

Dilations on Hilbert C^* - modules for C^* - dynamical systems

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Abstract

In this paper we investigate the dilations of a contractive covariant representation of a C^* - dynamical system on Hilbert C^* - modules.

1 Introdudere

Hilbert C^* -modules are generalizations of Hilbert spaces by allowing the inner- product to take values in a C^* - algebra rather than in the field of complex numbers, but there are some differences between these two classes. For example, each closed submodule of a Hilbert space is complemented, but a closed submodule of a Hilbert C^* -module is not complemented in general [1, chapter 3].

Definition 1.1. A **pre-Hilbert A -module** is a complex vector space E which is also a right A -module, compatible with the complex algebra structure, equipped with an A -valued inner product $\langle \cdot, \cdot \rangle : E \times E \rightarrow A$ which is $\mathbb{C}$ -and A -linear in its second variable and satisfies the following relations:

1. $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$;
2. $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$;
3. $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.

We say that E is a **Hilbert A -module** if E is complete with respect to the topology determined by the norm $\|\cdot\|$ given by $\|\xi\| = \sqrt{\|\langle \xi, \xi \rangle\|}$.

Given two Hilbert C^* -modules E and F over a C^* -algebra B , the Banach space of all bounded module homomorphisms from E to F is denoted by $\mathcal{B}_B(E, F)$. The subset of $\mathcal{B}_B(E, F)$ consisting of all adjointable module homomorphisms from E to F (that is, $T \in \mathcal{B}_B(E, F)$ such that there is $T^* \in \mathcal{B}_B(F, E)$ satisfying $\langle \eta, T\xi \rangle = \langle T^*\eta, \xi \rangle$ for all $\xi \in E$ and for all $\eta \in F$) is denoted by $\mathcal{L}_B(E, F)$. We will write $\mathcal{B}_B(E)$ for $\mathcal{B}_B(E, E)$ and $\mathcal{L}_B(E)$ for $\mathcal{L}_B(E, E)$.

In general $\mathcal{B}_B(E, F) \neq \mathcal{L}_B(E, F)$.

Given a countable family $\{E_n\}_n$ of Hilbert B -modules, the vector space $\bigoplus_n E_n = \{(\xi_n)_n / \xi_n \in E_n, \sum_n \langle \xi_n, \xi_n \rangle \text{ converges in } B\}$ becomes a Hilbert B -module with $(\xi_n)_n \cdot a = (\xi_n a)_n$ and $\langle (\xi_n)_n, (\eta_n)_n \rangle = \sum_n \langle \xi_n, \eta_n \rangle$.

A submodule E_0 of E is complemented if $E_0 \oplus E_0^\perp = E$, where

$$E_0^\perp = \{\eta \in E / \langle \eta, \xi \rangle = 0, \forall \xi \in E_0\}.$$

A representation of a C^* -algebra A on a Hilbert C^* -module E over B is a C^* -morphism Φ from A to $\mathcal{L}_B(E)$. The representation Φ is nondegenerate if $\Phi(A)E$ is dense in E .

Definition 1.2. Let A and B be two C^* -algebras and E a Hilbert B -module. Let $M_n(A)$ denote the $*$ -algebra of all $n \times n$ matrices over A with the algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of A . A **completely positive linear map** is a linear map $\rho : A \rightarrow \mathcal{L}_B(E)$ such that the linear map $\rho^{(n)} : M_n(A) \rightarrow M_n(\mathcal{L}_B(E))$ defined by

$$\rho^{(n)} \left([a_{ij}]_{i,j=1}^n \right) = [\rho(a_{ij})]_{i,j=1}^n$$

is positive for any positive integer n .

Definition 1.3. A completely positive linear map is **strict** if $(\rho(e_\lambda))_\lambda$ is strictly Cauchy in $\mathcal{L}_B(E)$, for some approximate unit $(e_\lambda)_\lambda$ of A .

In this paper we investigate the dilations of a contractive covariant representation of a C^* -dynamical system on Hilbert C^* -modules. We prove that if $\alpha : A \rightarrow A$ is an injective C^* -morphism of C^* -algebras which has a strict positive transfer operator τ , then any covariant representation (φ, T, E) of the pair (A, α) on the Hilbert C^* -module E admits a coisometric dilation

(Φ, V, F) adapted to τ and an isometric dilation (Ψ, W, G) . These extend some results of Muhly and Solel, arXIV:math.OA/0509506 V1/22 sept 2005 in the context of Hilbert C^* -modules and without assuming that the C^* -algebra A is unital.

2 The main results

Definition 2.1. Let A be a C^* -algebra and let $\alpha: A \rightarrow A$ be an injective C^* -morphism. A **strict transfer operator for α** is a strict completely positive linear map $\tau: A \rightarrow A$ such that $\tau \circ \alpha = \text{id}_A$.

Proposition 2.1. Let A be a C^* -algebra, let $\varphi: A \rightarrow \mathcal{L}_B(E)$ be a nondegenerate representation of A on the Hilbert C^* -module E over a C^* -algebra B and let $\alpha: A \rightarrow A$ be an injective C^* -morphism which has a strict transfer operator τ .

1. There is a Hilbert B -module E_τ , a representation Φ_τ of A on E_τ and an adjointable operator $V_\tau: E \rightarrow E_\tau$ such that

$$(a) \quad \varphi(a) = V_\tau^* \Phi_\tau(\alpha(a)) V_\tau, \text{ for all } a \in A$$

$$(b) \quad \varphi(\tau(a)) = V_\tau^* \Phi_\tau(a) V_\tau, \text{ for all } a \in A$$

$$(c) \quad \Phi_\tau(A) V_\tau E \text{ is dense in } E_\tau$$

2. If Φ is a representation of A on a Hilbert B -module F and $V: E \rightarrow F$ is an adjointable operator such that

$$(a) \quad \varphi(a) = V^* \Phi(\alpha(a)) V, \text{ for all } a \in A$$

$$(b) \quad \varphi(\tau(a)) = V^* \Phi(a) V, \text{ for all } a \in A$$

$$(c) \quad \Phi(A) V E \text{ is dense in } F$$

then there is a unitary operator $U: E_\tau \rightarrow F$ such that

$$(i) \quad U \Phi_\tau(a) = \Phi(a) U, \text{ for all } a \in A$$

and

$$(ii) \quad UV_\tau = V.$$

Proof. 1. Let $\rho = \varphi \circ \tau$. Since φ is a nondegenerate representation of A , there is a morphism $\bar{\varphi}: M(A) \rightarrow \mathcal{L}_B(E)$ which is strictly continuous on the unit ball and $\bar{\varphi}|_A = \varphi$ [1, Proposition 2.5], and since τ is strictly there is $\bar{\tau}: M(A) \rightarrow M(A)$ a completely positive linear map which is strictly continuous on the unit ball and $\bar{\tau}|_A = \tau$ [1, Corollary 5.7]. Then $\bar{\varphi} \circ \bar{\tau}: M(A) \rightarrow \mathcal{L}_B(E)$ is a completely positive map which is continuous on the unit ball and $\bar{\varphi} \circ \bar{\tau}|_A = \rho$. From these facts and [1, Corollary 5.7] we conclude that ρ is a strict completely positive linear map from A to $\mathcal{L}_B(E)$.

Let $(\Phi_\tau, V_\tau, E_\tau)$ be the KSGNS representation of A associated with ρ [1, theorem 5.6]. Thus, we showed that there is a representation Φ_τ of A on a Hilbert B -module E_τ and an isometry $V_\tau: E \rightarrow E_\tau$ which verify the relations (b) and (c).

Let $a \in A$. Then

$$\varphi(a) = \varphi(\tau(\alpha(a))) = \rho(\alpha(a)) = V_\tau^* \Phi_\tau(\alpha(a)) V_\tau$$

and the assertion is proved.

2. By [1, Theorem 5.6], there is a unitary operator $U: E_\tau \rightarrow F$ which verifies the relations (i) and (ii). $\square$

Remark 2.1. Taking into account the definitions of τ and Φ_τ [see proof of Theorem 5.6, 1] it is not difficult to check that $V_\tau V_\tau^*$ commutes with $\Phi_\tau(\alpha(a))$.

Definition 2.2. The representation $(\Phi_\tau, V_\tau, E_\tau)$ of A constructed above is called **the extension of (φ, E) adapted to τ** .

Remark 2.2. The extension of a representation φ adapted to τ is unique up to a unitary equivalence.

Definition 2.3. Let A be a C^* -algebra, let $\alpha: A \rightarrow A$ be an injective C^* -morphism. A **contractive** (resp. **isometric**, resp. **coisometric**, resp. **unitary**) **covariant representation** of the pair (A, α) on a Hilbert C^* -module is a triple (φ, T, E) consisting of a representation φ of A on a Hilbert C^* -module E over B and a contraction (resp. isometric, resp. coisometric, resp. unitary) operator T in $\mathcal{L}_B(E)$ such that

$$T\varphi(\alpha(a)) = \varphi(a)T$$

Definition 2.4. Let A be a C^* -algebra, let $\alpha: A \rightarrow A$ be an injective C^* -morphism and let (φ, T, E) be a contractive covariant representation of (A, α) . A **coisometric** (resp. **isometric**, resp. **unitary**) **dilation**

is a coisometric (resp. isometric, resp. unitary) covariant representation (Φ, V, F) on a Hilbert C^* -module F over B containing E as a complemented submodule such that $\Phi(a)E \subseteq E$ and $\Phi(a)|_E = \varphi(a)$ for all $a \in A$, $VE \subseteq E$ and $P_E V^n|_E = T^n$, where P_E is the projection of F on E .

Given a contractive covariant representation (φ, T, E) of a pair (A, α) such that α has a transfer operator τ , we construct a coisometric dilation of (φ, T, E) using the extension of φ adapted to τ .

The construction is done inductively and it is in the same manner as in [3, Theorem 1.2].

Given (φ, T, E) a contractive covariant representation of (A, α) , we can choose $(\Phi_\tau, V_\tau, E_\tau)$ as in Proposition 2.1. and we will use the following notations:

$$\Delta_{T^*} = (I - TT^*)^{\frac{1}{2}}$$

$$\mathcal{D}_{T^*} = \overline{\Phi_\tau(A)V_\tau\Delta_{T^*}E} \subseteq E_\tau$$

$$D_{T^*} = \Delta_{T^*}V_\tau^*|_{\mathcal{D}_{T^*}}$$

Δ_{T^*} is called the **defect operator** of T^* and $\mathcal{D}_{T^*}$ is called the associated **defect space**.

Theorem 2.1. *Let A be a C^* -algebra, let $\alpha: A \rightarrow A$ be an injective C^* -morphism which has a transfer operator τ and let (φ, T, E) be a nondegenerate contractive covariant representation of (A, α) on a Hilbert C^* -module E over a C^* -algebra B . Then (φ, T, E) has a coisometric dilation adapted to τ , (Φ, V, F) . Moreover, $VE \subseteq E$.*

Proof. Let $(\Phi_\tau, V_\tau, E_\tau)$ be the extension of φ adapted to τ constructed in Proposition 2.1. Clearly, $\Phi_\tau(A)\mathcal{D}_{T^*} \subseteq \mathcal{D}_{T^*}$ and $\widehat{\varphi}: A \rightarrow \mathcal{L}_B(\mathcal{D}_{T^*})$ defined by $\widehat{\varphi}(a) = \Phi_\tau(a)|_{\mathcal{D}_{T^*}}$ is a nondegenerate representation of A .

In the same manner as in the proof of [Theorem 1.2, 3], we conclude that $\left(\begin{bmatrix} \varphi & 0 \\ 0 & \widehat{\varphi} \end{bmatrix}, \begin{bmatrix} T & D_{T^*} \\ 0 & 0 \end{bmatrix}, E \oplus \mathcal{D}_{T^*} \right)$ is a contractive covariant representation of A such that $\begin{bmatrix} T & D_{T^*} \\ 0 & 0 \end{bmatrix}$ is a partial isometry which restricted to E gives (φ, T, E) .

This was the first step in the inductive construction.

Now, applying Proposition 2.1 to the representation $\widehat{\varphi}$ of A on $\mathcal{D}_{T^*}$, we obtain a Hilbert C^* -module E_1 over B , an adjointable operator $V_1: \mathcal{D}_{T^*} \rightarrow E_1$ and a representation $\Phi_1: A \rightarrow \mathcal{L}_B(E_1)$ such that $V_1^* \Phi_1(\alpha(a)) V_1 = \widehat{\varphi}(a)$, for all $a \in A$.

Set :

$$\mathcal{D}_1 = \overline{\Phi_1(A) V_1 \mathcal{D}_{T^*}} \subseteq E_1$$

$$D_1 = V_1^*|_{\mathcal{D}_1}$$

and

$$\widehat{\Phi}_1: A \rightarrow \mathcal{L}_B(\mathcal{D}_1),$$

defined by

$$\widehat{\Phi}_1(a) = \Phi_1(a)|_{\mathcal{D}_1}$$

Inductively, we obtain the sequences $(\mathcal{D}_k)_{k \geq 1}$, $(E_k)_{k \geq 1}$, $(V_k)_{k \geq 1}$, $(\Phi_k)_{k \geq 1}$, $(\widehat{\Phi}_k)_{k \geq 1}$, $(D_k)_{k \geq 1}$, where for $k > 1$:

$\mathcal{D}_k$ and E_k are Hilbert C^* -modules such that $\mathcal{D}_k = \overline{\Phi_k(A) V_k \mathcal{D}_{k-1}} \subseteq E_k$,
 $V_k: \mathcal{D}_{k-1} \rightarrow E_k$ is an adjointable operator and $D_k = V_k^*|_{\mathcal{D}_k}$,
 $\Phi_k: A \rightarrow \mathcal{L}_B(E_k)$ is a representation such that $V_k^* \Phi_k(\alpha(a)) V_k = \Phi_{k-1}(a)$,
 $\widehat{\Phi}_k: A \rightarrow \mathcal{L}_B(\mathcal{D}_k)$ is a representation such that $\widehat{\Phi}_k(a) = \Phi_k(a)|_{\mathcal{D}_k}$.

$$\text{Set } F = E \oplus \mathcal{D}_{T^*} \oplus \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \dots, \Phi = \begin{bmatrix} \varphi & & & & \\ & \widehat{\varphi} & & & \\ & & \widehat{\Phi}_1 & & \\ & & & \widehat{\Phi}_2 & \\ & & & & \ddots \end{bmatrix} \text{ and}$$

$$V = \begin{bmatrix} T & D_{T^*} & & & \\ & 0 & D_1 & & \\ & & 0 & D_2 & \\ & & & 0 & \ddots \\ & & & & \ddots \end{bmatrix}.$$

Clearly, F is a Hilbert B -module which contains E as a complemented submodule, Φ is a representation of A on F such that $\Phi(A)E \subseteq E$ and $\Phi(a)|_E = \varphi$ and V is an adjointable operator in $\mathcal{L}_B(F)$ such that $VE \subseteq E$ and $V|_E = T$.

Simple calculations show that $V\Phi(\alpha(a)) = \Phi(a)V$ for all $a \in A$, $VV^* = \text{id}_F$ and $P_EV^n|_E = T^n$. Therefore (Φ, V, F) is a coisometric dilation adapted to τ of (φ, T, E) . $\square$

Proposition 2.2. *Let A be a C^* - algebra, let $\alpha: A \rightarrow A$ be an injective C^* - morphism and let (φ, T, E) be a contractive covariant representation of (A, α) on a Hilbert C^* - module E over a C^* - algebra B . Then (φ, T, E) has an isometric dilation (Ψ, W, F) . Further, if T is coisometric, then W is coisometric.*

Proof. Let $\Delta_T = (I - T^*T)^{\frac{1}{2}}$, the defect operator of T .

From $T\varphi(\alpha(a)) = \varphi(a)T$, for all $a \in A$ and taking into account that φ and α are C^* - morphisms, we deduce that $T^*T\varphi(\alpha(a)) = \varphi(\alpha(a))T^*T$ for all $a \in A$ and so $\Delta_T\varphi(\alpha(a)) = \varphi(\alpha(a))\Delta_T$ for all $a \in A$. Hence $(\varphi \circ \alpha)(A)\mathcal{D}_T \subseteq \mathcal{D}_T$, where $\mathcal{D}_T$ is the defect space of T .

Let $F = E \oplus \mathcal{D}_T \oplus \mathcal{D}_T \oplus \dots$. The map $\Psi: A \rightarrow \mathcal{L}_B(F)$ defined by

$$\Psi = \begin{bmatrix} \varphi & & & & \\ & \varphi \circ \alpha|_{\mathcal{D}_T} & & & \\ & & \varphi \circ \alpha^2|_{\mathcal{D}_T} & & \\ & & & \varphi \circ \alpha^3|_{\mathcal{D}_T} & \\ & & & & \ddots \end{bmatrix}$$

is a representation of A on F . Clearly, $\Psi(A)E \subseteq E$ and $\Psi(a)|_E = \varphi(a)$ for all $a \in A$.

Let

$$W = \begin{bmatrix} T & & & & \\ \Delta_T & 0 & & & \\ & I_{\mathcal{D}_T} & 0 & & \\ & & I_{\mathcal{D}_T} & & \\ & & & \ddots & \ddots \end{bmatrix}$$

Clearly, $P_E W^n|_E = T^n$ for all positive integer n .

A simple calculation shows that W is an isometry and moreover if T is a coisometry, W is a unitary.

It is not difficult to check that $W\Psi(\alpha(a)) = \Psi(a)W$ for all $a \in A$ and the proposition is proved. $\square$

As in the case of dilation on Hilbert space [3, Corollary 1.3] we obtain the following corollary.

Corollary 2.1. *Let A be a C^* - algebra, let $\alpha: A \rightarrow A$ be an injective C^* -morphism which has a strict transfer operator τ and let (φ, T, E) be a contractive covariant representation of (A, α) . Then (φ, T, E) has a unitary dilation (Ψ, W, F) adapted to τ .*

Proof. By Theorem 2.1, (φ, T, E) has a coisometric dilation (Φ, V, F) adapted to τ . Applying Proposition 2.2 to coisometric covariant representation (Φ, V, F) of (A, α) , we obtain a unitary dilation (Ψ, U, G) of (Φ, V, F) . It is not difficult to check that (Ψ, U, G) is a unitary dilation adapted to τ of (φ, T, E) . $\square$

3 References

1. E. C. Lance, *Hilbert C^* - modules. A toolkit for operator algebraists*, London Mathematical Society Lecture Note Series 210, Cambridge University Press, Cambridge (1995).
2. M. McAsey, P. Muhly, *Representations of nonselfadjoint crossed products*, Proc. London Math. Soc. (3) **47** (1983), 128-144.
3. P. Muhly, B. Solel, *Extensions and dilations for C^* - dynamical systems*, arXIV:math.OA/0509506 V1/22 sept 2005.
4. P. Muhly, B. Solel, *Tensor algebras over C^* - correspondences: representations, dilations and C^* - envelopes*, J. Funct. Anal. **158** (1998), 389-457.

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