

IRREDUCIBLE COVARIANT REPRESENTATIONS ASSOCIATED WITH AN R-DISCRETE GROUPOID

Roxana VIDICAN*

Unei perechi covariante pozitiv definite (T, ρ) relativ la un grupoid r -discret G , i se poate asocia, printr-o teoremă de tip Stinespring, o reprezentare covariantă $(U, \tilde{\rho})$. Vom stabili o condiție necesară și suficientă de ireductibilitate pentru aceasta din urmă.

Let (T, ρ) be a positive definite covariant pair with respect to an r -discrete groupoid G . Using a theorem of type Stinespring, we can associate to the pair (T, ρ) a covariant representation $(U, \tilde{\rho})$. In this article, we provide a necessary and sufficient condition of irreducibility for the representation $(U, \tilde{\rho})$.

Keywords: r -discrete groupoid, positive definite covariant pair, covariant representation

Introduction

For a suitable amenable, r -discrete, principal groupoid G with the unit space G^0 and the semigroup \mathfrak{S} of its compact and open G -sets, we define a **covariant representation of \mathfrak{S}** , to be a pair (T, ρ) , where

(i) ρ is a $*$ -representation of $C_0(G^0)$ (= the C^* -algebra of complex-valued functions on G^0 , that are continuous and vanishing at infinity) on a (complex and separable) Hilbert space H ;

(ii) $T = \{T(s) \mid s \in \mathfrak{S}\}$ is a family of operators on H such that $\|T(s)\| \leq 1$ and $T(s)T(t) = T(st)$, for $s, t \in \mathfrak{S}$;

* Assist., Department of Mathematics II, University POLITEHNICA of Bucharest, ROMANIA.

- (iii) $T(s)\rho(a) = \rho(a \circ s)T(s)$, for $a \in C_0(G^0)$ and $s \in \mathfrak{S}$; and
- (iv) $T(s) = \rho(\aleph_s)$, for $s \in \mathfrak{S}$ such that $s \subseteq G^0$ (where \aleph_s denote the characteristic function of s).

Definition If T is a function from \mathfrak{S} to $B(H)$ and ρ is a $*$ -representation of $C_0(G^0)$ on H , then (T, ρ) is a **positive definite covariant pair**, if the following conditions are satisfied:

- (i) $T(s)\rho(a) = \rho(a \circ s)T(s)$, for $a \in C_0(G^0), s \in \mathfrak{S}$;
- (ii) $T(s)T(t) = T(st)$, for $s, t \in \mathfrak{S}$;
- (iii) $T(s) = \rho(\aleph_s)$, for $s \in \mathfrak{S}$ with $s \subseteq G^0$; and
- (iv) for each finite collection of points $s_1, \dots, s_n \in \mathfrak{S}$, the operator matrix $(T(s_i^{-1}s_j))_{1 \leq i, j \leq n}$ acting on $H \oplus \dots \oplus H$, is non-negative (this means that, $\sum_{i,j} \langle T(s_i^{-1}s_j)\xi_j, \xi_i \rangle_H \geq 0$, for $(\xi_1, \dots, \xi_n) \in H \oplus \dots \oplus H$).

Theorem Let (T, ρ) be a positive definite covariant pair with values in $B(H)$. Then there are a Hilbert space \tilde{H} , a covariant representation $(U, \tilde{\rho})$ of \mathfrak{S} on \tilde{H} and a Hilbert space isomorphism V mapping H into \tilde{H} such that $\rho(a) = V^* \tilde{\rho}(a) V$, for $a \in C_0(G^0)$ and $T(t) = V^* U(t) V$, for $t \in \mathfrak{S}$.

We shall show a necessary and sufficient condition of irreducibility for the covariant representation $(U, \tilde{\rho})$ from the above theorem in terms of the pair (T, ρ) .

1. Preliminaries

The definitions for the notions of: amenable, r-discrete principal groupoid, G -set, C^* -algebra associated with a locally compact groupoid, can be found in [1], [2], [3], [4] or [5].

Now, to understanding more easily the content of this paper, we shall present, first of all, the most important stages of the proof of the theorem from “Introduction”.

Let \tilde{H}_0 be the set of all functions $f : \mathfrak{T} \rightarrow H$, such that

(1) there is a compact set K_f (depending of f) such that $K_f \subseteq G$ and $f(t) = 0$, for $t \cap K_f = \emptyset$; and

(2) if $t, t_1 \in \mathfrak{T}$ such that $t_1 \subseteq t$, then $f(t_1) = \rho(\mathfrak{N}_{d(t_1)})f(t)$.

We define on \tilde{H}_0 a sesquilinear functional $\langle \cdot, \cdot \rangle$ by the formula

$$\langle f, g \rangle = \sum \langle T(s^{-1}t)f(t), g(s) \rangle_H,$$

where the sum is over any two finite sets $\{t_i\}, \{s_j\}$ in \mathfrak{T} such that $K_f \subseteq \cup t_i$, $K_g \subseteq \cup s_j$, and such that $t_i \cap t_j = s_k \cap s_l = \emptyset$, if $i \neq j, k \neq l$.

Let $N = \{f \in \tilde{H}_0 \mid \langle f, f \rangle = 0\}$. The sesquilinear functional $\langle \cdot, \cdot \rangle$ from above passes to an inner product on \tilde{H}_0 / N . (For this inner product we keep the notation $\langle \cdot, \cdot \rangle$.)

Let \tilde{H} be the completion of \tilde{H}_0 / N in the associated norm. Next, we define

$$V : H \rightarrow \tilde{H} \quad \text{by} \quad V\xi = \tilde{\xi} + N,$$

where $\tilde{\xi}(t) = \rho(E(\mathfrak{N}_{t^{-1}}))\xi$ (E being the conditional expectation from $C^*(G)$ to $C^*(G^0) = C_0(G^0)$). This operator V is an isometry.

Note that, for $f \in \tilde{H}_0$ and $t \in \mathfrak{T}$, if we denote $f_t(s) = f(t^{-1}s)$, $\forall s \in \mathfrak{T}$, we obtain $f_t \in \tilde{H}_0$.

Now, for $q \in \mathfrak{T}$, the operator U on \tilde{H}_0 / N defined by $U(q)(f + N) = f_q + N, \forall f \in \tilde{H}_0$ satisfies the properties $U(q^{-1}) = U(q)^*$ and $U(st) = U(s)U(t)$, for $q, s, t \in \mathfrak{T}$.

Finally, for $a \in C_0(G^0)$, we define $\tilde{\rho}_0(a)$ on \tilde{H}_0 by the formula $(\tilde{\rho}_0(a)f)(t) = \rho(a \circ t^{-1})f(t)$. Denote $\tilde{\rho}(a)(f + N) = \tilde{\rho}_0(a)f + N$, $\tilde{\rho}$ is a *-representation of $C_0(G^0)$ on \tilde{H} .

The pair $(U, \tilde{\rho})$ and the operator V have the properties asked in the theorem.

Observations (i) $(U, \tilde{\rho})$ is just a positive definite covariant pair: let $(\xi_1, \dots, \xi_n) \in H \oplus \dots \oplus H$ and $s_1, \dots, s_n \in \mathfrak{T}$; then

$$0 \leq \sum_{i,j=1}^n \langle T(s_i^{-1}s_j)\xi_j, \xi_i \rangle = \sum_{i,j=1}^n \langle U(s_i^{-1}, s_j)V\xi_j, V\xi_i \rangle .$$

From here, using $VH = \tilde{H}$, we deduce that $(U(s_i^{-1}s_j))_{i,j=1}^n$ is a positive operator matrix.

(ii) As V is an isometry, it results $V^*V = I_H$. Since V is an isomorphism, there exists V^{-1} and $V^*V = V^{-1}V = I_H$, hence $(V^* - V^{-1})V = 0$. Then $VH = \tilde{H}$ implies $V^* = V^{-1}$. Consequently, $T(t) = V^*U(t)V \Leftrightarrow VT(t)V^* = U(t)$, $\forall t \in \mathfrak{S}$.

(iii) If ρ is a non-degenerate representation of $C^*(G^0)$, then the family $\{T(s)\}_{s \in \mathfrak{S}}$ is non-degenerate (i.e. $\overline{\text{Sp}\{T(s)\xi \mid s \in \mathfrak{S}, \xi \in H\}} = H$) and so is $\{U(s)\}_{s \in \mathfrak{S}}$ also: let $\eta \in H$ and $\{a_i\}_i \subseteq C_c(G^0)$, $\{\xi_i\}_i \subseteq H$ such that $\eta = \lim_i \rho(a_i)\xi_i$; fix $\{s_i\}_i \subseteq \mathfrak{S}$ such that $\forall i : \text{supp } a_i \subseteq s_i^{-1}s_i$. Then

$$\eta = \lim_i \rho(\mathfrak{N}_{s_i^{-1}s_i} a_i)\xi_i = \lim_i \rho(\mathfrak{N}_{s_i^{-1}s_i})\rho(a_i)\xi_i = \lim_i T(s_i^{-1}s_i)\rho(a_i)\xi_i .$$

For the second part of the assertion : since V is an isometry, V, V^* are continuous operators. Therefore,

$$\tilde{H} = V(H) = V(\overline{\text{Sp}T(\mathfrak{S})H}) \subseteq \overline{\text{Sp}(VT(\mathfrak{S})H)} = \overline{\text{Sp}(VT(\mathfrak{S})V^*\tilde{H})} = \overline{\text{Sp}(U(\mathfrak{S})\tilde{H})} .$$

(iv) If (T, ρ) is a positive definite covariant pair, then $T^*(s) = T(s^{-1})$, $\forall s \in \mathfrak{S}$: from the proof of the theorem, we know that $U^*(t) = U(t^{-1})$, $\forall t \in \mathfrak{S}$, hence $T^*(t) = V^*U(t^{-1})V = T(t^{-1})$, $\forall t \in \mathfrak{S}$.

(v) If (T, ρ) is a positive definite covariant pair, then $\|T(s)\| \leq 1, \forall s \in \mathfrak{S}$: let $s \in \mathfrak{S}$; then we have $\|T(s)\|^2 = \|T(s)^*T(s)\| = \|T(s^{-1})T(s)\| = \|T(s^{-1}s)\| = \|\rho(\mathfrak{N}_{s^{-1}s})\| \leq \|\mathfrak{N}_{s^{-1}s}\| = 1$.

(vi) For $a \in C_0(G^0)$ and $s \in \mathfrak{S}$, we have $a \circ s = \mathfrak{N}_s a \mathfrak{N}_{s^{-1}}$ (where s from the right side is a G -set, while s from the left side is a notation for the function $u \mapsto d(u \cdot s)$; $u \cdot s$ is the element $x \in s$ with $r(x) = u$), hence $a \circ s^{-1} \circ s =$

$$(a \circ s^{-1}) \circ s = (\mathfrak{N}_{s^{-1}} a \mathfrak{N}_s) \circ s = (\mathfrak{N}_s \mathfrak{N}_{s^{-1}}) a (\mathfrak{N}_s \mathfrak{N}_{s^{-1}}) = \mathfrak{N}_{ss^{-1}} a \mathfrak{N}_{ss^{-1}} = a \circ ss^{-1} . \text{ Now:}$$

$$\rho(a)T(s) = \rho(a)T(ss^{-1}s) = \rho(a)T(ss^{-1})T(s) = \rho(a)\rho(\mathfrak{N}_{ss^{-1}})T(s) = \rho(a \circ (ss^{-1}))T(s) = \rho(a \circ s^{-1} \circ s)T(s) = T(s)\rho(a \circ s^{-1}) .$$

2. Irreducible representations of $C_0(G^0)$

Lema 1 If $(T_1, \rho_1), (T_2, \rho_2)$ are two positive definite covariant pairs of \mathfrak{S} on $B(H)$ with ρ_1, ρ_2 non-degenerate representations of $C_0(G^0)$ such that

$$\forall n \in \mathbb{N}^* \text{ and } \forall s_1, \dots, s_n \in \mathfrak{S} : (T_1(s_i^{-1}s_j))_n \leq (T_2(s_i^{-1}s_j))_n,$$

while $(U_1, \tilde{\rho}_1, \tilde{H}_1)$, respectively $(U_2, \tilde{\rho}_2, \tilde{H}_2)$ are the corresponding covariant representations, then there exists a contraction $W \in B(\tilde{H}_2, \tilde{H}_1)$ (i.e. $\|W\| \leq 1$) satisfying:

(i) $WV_2 = V_1$;

(ii) $WU_2(s) = U_1(s)W, \forall s \in \mathfrak{S}$; and

(iii)

$$W\tilde{\rho}_2(a) = \tilde{\rho}_1(a)W, \forall a \in C_0(G^0) \Leftrightarrow T_1(t)\rho_2(a \circ t^{-1}) = T_1(t)\rho_1(a \circ t^{-1}), \forall t \in \mathfrak{S},$$

$$\forall a \in C_0(G^0).$$

Proof: (i) Let $h_1, \dots, h_n \in H$ and $s_1, \dots, s_n \in \mathfrak{S}$. Then:

$$\begin{aligned} \|\sum_j U_1(s_j)V_1h_j\|^2 &= \sum_{i,j} \langle V_1^*U_1(s_i^{-1}s_j)V_1h_j, h_i \rangle = \sum_{i,j} \langle T_1(s_i^{-1}s_j)h_j, h_i \rangle \leq \sum_{i,j} \langle T_2(s_i^{-1}s_j)h_j, h_i \rangle = \\ &= \|\sum_j U_2(s_j)V_2h_j\|^2. \end{aligned}$$

Defining $W : \tilde{H}_2 \rightarrow \tilde{H}_1$ by $WU_2(t)V_2h = U_1(t)V_1h$, we remarke that W is a contraction with $WV_2 = V_1$.

(ii) For $s, t \in \mathfrak{S}$ and $h \in H$, we have

$$WU_2(s)U_2(t)V_2h = WU_2(st)V_2h = U_1(st)V_1h = U_1(s)U_1(t)V_1h = U_1(s)WU_2(t)V_2h.$$

(iii) If $a \in C_0(G^0), t \in \mathfrak{S}$ and $h \in H$, it follows

$$W\tilde{\rho}_2(a)U_2(t)V_2h = WU_2(t)\tilde{\rho}_2(a \circ t^{-1})V_2h = WU_2(t)V_2\rho_2(a \circ t^{-1})h = U_1(t)V_1\rho_2(a \circ t^{-1})h$$

On the other hand,

$$\tilde{\rho}_1(a)WU_2(t)V_2h = \tilde{\rho}_1(a)U_1(t)V_1h = U_1(t)\tilde{\rho}_1(a \circ t^{-1})V_1h = U_1(t)V_1\rho_1(a \circ t^{-1})h.$$

$$\begin{aligned} \text{Hence, } W\tilde{\rho}_2(a) &= \tilde{\rho}_1(a)W \Leftrightarrow V_1^*U_1(t)V_1\rho_2(a \circ t^{-1})h = V_1^*U_1(t)V_1\rho_1(a \circ t^{-1})h \Leftrightarrow \\ T_1(t)\rho_2(a \circ t^{-1})h &= T_1(t)\rho_1(a \circ t^{-1})h \quad \blacksquare \end{aligned}$$

Definition 2 Let H be a Hilbert space and $U \in B(H)$. U is a **positive operator** (and we shall denote this by $U \geq 0$), if $U = U^*$ and $\langle U(x), x \rangle \geq 0, \forall x \in H$.

Notation 3 For a Hilbert space H and a set M such that $M \subseteq B(H)$, we write M' for the **commutant** of M , i.e.

$$M' = \{U \in B(H) \mid \forall V \in M : UV = VU\}$$

Lemma 4 Let (T, ρ) a covariant positive definite pair on $B(H)$ with ρ a non-degenerate representation of $C_0(G^0)$, $(U, \tilde{\rho})$ the corresponding covariant representation on $B(\tilde{H})$, and V the isomorphism between H and \tilde{H} such that

$$\rho(a) = V^* \tilde{\rho}(a) V, \forall a \in C_0(G^0) \text{ and } T(t) = V^* U(t) V, \forall t \in \mathfrak{S}.$$

For $W \in U(\mathfrak{S})'$ and $S \in \tilde{\rho}(C_0(G^0))'$, we define the applications

$$\Phi_W : \mathfrak{S} \rightarrow B(H) \text{ by } \Phi_W(t) = V^* W U(t) V, \forall t \in \mathfrak{S} \text{ and}$$

$$\Psi_S : C_0(G^0) \rightarrow B(H) \text{ by } \Psi_S(a) = V^* W \tilde{\rho}(a) V, \forall a \in C_0(G^0).$$

Under these conditions, the following assertions are true:

- (i) the maps $W \mapsto \Phi_W$ and $S \mapsto \Psi_S$ are linear and injective;
- (ii) if $W \in U(\mathfrak{S})'$ and $0 \leq W$, then for $n \in \mathbb{N}^*$ and $s_1, \dots, s_n \in \mathfrak{S}$, the operator matrix $(\Phi_W(s_i^{-1} s_j))_{i,j=1}^n$ is non-negative; and
- (iii) if $W \in U(\mathfrak{S})'$ such that $W = W^*$ and $\forall n \in \mathbb{N}^*, \forall s_1, \dots, s_n \in \mathfrak{S}$, the operator matrix $(\Phi_W(s_i^{-1} s_j))_{i,j=1}^n$ is non-negative, then $W \geq 0$.

Proof: (i) It suffices to show, that the linear map $W \mapsto \Phi_W$ is injective. For the injectivity of $S \mapsto \Psi_S$, the proof is analogous.

We assume that $\Phi_W = 0$. Let $s, t \in \mathfrak{S}$ and $h, k \in H$. Then:

$$\langle W U(t) V h, U(s) V k \rangle = \langle U(s^{-1}) W U(t) V h, V k \rangle = \langle V^* W U(s^{-1} t) V h, k \rangle = \langle \Phi_W(s^{-1} t) h, k \rangle = 0$$

hence, $W = 0$.

(ii) We use the observation (i) from ‘‘Preliminaries’’ and the following theorem:

‘‘ If A is an involutive algebra, H a Hilbert space, π a representation of A on H and $T \in B(H)$ such that $T \geq 0$ and $T \in \pi(A)'$, then exists $K \in B(H)$ with $K = K^*$, $K^2 = T$ and $K \in \pi(A)'$. ‘‘

Thus, for $W \in U(\mathfrak{S})'$ with $0 \leq W$, there is $K \in B(\tilde{H})$ such that $K = K^*$, $K^2 = W$ and $K \in U(\mathfrak{S})'$.

Let $s_1, \dots, s_n \in \mathfrak{S}$ and $(\xi_1, \dots, \xi_n) \in H \oplus \dots \oplus H$. The required conclusion follows by

$$\begin{aligned} \sum_{i,j} \langle \Phi_W(s_i^{-1}s_j)\xi_j, \xi_i \rangle &= \sum_{i,j} \langle V^* W U(s_i^{-1}s_j) V \xi_j, \xi_i \rangle = \sum_{i,j} \langle K^2 U(s_i^{-1}s_j) V \xi_j, V \xi_i \rangle = \\ &= \sum_{i,j} \langle U(s_i^{-1}s_j) K V \xi_j, K V \xi_i \rangle \geq 0. \end{aligned}$$

(iii) Let W as in the statement of the lemma. For $n \in N^*$, $s_1, \dots, s_n \in \mathfrak{S}$, $h_1, \dots, h_n \in H$ and $k = U_1(s_1)Vh_1 + \dots + U_n(s_n)Vh_n$, we get

$$\begin{aligned} \langle Wk, k \rangle &= \sum_{i,j} \langle V^* U(s_i^{-1}) W U(s_j) V h_j, h_i \rangle = \sum_{i,j} \langle V^* W U(s_i^{-1}s_j) V h_j, h_i \rangle = \\ &= \sum_{i,j} \langle \Phi_W(s_i^{-1}s_j) h_j, h_i \rangle \geq 0, \end{aligned}$$

hence, $W \geq 0$ ■

Lemma 5 Let (T_1, ρ_1) and (T, ρ) two positive definite covariant pairs with ρ_1, ρ non-degenerate representations of $C_0(G^0)$ such that

$$\begin{aligned} \forall n \in N^*, \forall s_1, \dots, s_n \in \mathfrak{S}: (T_1(s_i^{-1}s_j))_{i,j=1}^n &\leq (T(s_i^{-1}s_j))_{i,j=1}^n \text{ and} \\ \forall t \in \mathfrak{S}, \forall a \in C_0(G^0): T_1(t)\rho_1(a \circ t^{-1}) &= T_1(t)\rho(a \circ t^{-1}). \end{aligned}$$

If $(U, \tilde{\rho})$ is the covariant representation with respect to (T, ρ) , then there is $W \in U(\mathfrak{S})' \cap \tilde{\rho}(C_0(G^0))'$ with the properties: $0 \leq W \leq I$ ($I: \tilde{H} \rightarrow \tilde{H}$ is the identity operator, $I(h) = h, \forall h \in \tilde{H}$), $T_1 = \Phi_W$ and $\rho_1 = \Psi_W$.

Proof: Let $V: H \rightarrow \tilde{H}$ the isomorphism corresponding to the pair (T, ρ) , (U_1, ρ_1) the covariant representation and $V_1: H \rightarrow \tilde{H}_1$ the isomorphism associated with the pair (T_1, ρ_1) .

The lemma 1 assures the existence of a contraction $\tilde{W}: \tilde{H} \rightarrow \tilde{H}_1$ such that $\tilde{W}V = V_1$, $\tilde{W}U(s) = U_1(s)\tilde{W}$, $\forall s \in \mathfrak{S}$ and $\tilde{W}\tilde{\rho}(a) = \tilde{\rho}_1(a)\tilde{W}$, $\forall a \in C_0(G^0)$.

We shall denote $W = \tilde{W}^*\tilde{W}$. Then $W \geq 0$ and for $h \in \tilde{H}$, we have: $\langle Wh, h \rangle = \langle \tilde{W}h, \tilde{W}h \rangle \leq \langle h, h \rangle$, hence $0 \leq W \leq I$.

Choose $s \in \mathfrak{S}$. Then,

$$\begin{aligned} WU(s) &= \tilde{W}^*U_1(s)\tilde{W} = (U_1^*(s)\tilde{W})^*\tilde{W} = (U_1(s^{-1})\tilde{W})^*\tilde{W} = (\tilde{W}U(s^{-1}))^*\tilde{W} = \\ &= U(s^{-1})^*\tilde{W}^*\tilde{W} = U(s)W \end{aligned}$$

implies $W \in U(\mathfrak{S})'$. Analogously, $W \in \tilde{\rho}(C_0(G^0))'$.

Suppose that $t \in \mathfrak{S}$ and $h, k \in H$. Since

$$\begin{aligned} \langle \Phi_W(t)h, k \rangle &= \langle WU(t)Vh, Vk \rangle = \langle \tilde{W}U(t)Vh, \tilde{W}Vk \rangle = \langle U_1(t)\tilde{W}Vh, \tilde{W}Vk \rangle = \\ &= \langle U_1(t)V_1h, V_1k \rangle = \langle V_1^*U_1(t)V_1h, k \rangle = \langle T_1(t)h, k \rangle, \end{aligned}$$

$T_1 = \Phi_W$. Similarly, $\rho_1 = \Psi_W$ ■

Observations 6 (i) If s is a G -set, then $ss^{-1}, s^{-1}s$ are also G -sets and at the same time, they are subsets of G^0 . Moreover, $s = ss^{-1}s$. (If $x \in ss^{-1}s$, then $x = yu$, with $y \in s$ and $u \in s^{-1}s$, hence $x = y \in s$. Conversely, for $x \in s$, we have $x = xx^{-1}x \in ss^{-1}s$.)

(ii) The operator W from lemma 5 satisfies $W^2 = W$, as follows from

$$\begin{aligned} T_1 = \Phi_W &\Rightarrow T_1(st) = \Phi_W(st), \forall s, t \in \mathfrak{S} \Rightarrow \Phi_W(s)\Phi_W(t) = T_1(s)T_1(t) = T_1(st) = \\ &= \Phi_W(st) \Rightarrow V^*WU(s)VV^*WU(t)Vh = V^*WU(st)Vh, \forall s, t \in \mathfrak{S}, \forall h \in H \Rightarrow \\ &V^*W^2U(st)Vh = V^*WU(st)Vh, \forall s, t \in \mathfrak{S}, \forall h \in H \Rightarrow \\ W^2U(st)Vh &= WU(st)Vh, \forall s, t \in \mathfrak{S}, \forall h \in H \Rightarrow W^2U(ss^{-1}s)\tilde{h} = WU(ss^{-1}s)\tilde{h}, \\ \forall s \in \mathfrak{S}, \forall \tilde{h} \in \tilde{H} &\Rightarrow W^2U(s)\tilde{h} = WU(s)\tilde{h}, \forall s \in \mathfrak{S}, \forall \tilde{h} \in \tilde{H}. \end{aligned}$$

Proposition 7 Let (T, ρ) a positive definite covariant pair. Then there is a bijection between the set

$$A = \{W \in U(\mathfrak{S})' \cap \tilde{\rho}(C_0(G^0))' \mid 0 \leq W \leq I, W^2 = W\}$$

and the family B of the positive definite covariant pairs (S, θ) on $B(H)$, which are the property

$$\begin{aligned} \forall n \in N^*, \forall s_1, \dots, s_n \in \mathfrak{S} : (S(s_i^{-1}s_j))_{i,j=1}^n &\leq (T(s_i^{-1}s_j))_{i,j=1}^n \text{ and} \\ S(t)\rho(a \circ t^{-1}) &= S(t)\theta(a \circ t^{-1}), \forall a \in C_0(G^0), \forall t \in \mathfrak{S}. \end{aligned}$$

Proof: Assume $W \in A$. By lemma 4, for $n \in N^*$ and $t_1, \dots, t_n \in \mathfrak{S}$, the operator matrix $(\Phi_W(t_i^{-1}t_j))_{i,j=1}^n$ is non-negative. Because $W \leq I$, we can claim that $\forall n \in N^*$ and $\forall t_1, \dots, t_n \in \mathfrak{S} : (\Phi_{I-W}(t_i^{-1}t_j))_{i,j=1}^n$ is a non-negative matrix, hence $\Phi_W \leq T$.

Using $W^2 = W$, it is clear that $\Phi_W(st) = \Phi_W(s)\Phi_W(t)$, $\forall s, t \in \mathfrak{S}$ and that Ψ_W is a representation of $C_0(G^0)$.

For $s \in \mathfrak{T}$ such that $s \subseteq G^0$, we have $\Phi_w(s) = V^*WU(s)V = V^*W\tilde{\rho}(\mathfrak{N}_s)V = \Psi_w(\mathfrak{N}_s)$. Next, for $a \in C_0(G^0)$ and $s \in \mathfrak{T}$, it follows

$$\begin{aligned}\Phi_w(s)\Psi_w(a) &= V^*WU(s)VV^*W\tilde{\rho}(a)V = V^*W^2U(s)\tilde{\rho}(a)V = \\ &= V^*W^2\tilde{\rho}(a \circ s)U(s)V = V^*W\tilde{\rho}(a \circ s)VV^*WU(s)V = \Psi_w(a \circ s)\Phi_w(s).\end{aligned}$$

Finally, if $t \in \mathfrak{T}$ and $a \in C_0(G^0)$, then

$$\begin{aligned}\Phi_w(t)\rho(a \circ t^{-1}) &= V^*WU(t)V\rho(a \circ t^{-1}) \text{ and} \\ \Phi_w(t)\Psi_w(a \circ t^{-1}) &= V^*WU(t)V V^*W\tilde{\rho}(a \circ t^{-1})V = V^*WU(t)\tilde{\rho}(a \circ t^{-1})V = \\ &= V^*WU(t)V\rho(a \circ t^{-1}).\end{aligned}$$

Thus, $\Phi_w(t)\rho(a \circ t^{-1}) = \Phi_w(t)\Psi_w(a \circ t^{-1})$. All these imply $(\Phi_w, \Psi_w) \in B$.

By the lemma 4, the map $W \mapsto (\Phi_w, \Psi_w)$ is injective. Its surjectivity follows from lemma 5 and observation 6 (ii) ■

Proposition 8 If (T, ρ) is a non-null, positive definite covariant pair such that, for every $(S, \theta) \in B$, there is $\lambda \in C$ with $S = \lambda T$, then the covariant representation $(U, \tilde{\rho})$ associated with (T, ρ) is irreducible (in the sense that, only the subspaces 0 and \tilde{H} are closed and invariant with respect to $U(\mathfrak{T})$ and $\tilde{\rho}(C_0(G^0))$)

Proof: Let (T, ρ) and $(S, \theta) \in B$ as in the statement of the proposition. By lemma 5, we can find $W \in A$ such that $S = \Phi_w$ and $\theta = \Psi_w$. Consequently, there is $\lambda \in C$ such that $\Phi_w = \lambda T$. It results:

$$\begin{aligned}V^*WU(t)V = \lambda T(t), \forall t \in \mathfrak{T} &\Leftrightarrow WU(t) = \lambda VT(t)V^*, \forall t \in \mathfrak{T} \Leftrightarrow \\ WU(t) = \lambda U(t), \forall t \in \mathfrak{T} &\Leftrightarrow W = \lambda I\end{aligned}$$

Let K a closed subspace of \tilde{H} , such that K is invariant with respect to $U(\mathfrak{T})$ and $\tilde{\rho}(C_0(G^0))$. For $h \in \tilde{H}$, we consider the writting $h = h_1 + h_2$, where $h_1 \in K$ and $h_2 \in K^\perp$. We shall denote with P_K , the projection on K , $P_K : \tilde{H} \rightarrow \tilde{H}$, $P_K h = h_1$. It remains to show that, $P_K \in A$ (A from proposition 7). Then it will exist $\lambda \in C$ such that $P_K = \lambda I$. This is equivalent with $K = 0$ or $K = \tilde{H}$.

For $k \in K, h \in K^\perp$ and $s \in \mathfrak{T}$, we have $U(s^{-1})k \in K$, hence $\langle U(s)h, k \rangle = \langle h, U(s^{-1})k \rangle = 0$, from where $U(s)h \in K^\perp$. Thus, if $h \in \tilde{H}$ such that $h = h_1 + h_2$, with $h_1 \in K$ and $h_2 \in K^\perp$, we can write

$$P_K U(s)h = P_K (U(s)h_1 + U(s)h_2) = U(s)h_1 = U(s)P_K h,$$

i.e. $P_K \in U(\mathfrak{S})'$. Analogously, it results that $P_K \in \tilde{\rho}(C_0(G^0))'$.

For $h \in \tilde{H}$ with the same decomposition as above, we find

$$P_K^2 h = P_K h_1 = h_1 = P_K h, \quad \langle P_K h, h \rangle = \langle h_1, h_1 + h_2 \rangle = \langle h_1, h_1 \rangle \geq 0 \quad \text{and} \\ \langle (P_K - I)h, h \rangle = \langle h_1 - h, h \rangle = -\langle h_2, h \rangle = -\langle h_2, h_2 \rangle \leq 0.$$

Hence $P_K^2 = P_K$ and $0 \leq P_K \leq I$ ■

Proposition 9 If (T, ρ) is a non-null, positive definite covariant pair such that ρ is a non-degenerate representation of $C_0(G^0)$ and $(U, \tilde{\rho})$ is irreducible, then for $\forall (S, \theta) \in B$, there is $\lambda \in C$ such that $S = \lambda T$.

Proof: Choose (T, ρ) and $(U, \tilde{\rho})$ with the properties from the statement and $(S, \theta) \in B$. By lemma 5, there is an operator $W \in A$ such that $(S, \theta) = (\Phi_W, \Psi_W)$.

Let $K = W(\tilde{H})$. Then

$$W(\tilde{H}) = \{h \in \tilde{H} \mid \exists k \in \tilde{H} \text{ a.î. } h = Wk\} = \{h \in \tilde{H} \mid h = Wk\}$$

is a linear closed subspace of \tilde{H} . Moreover, it is invariant with respect to $U(\mathfrak{S})$ and $\tilde{\rho}(C_0(G^0))$. Because $(U, \tilde{\rho})$ is irreducible, we deduce that $W(\tilde{H})$ is 0 or \tilde{H} , hence $W = \lambda I$, for some $\lambda \in C$. Finally, $\forall t \in \mathfrak{S}: S(t) = \Phi_W(t) = V^* \lambda U(t) V = \lambda T(t)$ ■

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