1. $\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$

$$misal: u^2 = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

$$u^2 = \frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}$$

$$2u\,du = \frac{x-\mu}{\sigma}.\frac{1}{\sigma}dx$$

$$= \frac{x-\mu}{\sigma^2}dx$$

$$2u^2\sigma^2 = (x-\mu)^2$$

$$\sqrt{2}\,u\,\sigma = x-\mu$$

$$\frac{2u\,\sigma^2}{x-\mu}du = dx$$

$$\frac{2u\,\sigma^2}{\sqrt{2}\,u\,\sigma}du = dx$$

$$\frac{2}{\sqrt{2}}\sigma\,du = dx$$

$$= 2\int_0^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-u^2} du$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{0} \frac{1}{\sqrt{\pi}} e^{-u^2} du$$

$$= \frac{2}{\sqrt{\pi}}.(-2)\left[u\,.\,e^{-u^2}\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{\pi}}.\frac{1}{2}\sqrt{\pi}$$

$$= 1$$

2. $V_{ar}(x) = \int_{-\infty}^{\infty} \frac{(x-\mu)}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}} \frac{(x-\mu)^2}{\sigma} dx = \sigma^2$

Bukti :

$$misal : u^2 = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

$$= \frac{1}{2} . \frac{(x-\mu)^2}{\sigma^2}$$

$$\mu^2\sigma^2 = \frac{1}{2}(x-\mu)^2$$

$$V_{ar}(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 . e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} 2u^2\sigma^2 e^{-u^2} . \frac{2}{\sqrt{2}} \sigma \, du$$

$$= \frac{8\sigma^3}{2\sigma\sqrt{\pi}} \int_{-\infty}^{\infty} u^2 e^{-u^2} du$$

$$= \frac{4\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} u^2 e^{-u^2} du$$

$$= \frac{4\sigma^2}{\sqrt{\pi}} \left\{ \left[-\frac{1}{2} u \, e^{-u^2} \right]_0^{\infty} - \int_{0}^{\infty} -\frac{1}{2} e^{u^2} du \right\}$$

$$= \frac{4\sigma^2}{\sqrt{\pi}} \left\{ 0 + \frac{1}{2} \int_{0}^{\infty} e^{-u^2} du \right\}$$

$$= \frac{4\sigma^2}{\sqrt{\pi}} \left(\frac{1}{2} . \frac{1}{2} \sqrt{\pi} \right) = \frac{4\sigma^2}{\sqrt{\pi}} \left(\frac{1}{4} \sqrt{\pi} \right) = \sigma^2$$

3. $E(x) = \int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2}.u\sigma.e^{-u^2} du + \frac{1}{\sqrt{\pi}} \mu \int_{-\infty}^{\infty} e^{-u^2} du$$

$$= \frac{1}{\sigma\sqrt{\pi}} \int_{-\infty}^{\infty} \sqrt{2}.u\sigma^{-\mu^2} du + \frac{1}{\sqrt{\pi}} \mu \int_{-\infty}^{\infty} e^{-u^2} du$$

$$= \frac{\sqrt{2}\,\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} u\, e^{-u^2} du + \frac{2\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u^2} du$$

$$= \frac{\sqrt{2}\,\sigma}{\sqrt{\pi}} \left[-\frac{1}{2} e^{-u^2} \right]_{-\infty}^{\infty} + \frac{2\mu}{\sqrt{\pi}} . \frac{1}{2}\sqrt{\pi}$$

$$= \frac{\sqrt{2}\,\sigma}{\sqrt{\pi}} [0-0] + \mu$$

$E(x) = 0 + \mu = \mu$

4. Buktikan bahwa $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

Belok pada x = μ + σ dan x = μ - σ

Bukti : Fungsi f(x) mempunyai titik belok pada $f''(x) = 0$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} . \left(-\frac{1}{2}\right) . 2\left(\frac{x-\mu}{\sigma}\right) . \frac{1}{\sigma}$$

$$= \frac{-(x-\mu)}{\sigma^3\sqrt{2\pi}} . e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}} . e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} + \frac{-(x-\mu)}{\sigma^3\sqrt{2\pi}} . e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} . \left(-\frac{1}{2}\right)(2)\left(\frac{x-\mu}{\sigma}\right)\frac{1}{\sigma}$$

$$= -\frac{1}{\sigma^3\sqrt{2\pi}} . e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} + \frac{(x-\mu)^2}{\sigma^5\sqrt{2\pi}} . e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Syarat titik belok, F''(x) = 0

$$-\frac{1}{\sigma^3\sqrt{2\pi}}.e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}+\frac{(x-\mu)^2}{\sigma^5\sqrt{2\pi}}.e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}=0$$

$$\frac{(x-\mu)^2}{\sigma^5\sqrt{2\pi}}.e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}=\frac{1}{\sigma^3\sqrt{2\pi}}.e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$(x-\mu)^2=\sigma^2$$

$$(x-\mu)=\pm\sigma$$

$$x=\mu\pm\sigma$$

Titik belok adalah di $x = \mu \pm \sigma$ (Karena sumbu x adalah asymtot datar)

$$f(\mu+\sigma)=\frac{1}{\sigma\sqrt{2\pi}}.e^{-\frac{1}{2}\left(\frac{\mu+\sigma-\mu}{\sigma}\right)^2}=\frac{1}{\sigma\sqrt{2\pi}}.e^{-\frac{1}{2}}=\frac{1}{\sigma\sqrt{2\pi}.\sqrt{e}}=\frac{1}{\sigma\sqrt{2e\pi}}$$

$$f(\mu-\sigma)=\frac{1}{\sigma\sqrt{2\pi}}.e^{-\frac{1}{2}\left(\frac{\mu-\sigma-\mu}{\sigma}\right)^2}=\frac{1}{\sigma\sqrt{2\pi}}.e^{-\frac{1}{2}(-1)^2}=\frac{1}{\sigma\sqrt{2\pi}.\sqrt{e}}=\frac{1}{\sigma\sqrt{2e\pi}}$$

5. Diketahui : $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

Jika $z=\frac{x-\mu}{\sigma}$, buktikan $g(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z)^2}$

Bukti : Gunakan teorema jacobian transformasi yaitu:
Misalkan x variable random kontinu dengan fungsi peluang f(x), misalkan z = g(x) adalah transformasi 1-1 antara nilai-nilai x dan z sedemikian hingga persamaan z = g(x) mempunyai penyelesaian tunggal yakni x = g^{-1}(z), maka fungsi probabilitas z adalah f(z) = $f\left[g^{-1}(z)\right].\left|\left(g^{-1}(z)'\right)\right|$ adalah jacobian transformasi.

$z=\frac{x-\mu}{\sigma}$ maka x = $\mu+z\sigma$ atau g^{-1}(z) = $\mu+z\sigma$

$$f\left[g^{-1}(z)\right]=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(z)^2}$$

$g^{-1}(z) = \mu + z\sigma$

$$\frac{\partial}{\partial z}\left(g^{-1}(z)\right) = \frac{\partial}{\partial z}(\mu + z\sigma)$$

$$\left|(g^{-1}(z))'\right| = \sigma$$

$$f(z) = f\left[g^{-1}(z)\right].\left|\left(g^{-1}(z)'\right)\right|$$

$$= f\left[\mu + z\sigma\right].\sigma$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2} \,.\, \sigma$$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2}$$