+2} = 1 - p_{\min}\,.$$

The largest probability to generate the optimum is given by

$$\hat{p}(\mathbf{x}_{\text{opt}}) = \prod_{i=1}^{n} \left(1 - \frac{1}{N+2}\right) \approx e^{-\frac{n}{N+2}} .$$

If $N = O(n^{1-\alpha})$ with $\alpha > 0$, then $p(\mathbf{x}_{\text{opt}})$ becomes arbitrarily small for large n. For $N = n$ we obtain $\hat{p}(\mathbf{x}_{\text{opt}}) \approx e^{-1}$. This results in the following guideline, which is actually a lower bound of the population size.

Rule of Thumb. *For UMDA the size of the population should be at least equal to the size of the problem, if a Bayesian prior of $r = 1$ is used.*

Bayesian priors are also defined for conditional distributions. The above heuristic derivation can also be used for general Bayesian factorizations. For binary variables, the Bayesian estimator is

$$\hat{p}(x_i|\text{pa}_i) = \frac{m + r}{P + 2r} ,$$

where P is the number of occurrences of pa_i. We assume that in the best case the optimum constitutes 25% of the population. This yields $P \geq N/4$. For $r = 1$ we compute as before

$$\hat{p}(x_{\text{opt}}) = \prod_{i=1}^{n} \hat{p}(x_{\text{opt}_i}|\text{pa}_{\text{opt}_i}) = \prod_{i=1}^{n} \left(1 - \frac{1}{N/4+2}\right) \approx e^{-\frac{n}{N/4+2}} .$$

If we set $N = 4n$ we obtain $\hat{p}(x_{\text{opt}}) \approx e^{-1}$. Thus we obtain a lower bound for the population size.

Rule of Thumb. *For FDA using a factorization with many conditional distributions and Bayesian prior of $r = 1$, the size of the population should be about four times the size of the problem.*

These rules of thumb have been derived heuristically and they have to be confirmed by numerical studies. Our FDA estimate is a crude lower bound. There exist more general estimates, e.g. Vapnik (1998). In order to approximate a distribution with a reasonable accuracy, he proposes to use a sample size that is about 20 times larger than the number of free parameters of the distribution. For UMDA this means $20n$, i.e. 20 times our estimate.

We demonstrate the importance of using a Bayesian prior by an example. It is a deceptive function of order 4 and problem size of $n = 32$. Our convergence theorem gives convergence of FDA with Boltzmann selection and an exact factorization. The exact factorization consists of marginal distributions of size 4. We compare in figure 14.1 FDA with SDS Boltzmann selection and truncation selection without Bayesian prior. We also show a run with SDS Boltzmann selection and Bayesian prior.

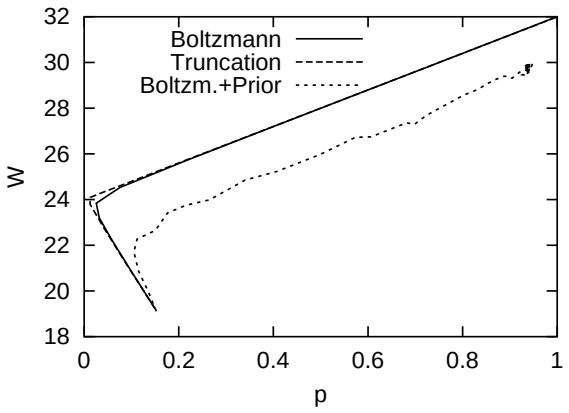

Figure 14.1. Average fitness $W(p)$ for FDA for Decep$(32, 4)$; population size $N = 20000$ without prior and $N = 200$ with prior $r = 1$.

The simulation was started at $p = 0.15$, i.e. near the local optimum $p = 0$. Nevertheless, FDA converges to the global optimum at $p = 1$. Note that FDA moves at first in the direction of the local optimum. At the very last moment the direction of the curve changes dramatically. SDS Boltzmann selection behaves almost identically to truncation selection with threshold $\tau = 0.35$. But both methods require a huge population size in order to converge to the optimum. In this example it is $N = 20000$. If a prior of $r = 1$ is used the population size can be reduced to $N = 200$. With this prior the curve changes direction earlier. Because of the prior, the univariate marginal probabilities never reach $p = 0$ or $p = 1$: in this example it stops at about $p = 0.975$.

To summarize the results: because FDA uses finite samples of points to estimate the conditional probabilities, convergence to the optimum depends on the size of the samples (the population size). FDA has proven to be very successful experimentally on a number of functions where standard genetic algorithms fail to find the global optimum. Mühlenbein and Mahnig (1999b) studies the scaling behavior for various test functions. The estimation of the probabilities and the generation of new points can be performed in polynomial time. Using a Bayesian prior reduces the influence of the population size. But there is a tradeoff: if no prior is used then convergence is fast, although a large population size might be needed. If a prior is used, the population size can be much smaller, but the number of generations until convergence increases. We do not have enough numerical results yet, and so we can only conjecture:

Conjecture. *FDA, with a finite population of size $N = 4n$, SDS Boltzmann selection, Bayesian prior, and a Bayesian factorization where the number of parents is restricted by k independent of n, will converge to the optimum in polynomial time with high probability.*

14.6 Constraint Optimization Problems

One advantage of FDA over genetic algorithms is that it can handle optimization problems with constraints. Mendelian recombination or crossover in genetic algorithms often creates points that violate the constraints. If the structure of the constraints and the structure of the ADF are compatible, then FDA generates only legal points.

Definition 14.9. *A constraint optimization problem is defined by*

$$\max f(\mathbf{x}) = \sum_{i=1}^{m} f_i(\mathbf{x}_{s_i}) \tag{14.53}$$

such that $C_i(\mathbf{x}_{u_i})$.

$C_i(\mathbf{x}_{u_i})$ stands for the ith constraint function. $\mathbf{x}_{s_i}, \mathbf{x}_{u_i} \subseteq X$ are sets of variables. The constraints are defined locally. Thus they can be used to test which marginal probabilities are 0. This is technically somewhat complicated, but nevertheless straightforward. For instance, if we have $C_1(x_1, x_2) = \{x_1 + x_2 \leq 1\}$, then $p(X_1 = 1, X_2 = 1) = 0$. Thus the constraints are mapped to marginal distributions: if $C_i(\mathbf{x}_{u_i})$ is violated then we set $p_i(x_{u_i}) = 0$.

We can now factorize $f(\mathbf{x})$ as before. But we can also factorize the graph defined by $C_i(x_{u_i})$. Our theory can handle both cases: the factorization of the constraints is contained in the factorization of the function, i.e. $x_{u_i} \subseteq x_{s_i}$, or the factorization of the function is contained in the factorization of the constraints, i.e. $x_{s_i} \subseteq x_{u_i}$.

Let Ω_c be the set of *feasible* solutions. Then the Boltzmann distribution on Ω_c is defined as

$$p_{b,f,c}(\mathbf{x}) = \frac{p_0(x)e^{\beta f(\mathbf{x})}}{\sum_{y \in \Omega_c} p_0(\mathbf{y})e^{\beta f(\mathbf{y})}} \,. \tag{14.54}$$

Then the following convergence theorem holds.

Theorem 14.6 (Convergence). *Let (1) the initial population be feasible. Let (2) the factorization of the constraints and the factorization of the function be contained in the FDA factorization. Let (3) $\Delta\beta(t)$ be an annealing schedule. Then for FDA the distribution at time t is given by*

$$p(\mathbf{x}, t) = \frac{p_0(\mathbf{x})e^{\beta(t) f(\mathbf{x})}}{\sum_{y \in \Omega_c} p_0(\mathbf{y})e^{\beta(t) f(y)}} \tag{14.55}$$

with the inverse temperature

$$\beta(t) = \sum_{\tau=1}^{t} \Delta\beta(\tau)\,. \tag{14.56}$$

Let $\mathcal{M}$ be the set of global optima. If $\beta(t) \to \infty$, then

$$\lim_{t\to\infty} p(x,t) = \begin{cases} 1/|\mathcal{M}| & x \in \mathcal{M} \\ 0 & \text{otherwise.} \end{cases} \tag{14.57}$$

Proof. The proof is almost identical to the proof of theorem 14.1. We have to show only that the factorization generates only feasible solutions, if the probabilities are computed from a set of feasible solutions. The proof is indirect. Suppose there exists an $\mathbf{x}$ that does not satisfy the kth constrain $C_k(\mathbf{x}_{u_k})$. Then

$$0 \neq p(\mathbf{x}, t+1) = \prod_{i=1}^{n} p^s(\mathbf{x}_{b_i}|\mathbf{x}_{c_i}, t)\,.$$

Thus we have $p^s(\mathbf{x}_{u_k}) \neq 0$. This means that there exists at least one individual in generation t that violates the constraint. But this contradicts assumption (1). □

The factorization theorem requires an analytical description of the function. But it is also possible to determine the factorization from the sampled data points. This is described next.

15 Computing a Bayesian Network from Data

The FDA factorization is based on the decomposition of the fitness function. This has two drawbacks: first, the structure of the function has to be known. Second, for a given instance of the fitness function, the structure might not give the smallest possible factorization. In other words: Complex structures do not necessarily lead to complex dependency structures for a given instance of a fitness function. The actual dependencies are determined by the function values. This problem can be circumvented by computing the dependency structure from the data.

Computing the structure of a Bayesian network from data is called *learning*. Learning gives an answer to the question: given a population of selected points $M(t)$, what is a good Bayesian factorization fitting the data? The most difficult part of the problem is to define a quality measure also called a *scoring measure*.

A Bayesian network with more arcs fits the data better than one with less arcs. Therefore a scoring metric should give the best score to the

minimal Bayesian network that fits the data. It is outside the scope of this paper to discuss this problem in more detail. The interested reader is referred to the two papers by Heckerman and Friedman et al. in Jordan (1999).

For Bayesian networks two quality measures are used most frequently—the *Bayes Dirichlet* (BDe) score and the *minimal description length* (MDL) score. We concentrate on the MDL principle. This principle is motivated by universal coding. Suppose we are given a set D of instances, which we would like to store. Naturally, we would like to conserve space and save a compressed version of D. One way of compressing the data is to find a suitable model for D that the encoder can use to produce a compact version of D. In order to be able to recover D we must also store the model used by the encoder to compress D. Thus the total description length is defined as the sum of the length of the compressed version of D and the length of the description of the model. The MDL principle postulates that the optimal model is the one that minimizes the total description length.

15.1 LFDA—Learning a Bayesian Factorization

In the context of learning Bayesian networks, the model is a network B describing a probability distribution p over the instances appearing in the data. Several authors have approximately computed the MDL score. Let $M = |D|$ denote the size of the data set. Then MDL is approximately given by

$$\text{MDL}(B, D) = -\log_2(P(B)) + M \cdot H(B, D) + \tfrac{1}{2}\text{PA} \cdot \log_2(M) \tag{15.1}$$

where $P(B)$ denotes the prior probability of network B and $\text{PA} = \sum_i 2^{|\text{pa}_i|}$ gives the total number of probabilities to compute. $H(B, D)$ is defined by

$$H(B, D) = -\sum_{i=1}^{n} \sum_{\text{pa}_i} \sum_{x_i} \frac{m(x_i, \text{pa}_i)}{M} \log_2 \frac{m(x_i, \text{pa}_i)}{m(\text{pa}_i)}\,, \tag{15.2}$$

where $m(x_i, \text{pa}_i)$ denotes the number of occurrences of x_i given configuration pa_i and $m(\text{pa}_i) = \sum_{x_i} m(x_i, \text{pa}_i)$. If $\text{pa}_i = \emptyset$, then $m(x_i, \emptyset)$ is set to the number of occurrences of x_i in D.

The formula has an interpretation that can be easily understood. If no prior information is available, $P(B)$ is identical for all possible networks. For minimizing, this term can be left out. $0.5\,\text{PA} \cdot \log_2(M)$ is the length required to code the parameter of the model with precision 1/M. Normally one would need $\text{PA} \cdot \log_2(M)$ bits to encode the parameters. However, the central limit theorem says that these frequencies are roughly normally distributed with a variance of $M^{-1/2}$. Hence the higher

$0.5 \log_2(M)$ bits are not very useful and can be left out. $-M \cdot H(B, D)$ has two interpretations. First, it is identical to the logarithm of the maximum likelihood ($\log_2(L(B|D))$). Thus we arrive at the following principle:

Choose the model that maximizes $\log_2(L(B|D)) - \frac{1}{2}\mathrm{PA} \cdot \log_2(M)$.

The second interpretation arises from the observation that $H(B, D)$ is the conditional entropy of the network structure B, defined by PA_i, and the data D. The above principle is appealing, because it has no parameter to be tuned. But the formula has been derived under many simplifications. In practice, one needs more control about the quality vs. complexity tradeoff. Therefore we use a weight factor α. Our measure is defined by BIC:

$$\mathrm{BIC}(B, D, \alpha) = -M \cdot H(B, D) - \alpha \mathrm{PA} \cdot \log_2(M)\,. \tag{15.3}$$

This measure with $\alpha = 0.5$ has been first derived by Schwarz (1978) as *Bayesian information criterion.* Therefore we abbreviate our measure as $\mathrm{BIC}(\alpha)$.

To compute a network B^* that maximizes BIC requires a search through the space of all Bayesian networks. Such a search is more expensive than a search for the optima of the function. Therefore we use the following greedy algorithm. $k_{\max}$ is the maximum number of incoming edges allowed.

$$\mathbf{BN}(\alpha, \mathbf{k}_{\max})$$

- STEP 0: Start with an arc-less network.
- STEP 1: Add the arc (x_i, x_j) which gives the maximum increase of $\mathrm{BIC}(\alpha)$ if $|\mathrm{PA}_j| \leq k_{\max}$ and adding the arc does not introduce a cycle.
- STEP 2: Stop if no arc is found.

Whether an arc would introduce a cycle can be checked easily by maintaining for each node a list of parents and ancestors, i.e. parents of parents etc. Then $(x_i \to x_j)$ introduces a cycle if x_j is ancestor of x_i.

The BOA algorithm of Pelikan (Pelikan et al., 2000) uses the BDe score. This measure has as a drawback that it is more sensitive to coincidental correlations implied by the data than the MDL measure. As a consequence, the BDe measure prefers network structures with more arcs over simpler networks (Bouckaert, 1994). The BIC measure with $\alpha = 1$ has also been proposed by Harik (1999). But Harik allows only factorizations without conditional distributions. This distribution is correct only for separable functions.

Table 15.1
Numerical Results for Different Algorithms, LFDA with BN($\alpha, 8$)

Function	n	α	N	τ	Succ.%	SDev
OneMax	30	UMDA	30	0.3	75	4.3
	30	0.25	100	0.3	2	1.4
	30	0.50	100	0.3	38	4.9
	30	0.75	100	0.3	80	4.0
	30	0.25	200	0.3	71	4.5
Saw $(32, 2, 0.5)$	32	UMDA	50	0.5	71	4.5
	32	UMDA	200	0.5	100	0.0
	32	0.25	200	0.5	41	2.2
	32	0.50	200	0.5	83	1.7
	32	0.75	200	0.5	96	0.9
	32	0.25	400	0.5	84	3.7
Decep-4	32	UMDA	800	0.3	0	0.0
	32	FDA	100	0.3	81	3.9
	32	0.25	800	0.3	92	2.7
	32	0.50	800	0.3	72	4.5
	32	0.75	800	0.3	12	3.2

Given the BIC score we have several options to extend FDA to LFDA, which learns a factorization. Given limitations of space, we can show results only of an algorithm that computes a Bayesian network at each generation using algorithm BN($0.5, k_{\max}$). FDA and LFDA should behave fairly similarly, if LFDA computes factorizations that are probabilistically very similar to the FDA factorization. FDA uses the same factorization for all generations, whereas at each step LFDA computes a new factorization that depends on the given data M.

We have applied LFDA to many problems (Mühlenbein and Mahnig, 1999b). The results are encouraging. Here we discuss only the functions introduced in section 13.4. Recall that UMDA finds the optimum of the multimodal function Saw. UMDA uses univariate marginal distributions only. Therefore its Bayesian network has no arcs.

Table 15.1 summarizes the results. For LFDA we used three values of α, namely $\alpha = 0.25, 0.5, 0.75$. The smaller α, the less penalty for the size of the structure. Let us discuss the results in more detail. $\alpha = 0.25$ gives by far the best results when a network with many arcs is required. This is the case for Decep $-$ 4. Here a Bayesian network with three parents is optimal. $\alpha = 0.25$ performs poorly on problems where a network with no arcs defines a good search distribution. For the linear function OneMax BIC(0.25) has a success rate of only 2%. The success rate can be improved if a larger population size N is used, for the following rea-

son. BIC(0.25) allows denser networks; but if a small population is used, spurious correlations may arise. These correlations have a negative impact on the search distribution. The problem can be solved by using a larger population. Increasing the value from $N = 100$ to $N = 200$ increases the success rate from 2% to 71% for OneMax.

For Saw a Bayesian network with no arcs is able to generate the optimum. An exact factorization requires a factor with n parameters. We used the heuristic BN with $k_{\max} = 8$. Therefore the exact factorization cannot be found. In all these cases $\alpha = 0.75$ gives the best results. BIC(0.75) enforces smaller networks. But BIC(0.75) performs very poorly on Decep-4. Taking all results together, BIC(0.5) gives promising results. These numerical results support the theoretical estimate.

The numerical result indicates that control of the weight factor α can substantially reduce the amount of computation. For Bayesian networks we have not yet experimented with control strategies. We have studied the problem extensively in the context of neural networks (Zhang et al., 1997).

UMDA is most efficient at optimizing the functions OneMax and Saw. FDA is efficient if the exact factorization requires a small number of parents in the Bayesian graph ($k \leq 5$). LFDA also finds the optimum most of the time. From the functions considered it has the greatest difficulty with the function Saw. The performance of LFDA can be improved substantially, if for each fitness function a suitable value of α is chosen. Recall that a small value of α leads to more complex Bayesian factorizations. The BIC score uses $\alpha = 0.5$, a good compromise. But $\alpha = 0.75$ results in a much better performance for the functions OneMax and Saw, whereas $\alpha = 0.25$ yields the best results for the function Decep$(36, 4)$. These results are explained next.

15.2 Optimization, Dependencies, and Search Distributions

We have proven in section 14.1 convergence of FDA with Boltzmann selection to the set of global maxima. If the Boltzmann distribution can be factorized, the computational complexity for one generation is bounded by $O(n \cdot N \cdot 2^k)$. k denotes the maximum number of parents. A factorization can be determined if the fitness function is decomposed. It can also be obtained from the data sampled. Unfortunately, for many interesting applications k is very large. If the fitness function is decomposed additively on a 2-D grid of size n, then k scales like $O(\sqrt{n})$. It is easy to show that k even scales like $O(n)$ for the function Saw.

But we have demonstrated that the simple search distribution used by UMDA guides the search to the optimum of Saw. The reason is that Saw has the following tendency: the more bits on, the higher the fitness value.

Therefore an exact Boltzmann factorization is not needed for optimization. The problem of finding a good approximation of the Boltzmann distribution that generates the optima with high probability cannot be solved theoretically. Therefore we propose the following heuristic:

Multi-Factorization LFDA. *Use different values of α in order to obtain factorizations of various complexity. In a standard setting, use $\alpha = 0.25$, 0.5, and 0.75. Generate new search points using the different factorizations for a certain percentage of the population.*

16 The System Dynamics Approach to Optimization

So far we have transformed genetic algorithms that use a population of strings and probabilistic recombination of strings into algorithms that use probability distributions to generate new strings. We have derived Wright's equation and used it to analyze the behavior of UMDA. We have shown that UMDA converges to local attractors of the average fitness. These attractors are local optima of the fitness function.

We can go even one step further: We can use the difference equations directly, without generating a population of strings at all. This approach seems surprising at first, but it has been used in mathematics a number of times. The approach is called called the *systems dynamics approach to optimization*. We discuss just a few examples that are related to our previous algorithms.

16.1 The Replicator Equation

In this section we investigate the relation between Wright's equation and a popular equation called the *replicator equation.* Replicator dynamics is a standard model in evolutionary biology used to describe the dynamics of growth and decay of a number of species under selection. Let $S = \{1, 2, \ldots, s\}$ be a set of species and p_i the frequency of species i in a fixed population of size N. Then the replicator equation is defined on a simplex $S^s = \{p : \sum p_i = 1, 0 \leq p_i \leq 1\}$:

$$\frac{dp_i}{dt} = p_i(t) \left(f_i(\mathbf{p}) - \sum_{i=1}^{s} p_i(t) f_i(\mathbf{p}) \right), \tag{16.1}$$

where f_i gives the fitness of species i in relation to the others. The replicator equation is discussed in detail in Hofbauer and Sigmund (1998). For the replicator equation a maximum principle can be shown.

Theorem 16.1. *If there exists a potential V with $\partial V/\partial p_i = f_i(\mathbf{p})$, then $dV/dt \geq 0$, i.e. the potential V increases using the replicator dynamics.*

If we want to apply the replicator equation to a binary optimization problem of size n, we have to set $s = 2^n$. Thus the number of species is exponential in the size of the problem. The replicator equation can be used only for small size problems.

Voigt (1989) had the idea to generalize the replicator equation by introducing continuous variables $0 \leq p_i(x_k) \leq 1$ with $\sum_k p_i(x_k) = 1$. Thus $p_i(x_k)$ can be interpreted as univariate probabilities. Voigt (1989) proposed the following discrete equation.

Definition 16.1. *The* discrete diversified replicator equation (DDRP) *is given by*

$$p_i(x_k)(t+1) - p_i(x_k)(t) = p_i(x_k)(t)\frac{f_{ik}(\mathbf{p}) - \sum_{x_k} p_i(x_k) f_{ik}(\mathbf{p})}{\sum_{x_k} p_i(x_k) f_{ik}(\mathbf{p})}\,. \tag{16.2}$$

The name discrete diversified replicator equation was not a good choice, as the DDRP is more similar to Wright's equation than to the replicator equation. This is the content of the next theorem.

Theorem 16.2. *If the average fitness $W(\mathbf{p})$ is used as potential, then Wright's equation and the discrete diversified replicator equation are identical.*

Proof. The average fitness is defined as

$$W(\mathbf{p}) = V(\mathbf{p}) = \sum_x a_x \prod_{i=1}^{n} p_i(x_i)\,.$$

We compute the derivatives

$$\frac{\partial V(\mathbf{p})}{\partial p_i(1)} = \sum_{x|x_i=1} a_x \prod_{j\neq i}^{n} p_j(x_j)$$

$$\frac{\partial V(\mathbf{p})}{\partial p_i(0)} = \sum_{x|x_i=0} a_x \prod_{j\neq i}^{n} p_j(x_j)\,.$$

Then, obviously:

$$p_i(1)\frac{\partial V}{\partial p_i(1)} + p_i(1)\frac{\partial V}{\partial p_i(1)} = V(\mathbf{p})\,.$$

The conjecture now follows from the proof of Wright's equation. □

We recently discovered that Baum and Eagon (1967) have proven a discrete maximum principle for certain instances of the DDRP.

Theorem 16.3 (Baum-Eagon). *Let $V(\mathbf{p})$ be a polynomial with non-negative coefficients homogeneous of degree d in its variables $p_i(x_j)$ with $p_i(x_j) \geq 0$ and $\sum_{x_j} p_i(x_j) = 1$. Let $\mathbf{p}(t+1)$ be the point given by*

$$p_i(x_j, t+1) = \frac{p_i(x_j, t)\frac{\partial V}{\partial p_i(x_j)}}{\sum_{x_k} p_i(x_k)\frac{\partial V}{\partial p_i(x_k)}} \,. \tag{16.3}$$

The derivatives are taken at $\mathbf{p}(t)$. Then $V(\mathbf{p}(t+1)) > V(\mathbf{p}(t)$ unless $\mathbf{p}(t+1) = \mathbf{p}(t)$.

Equation (16.3) is exactly the DDRP with a potential V. Thus the DDRP could be called the Baum-Eagon equation. From the above theorem the discrete maximum principle for Wright's equation follows by setting $V = W$ and $d = n$. Thus the potential is the average fitness, which is homogeneous of degree n.

16.2 Boltzmann Selection and the Replicator Equation

For FDA with Boltzmann selection we have a closed solution for the probability $p(\mathbf{x}, t)$. It is given by

$$p_{\beta,p_0}(\mathbf{x}, t) = \frac{p_0(\mathbf{x})e^{\beta f(\mathbf{x})}}{\sum_y p_0(\mathbf{y})e^{\beta f(\mathbf{y})}} \,. \tag{16.4}$$

If we differentiate this equation we obtain after some computation:

$$\frac{dp_{\beta,p_0}(\mathbf{x}, t)}{dt} = \frac{d\beta}{dt} p_{\beta,p_0}(\mathbf{x}, t) \left(f(\mathbf{x}) - \sum_y p_{\beta,p_0}(\mathbf{y}, t) f(y) \right) . \tag{16.5}$$

For $\beta' = 1$ we obtain a special case of the replicator equation (16.1). We have only to set $f(\mathbf{p}) = f_i$.

Theorem 16.4. *The dynamics of Boltzmann selection with $\Delta\beta(t) = 1$ is given by the replicator equation.*

From the convergence theorem 14.1 we know that the global optima are the only stable attractors of the replicator equation. Thus the replicator equation is an ideal starting point for a system dynamics approach to optimization discussed in section 16. Unfortunately, the replicator equation consists of 2^n different equations for a problem of size n.

Thus we are led to the same problem encountered when analyzing the Boltzmann distribution: We have to factorize the probability $p(\mathbf{x})$ if we want to use the equation numerically.

Example 16.1. *Linear function $f(\mathbf{x}) = \sum_i^n \alpha_i x_i$. In this case the UMDA factorization is valid $p(\mathbf{x}) = \prod_{i=1}^n p_i(x_i)$. By summation we ob-*

tain from equation (16.5)*, after some manipulation:*

$$\frac{dp_i}{dt} = \frac{d\beta}{dt} p_i(1-p_i)\alpha_i \,. \tag{16.6}$$

For $d\beta/dt = 1$ *this is just Wright's equation without the denominator* $\tilde{W}$*.*

If we extend this equation we obtain another proposal for the systems dynamics approach to optimization:

$$\frac{dp_i}{dt} = \frac{d\beta}{dt} p_i(1-p_i)\frac{\partial \tilde{W}}{\partial p_i} \,. \tag{16.7}$$

This leads to the difference equation:

$$\Delta p_i = \Delta\beta p_i(1-p_i)\frac{\partial \tilde{W}}{\partial p_i} \,. \tag{16.8}$$

We just recently derived this equation, and so we have not yet used it for numerical experiments.

16.3 Some System Dynamics Equations for Optimization

Theorem 16.3 shows that both Wright's equation and the DDRP maximize some potential. This means that both equations can be used for maximization. But there is a problem: both equations are deterministic. For difficult optimization problems, there exists a large number of attractors, each with a corresponding attractor region. If the iteration starts at a point within the attractor region, it will converge to the corresponding attractor at the boundary. But if the iteration starts at points that lie at the boundary of two or more attractors, i.e. on the separatrix, the iteration will be confined to the separatrix. The deterministic system cannot decide on one of the attractors.

UMDA with a finite population does not have a sharp boundary between attractor regions. We model this behavior by introducing randomness. The new value $p_i(x_j, t+1)$ is randomly chosen from the interval

$$[(1-c)p'_i(x_j, t+1), (1+c)p'_i(x_j, t+1)] \,,$$

where $p'_i(x_j, t+1)$ is determined by the deterministic equation. c is a small number. For $c = 0$ we obtain the deterministic equation. In order to use the difference equation optimally, we do not allow the boundary values $p_i = 0$ or $p_i = 1$, but use instead $p_i = p_{\min}$ and $p_i = 1 - p_{\min}$.

A second extension concerns the determination of the solution. All dynamic equations presented use variables, which can be interpreted as probabilities. Thus instead of waiting that the dynamic system converges to some boundary point, we terminate the iteration at a suitable time

and generate a set of solutions. Thus, given the values for $p_i(x_j)$, we generate points x according to the UMDA distribution $p(\mathbf{x}) = \prod_{i=1}^{n} p_i(x_i)$.

We can now formulate a family of optimization algorithms, based on difference equations (DIFFOPT).

DIFFOPT

- STEP 0: Set $t \Leftarrow 0$ and $p_i(x_j, 0) = 0.5$. Input $p_{\min}$.
- STEP 1: Compute $p_i'(x_j, t+1)$ according to a dynamic difference equation. If $p_i'(x_j, t+1) < p_{\min}$ then $p_i'(x_j, t+1) = p_{\min}$. If $p_i'(x_j, t+1) > 1 - p_{\min}$ then $p_i'(x_j, t+1) = 1 - p_{\min}$
- STEP 2: Compute randomly $p_i(x_j, t+1)$ in the interval $(1-c)p_i'(x_j, t+1), (1+c)p_i'(x_j, t+1)$. Set $t \Leftarrow t+1$
- STEP 3: If termination criteria are not met, go to STEP 1.
- STEP 4: Generate N solutions according to $p(\mathbf{x}, t) = \prod_{i=1}^{n} p_i(x_i, t)$ and compute $\max f(\mathbf{x})$ and $\operatorname{argmax} f(\mathbf{x})$.

DIFFOPT is not restricted to Wright's equation or to DDRP. We propose a third one. Its rationale is as follows: from the analysis of UMDA we know that Wright's equation models proportionate selection. But this method converges very slowly when approaching the boundary. We have not been able to derive dynamic equations for truncation selection. Therefore we experimented with a number of faster versions of Wright's equation and finally chose the following difference equation.

Definition 16.2. *F-Wright(α) (Fast Wright) is defined by the following difference equation:*

$$p_i(x_i, t+1) = p_i(x_i, t) + \operatorname{sign}(\Delta) \cdot \exp\left(\alpha \ln |\Delta|\right) \tag{16.9}$$

$$\Delta = p_i(x_i, t) \frac{\frac{\partial \tilde{W}}{\partial p_i(x_i)} - \sum_{y_i \in \bar{\Lambda}_i} p_i(y_i, t) \frac{\partial \tilde{W}}{\partial p_i(y_i)}}{\tilde{W}(\mathbf{p})} . \tag{16.10}$$

If a value outside the interval $[p_{\min}, 1 - p_{\min}]$ is generated, we just set the value to the corresponding boundary value of the interval. For $\alpha = 1$ we obtain Wright's equation. We usually set $\alpha = 0.5$. The reason for this choice is that we wanted a difference equation that resembles as much as possible *truncation selection*. If we take the fitness function OneMax, we obtain for F-Wright(0.5) the difference equation:

$$p(t+1) - p(t) = \frac{\sqrt{p(t)(1-p(t))}}{np(t)} = \frac{\sqrt{1-p(t)}}{n} . \tag{16.11}$$

This equation is similar to the approximate equation we have computed for UMDA with truncation selection: only the multiplication by p is missing. This means that F-Wright normally will converge faster than UMDA with truncation selection.

We next evaluate the three difference equations with optimization problems.

16.4 Optimization of Binary Functions

The DDRP opens the possibility of using an arbitrary potential. If the potential is not a representation of the average fitness, Wright's equation and DDRP are different. We demonstrate this with a simple example, a quadratic potential.

Example 16.2. $V(\mathbf{p}) = \sum_{ij} a_{ij} p_i(1) p_j(0) + c$. *$c$ is chosen such that* $V(\mathbf{p}) > 0$. *We assume* $a_{ii} = 0$.

We obtain

$$\begin{aligned} \frac{\partial V}{\partial p_i(1)} &= \sum_j a_{ij} p_j(0) \\ \frac{\partial V}{\partial p_i(0)} &= \sum_j a_{ji} p_j(1) \\ V_i(\mathbf{p}) &= p_i(1) \frac{\partial V}{\partial p_i(1)} + p_i(0) \frac{\partial V}{\partial p_i(0)} \,. \end{aligned}$$

Obviously $\sum_i p_i(1) \sum_j a_{ij} p_j(0) = \sum_i p_i(0) \sum_j a_{ji} p_j(1)$. Therefore we get the following proposition.

Proposition. $V(\mathbf{p}) = \frac{1}{2} \sum_i V_i(\mathbf{p})$ *if c is suitably chosen.*

The DDRP is given by

$$\Delta p_i(1) = p_i(1) \frac{\sum_{j \neq i} a_{ij} p_j(0) - V_i}{V_i + c_i} \,.$$

c_i has to be chosen such that $V_i(\mathbf{p}) + c_i > 0$. If we eliminate $p_i(0) = 1 - p_i(1)$ and abbreviate $p_i := p_i(1)$ we obtain

$$\Delta p_i = p_i(1 - p_i) \frac{\sum_{j \neq i} a_{ij}(1 - p_j) - \sum_{j \neq i} a_{ji} p_j}{V_i + c_i} \,. \tag{16.12}$$

We now determine Wright's equation for the same problem. This means we have to find a fitness function, which will give $V(\mathbf{p}) = \tilde{W}(\mathbf{p})$.

Example 16.3. $f(\mathbf{x}) = \sum_{ij} a_{ij} x_i (1 - x_j) + c$. *$c$ is chosen such that* $f(\mathbf{x}) > 0$.

We compute $\tilde{W(\mathbf{p})}$ using our lemma

$$\tilde{W}(\mathbf{p}) = \sum_{ij} a_{ij} p_i (1 - p_j) + c \,. \tag{16.13}$$

Obviously $\tilde{W}(\mathbf{p}) = V(\mathbf{p})$. Wright's equation is given by

$$\Delta p_i = p_i(1-p_i)\frac{\sum_{j\neq i} a_{ij}(1-p_j) - \sum_{j\neq i} a_{ji}p_j}{\tilde{W}(\mathbf{p})}\,. \tag{16.14}$$

We now compare the two difference equations. We assume that $c = c_i = 0$ and obtain

$$\begin{aligned}
\Delta p_i &= p_i(1-p_i)\frac{\sum_{j\neq i} a_{ij}(1-p_j) - \sum_{j\neq i} a_{ji}p_j}{\tilde{W}(\mathbf{p})} \\
\Delta p_i &= p_i(1-p_i)\frac{\sum_{j\neq i} a_{ij}(1-p_j) - \sum_{j\neq i} a_{ji}p_j}{p_i\sum_j a_{ij}(1-p_j) + (1-p_i)\sum_j a_{ji}p_j}\,.
\end{aligned}$$

The two equations differ in the denominator only. The denominator of DDRP is normally smaller than the denominator of Wright's equation, and thus DDRP will converge faster. We compare three different examples.

Problem 1. $a_{i,i+1} = 1, a_{i,i-1} = 1$. All other values are set to 0.

The two global optima of this problem are $1,0,1,0,\ldots$ and $0,1,0,1,\ldots$ with a fitness value of $n-1$. The fitness function is symmetric. $f(\mathbf{x})$ and $f(\bar{\mathbf{x}})$ have the same fitness value where $\bar{\mathbf{x}}$ is the string with all bits inverted. We have an unstable attractor at $p_i = 0.5$.

Problem 2. $a_{i,i+1} = 1, a_{i,i-1} = 2, a_{n-1,n-2} = 3$. All other values are set to 0.

Here the matrix a is not symmetric. The value $a_{n-1,n-2} = 3$ deceives the system to set $x_{n-1} = 1$. But the optimal solution is $x_{\max} = (0,1,0,1,\ldots)$ with $x_{n-1} = 0$ for n even. The optimum fitness value is $1.5n - 1$.

Problem 3. $a_{i,j} = 1, \quad j < i$. All other values are set to 0.

Here the maximum is $x_{\max} = (0,0,\ldots,0,1,\ldots,1,1)$, i.e. the first half of the bits are 0, the second half of the bits are 1. For $n = 30$ the optimal value is 225.

Table 16.1 displays the numerical results. In problem 1 with $n = 30$ the optimum is found at least once by all three methods. On the average one bit is wrong. This behavior can be understood because of the parallel search and the symmetry of the problem. For $n = 60$ we have 3 bits wrong on the average. In problem 2 bit $n-1$ is always set to 1 (because $a_{n-1,n-3} = 3$). Therefore the optimum is missed, which has a 0 at this place. The same behavior is observed for $n = 60$. The optimum is missed by one point. A large difference in the performance can be seen in problem 3. Here the results for the more local DDRP are really poor. DDRP is not able to set the bits correct in the area where

Table 16.1
Numerical Results (Average over 10 runs)
The number in brackets give the number of times S
a global optimum has been found.

Algorithm	Prob.	n	Iter.	Maximum (S)
Wright	1	30	250	28.2 (2)
DDRP	1	30	70	27.8 (1)
F-Wright(0.5)	1	30	20	27.6 (1)
Wright	1	60	500	55.6 (0)
DDRP	1	60	140	55.6 (0)
F-Wright(0.5)	1	60	20	54.5 (0)
Wright	2	30	60	43.0 (0)
DDRP	2	30	70	43.1 (1)
F-Wright(0.5)	2	30	20	43.0 (0)
Wright	2	60	500	88.0 (0)
DDRP	2	60	50	87.7 (0)
F-Wright(0.5)	2	60	20	88.0 (0)
Wright	3	30	250	225.0 (10)
DDRP	3	30	250	204.4 (00)
F-Wright(0.5)	3	30	20	225.0 (10)

Table 16.2
Numerical Results for UMDA with
Proportionate Selection (p) and Truncation Selection (tr) and
a Genetic Algorithm with Uniform Crossover (uc).

Algorithm	Prob.	n	N	Iter.	Maximum (S)
UMDA p.	1	30	300	230	27.2 (4)
UMDA tr.	1	30	300	90	26.9 (2)
GA uc	1	30	300	100	27.4 (1)
UMDA p.	1	60	600	400	53.0 (0)
UMDA tr.	1	60	600	150	53.3 (0)
GA uc	1	60	600	150	55.3 (0)
UMDA p.	3	30	300	200	225.0 (10)
UMDA tr.	3	30	300	10	225.0 (10)
GA uc	3	30	300	30	225.0 (10)

all 1 meets all 0. This problem is the simplest for Wright's equation and F-Wright.

All three examples taken together show that F-Wright(0.5) is the fastest and most efficient algorithm.

Table 16.2 shows the numerical results for a genetic algorithm GA and UMDA. The results of UMDA with proportionate selection and Wright's

Table 17.1
Three Classes of Evolutionary Algorithms

Algorithm	Selection	Reproduction
Genetic Algorithm	microscopic	microscopic
UMDA	microscopic	macroscopic
System Dynamics	macroscopic	macroscopic

equation are fairly similar. The results for problem 2 are omitted because they are similar to those for problem 1. Note that no algorithm is able to locate the global optimum for problem 1 with size $n = 60$. For this problem FDA has to be used.

17 Three Royal Roads to Optimization

In this section we will classify the different approaches presented. Population search methods are based on two components at least: selection and reproduction with variation. In our research we have transformed genetic algorithms to a family of algorithms using search distributions instead of recombination/mutation of strings. The simplest algorithm of this family is the univariate marginal distribution algorithm UMDA.

Wright's equation describes the behavior of UMDA using an infinite population and proportionate selection. The equation shows that UMDA does *not* primarily optimize the *fitness function* $f(\mathbf{x})$, but the *average fitness* of the population $W(\mathbf{p})$, which depends on the continuous marginal frequencies $p_i(x_i)$. Thus the important landscape for population search is *not* the landscape defined by the fitness function $f(\mathbf{x})$, but, rather the landscape defined by $W(\mathbf{p})$.

The two components of population-based search methods—selection and reproduction with variation—can work on a microscopic (individual) or a macroscopic (population) level. The level can be different for selection and reproduction. It is possible to classify the various approaches according to the level at which the components work. Table 17.1 shows three classes of evolutionary algorithms, each with a representative member.

A genetic algorithm uses a population of individuals. Selection and recombination are performed by manipulating individual strings. UMDA uses marginal distributions to create individuals. These are macroscopic variables. Selection is performed on a population of individuals, as with genetic algorithms. In the system dynamics approach, selection is modeled by a specific dynamic difference equation for macroscopic variables.

We believe that a fourth class—macroscopic selection and microscopic reproduction—makes no sense.

Each of the approaches has its specific pros and cons. Genetic algorithms are very flexible, but the standard recombination operator has limited capabilities. UMDA can use any kind of selection method that is used by a genetic algorithm. UMDA can also be extended to an algorithm that uses a more complex factorization of the distribution, namely the factorized distribution algorithm FDA. Selection is very difficult to model on a macroscopic level. Wright's equations are valid only for proportionate selection. Other selection schemes lead to very complicated system dynamics equations.

Thus for proportionate selection and gene pool recombination all methods behave similarly. But each of the methods allows extensions that cannot be modeled with an approach using a different level.

Especially interesting mathematically is the extension of UMDA to FDA with an adaptive Boltzmann annealing schedule. For this algorithm convergence for a large class of discrete optimization problems has been shown.

18 Conclusion and Outlook

This chapter describes a complete mathematical analysis of evolutionary methods for optimization. The optimization problem is defined by a fitness function with a given set of variables. Part of the theory consists of an adaptation of classical population genetics and the science of breeding to optimization problems. The theory is extended to general population-based search methods by introducing search distributions instead of performing string recombination. This theory also can be used for continuous variables, a mixture of continuous and discrete variables as well as constraint optimization problems. The theory combines *learning* and *optimization* into a common framework based on *graphical models*.

We have presented three approaches to optimization. We believe that the optimization methods based on search distributions (UMDA, FDA, LFDA) have the greatest optimization power. The dynamic equations derived for UMDA with proportionate selection are fairly simple. For UMDA with truncation or tournament selection and FDA with conditional marginal distributions, the dynamic equations can become very complicated. FDA with Boltzmann selection SDS is an extension of simulated annealing to a population of points. It shares with simulated annealing the convergence property, but convergence is much faster.

Ultimately our theory leads to a synthesis problem: finding a good factorization for a search distribution defined by a finite sample. This is a central problem in probability theory. One approach to this problem uses Bayesian networks. For Bayesian networks, researchers have developed numerically efficient algorithms. Our LFDA algorithm computes a Bayesian network by minimizing the Bayesian information criterion.

The computational effort of both FDA and LFDA is substantially higher than that of UMDA. Thus UMDA should be the first algorithm tried in a practical problem. Next the multi-factorization LFDA should be applied.

Our theory is defined for optimization problems that are defined by quantitative variables. The optimization problem can be defined by a cost function or a complex process to be simulated. The theory is not applicable directly if either the optimization problem is qualitatively defined or the *problem solving method is nonnumeric*. A popular example of a nonnumeric problem solving method is *genetic programming*. In genetic programming we try to find a program that optimizes the problem, not an optimal solution. Understanding these kinds of problem-solving methods will be a challenge for the next decade. We are convinced that the theory presented here can be extended to design and analyze those methods.

References

Asoh, H. and Mühlenbein, H. (1994a). Estimating the heritability by decomposing the genetic variance. In Davidor, Y., Schwefel, H.-P., and Männer, R. (eds.), *Parallel Problem Solving from Nature* (pp. 98–107). Lecture Notes in Computer Science 866. Berlin: Springer-Verlag.

Asoh, H. and Mühlenbein, H. (1994b). On the mean convergence time of evolutionary algorithms without selection and mutation. In Davidor, Y., Schwefel, H.-P., and Männer, R. (eds.), *Parallel Problem Solving from Nature* (pp. 88–97). Lecture Notes in Computer Science 866. Berlin: Springer-Verlag.

Ballard, D. (1997). *An Introduction to Natural Computation*. Cambridge, MA: MIT Press.

Baum, L. and Eagon, J. (1967). An inequality with applications to statistical estimation for probabilistic functions of markov processes and to a model for ecology. *Bull. Am. Math. Soc.* 73:360–363.

Bouckaert, R. (1994). Properties of bayesian network learning algorithms. In de Mantaras, R. L. and Poole, D. (eds.), *Proc. Tenth Conference on Uncertainty in Artificial Intelligence* (pp. 102–109). San Francisco: Morgan Kaufmann.

Christiansen, F. and Feldman, M. (1998). Algorithms, genetics, and populations: The schemata theorem revisited. *Complexity* 3:57–64.

de la Maza, M. and Tidor, B. (1993). An analysis of selection procedures with particular attention paid to proportional and boltzmann selection. In Forrest, S. (ed.), *Proc. of the Fifth Int. Conf. on Genetic Algorithms* (pp. 124–131). San Mateo, CA: Morgan Kaufmann.

Falconer, D. S. (1981). *Introduction to Quantitative Genetics*. London: Longman.

Freedman, D., R.Pisani, Purves, R., and Adhikkari, A. (1991). *Statistics*, second edition. New York: W. W. Norton.

Frey, B. (1998). *Graphical Models for Machine Learning and Digital Communication*. Cambrigde, MA: MIT Press.

Geiringer, H. (1944). On the probability theory of linkage in mendelian heredity. *Annals of Math. Stat.* 15:25–57.

Goldberg, D. (1989). *Genetic Algorithms in Search, Optimization, and Machine Learning*. Reading: Addison-Wesley.

Goldberg, D. and Deb, K. (1991). A comparative analysis of selection schemes used in genetic algorithms. In Rawlins, G. (ed.), *Foundations of Genetic Algorithms* (pp. 69–93). San Mateo, CA: Morgan-Kaufmann.

Harik, G. (1999). Linkage learning via probabilistic modeling in the ecga. Technical Report IlliGal 99010, University of Illinois, Urbana-Champaign.

Harik, G., Cantu-Paz, E., Goldberg, D., and Miller, B. (1999). The gambler's ruin problem, genetic algorithms, and the sizing of populations. *Evolutionary Computation* 7:231–255.

Hofbauer, J. and Sigmund, K. (1998). *Evolutionary Games and Population Dynamics*. Cambridge: Cambridge University Press.

Holland, J. (1975/1992). *Adaptation in Natural and Artificial Systems*. Ann Arbor, MI: Univ. of Michigan Press.

Jordan, M. (1999). *Learning in Graphical Models*. Cambridge, MA: MIT Press.

Kirkpatrick, S., Gelatt, C., and Vecchi, M. (1983). Optimization by simulated annealing. *Science* 220:671–680.

Lauritzen, S. L. (1996). *Graphical Models*. Oxford: Clarendon Press.

Mitchell, M., Holland, J., and Forrest, S. (1994). When will a genetic algorithm outperform hill climbing? *Advances in Neural Information Processing Systems* 6:51–58.

Mühlenbein, H. (1991). Evolution in time and space—The parallel genetic algorithm. In Rawlins, G. (ed.), *Foundations of Genetic Algorithms* (pp. 316–337). San Mateo, CA: Morgan-Kaufmann.

Mühlenbein, H. (1997). The equation for the response to selection and its use for prediction. *Evolutionary Computation* 5(3):303–346.

Mühlenbein, H. and Mahnig, T. (1999a). Convergence theory and applications of the factorized distribution algorithm. *Journal of Computing and Information Technology* 7:19–32.

Mühlenbein, H. and Mahnig, T. (1999b). FDA—A scalable evolutionary algorithm for the optimization of additively decomposed functions. *Evolutionary Computation* 7(4):353–376.

Mühlenbein, H. and Mahnig, T. (2000). Evolutionary algorithms: From recombination to search distributions. In Kallel, L., Naudts, B., and Rogers, A. (eds.), *Theoretical Aspects of Evolutionary Computing* (pp. 137–176). Natural Computing series. Berlin: Springer Verlag.

Mühlenbein, H., Mahnig, T., and Ochoa, A. R. (1999). Schemata, distributions and graphical models in evolutionary optimization. *Journal of Heuristics* 5:215–247.

Mühlenbein, H. and Schlierkamp-Voosen, D. (1994). The science of breeding and its application to the breeder genetic algorithm. *Evolutionary Computation* 1:335–360.

Mühlenbein, H. and Voigt, H.-M. (1996). Gene pool recombination in genetic algorithms. In Kelly, J. and Osman, I. (eds.), *Metaheuristics: Theory and Applications* (pp. 53–62). Norwell: Kluwer Academic Publishers.

Nagylaki, T. (1992). *Introduction to Theoretical Population Genetics*. Berlin: Springer.

Pelikan, M., Goldberg, D., and Cantu-Paz, E. (2000). Linkage problem, distribution estimation, and Bayesian networks. *Evolutionary Computation* 8: 311–341.

Prügel-Bennet, A. and Shapiro, J. (1997). An analysis of a genetic algorithm for simple random ising systems. *Physica D* 104:75–114.

Rao, C. (1973). *Linear Statisticcal Inference and Its Application*. New York: Wiley.

Rattray, L. M. and Shapiro, J. (in press). Cumulant Dynamics of a Population under Multiplicative Selection, Mutation and Drift. *Theoretical Population Biology*.

Schwarz, G. (1978). Estimating the dimension of a model. *Annals of Statistics* 7:461–464.

Vapnik, V. (1998). *Statistical Learning Theory*. New York: Wiley.

Voigt, H.-M. (1989). *Evolution and Optimization*. Berlin: Akademie-Verlag.

Vose, M. (1999). *The Simple Genetic Algorithm: Foundations and Theory*. Cambridge: MIT Press.

Wright, S. (1970). Random drift and the shifting balance theory of evolution. In Kojima, K. (ed.), *Mathematical Topics in Population Genetics* (pp. 1–31). Berlin: Springer Verlag.

Zhang, B.-T., Ohm, P., and Mühlenbein, H. (1997). Evolutionary induction of sparse neural trees. *Evolutionary Computation* 5:213–236.

Index