REPRESENTATIONS ASSOCIATED WITH COMPLETELY n-POSITIVE LINEAR MAPS BETWEEN C^*-ALGEBRAS

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Abstract. Let A and B be two C^* -algebras. A completely n -positive linear map from A to B is an $n \times n$ matrix $[\rho_{ij}]_{i,j=1}^n$ whose elements are bounded linear maps from A to B and the linear map $\rho : M_n(A) \rightarrow M_n(B)$ ($M_n(A)$ denotes the C^* -algebra of all $n \times n$ matrices with elements in A) defined by $\rho\left([a_{ij}]_{i,j=1}^n\right) = [\rho_{ij}(a_{ij})]_{i,j=1}^n$ is completely positive.

Suen [Proc. Amer. Math. Soc. 112(1991),3, 709-712] showed that any completely n -positive linear map $\rho = [\rho_{ij}]_{i,j=1}^n$ from A to $L(H)$, the C^* -algebra of all bounded linear operators on the Hilbert space H, has the form $\rho_{ij}(\cdot) = V^* T_{ij} \Phi(\cdot) V$, $i, j = 1, 2, ..., n$, where Φ is a representation of A on a Hilbert space K, $V : H \rightarrow K$ is an isometry and $[T_{ij}]_{i,j=1}^n$ is a positive element in the C^*-algebra of $n \times n$-matrices with elements in $\Phi(A)'$, the commutant of $\Phi(A)$.

In this work, we extend the result of Suen for completely n -positive linear maps between C^* -algebras.

1. Introduction and preliminaries

Let A and B be two unital C^*-algebras.

Let $M_n(A)$ denote the $*$ -algebra of all $n \times n$ matrices over A with the algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of A.

Definition 1.1. *A completely positive linear map from A to B is a linear map $\rho : A \rightarrow B$ such that the linear map $\rho^{(n)} : M_n(A) \rightarrow M_n(B)$ defined by*

$$\rho^{(n)}\left([a_{ij}]_{i,j=1}^n\right) = [\rho(a_{ij})]_{i,j=1}^n$$

is positive for any positive integer n. We say that ρ is unital if $\rho(1_A) = 1_B$, where 1_A is the unity of A and 1_B is the unity of B.

Definition 1.2. *A completely n -positive linear map from A to B is an $n \times n$ matrix $\left[\rho_{ij}\right]_{i,j=1}^n$ of linear maps from A to B such that the map $\rho : M_n(A) \rightarrow M_n(B)$ defined by*

$$\rho\left([a_{ij}]_{i,j=1}^n\right) = \left[\rho_{ij}(a_{ij})\right]_{i,j=1}^n$$

is completely positive. If ρ is a unital completely positive linear map, we say that $\left[\rho_{ij}\right]_{i,j=1}^n$ is a unital completely n -positive linear map.

Mathematical Subject Classification: 46L05; 46L08

This research was supported by grant CNCSIS (Romanian National Council for Research in High Education)-code A 1065/2006.

In [**8**], Suen showed that given a unital completely n -positive linear map $\left[\rho_{ij}\right]_{i,j=1}^{n}$ from A to $L(H)$, the C^{*}-algebra of all bounded linear operators on the Hilbert space H, there is a representation Φ of A on a Hilbert space K, an isometry $V : H \rightarrow K$ and a positive element $[T_{ij}]_{i,j=1}^{n}$ in $M_n(\Phi(A)')$, where $\Phi(A)'$ is the commutant of $\Phi(A)$ in $L(K)$ such that

$$\rho_{ij}(a) = V^{*}T_{ij}\Phi(a)V$$

for all $a \in A$ and for all $i, j = 1, ..., n$.

In this paper, we extend the result of Suen for unital completely n -positive linear maps from a C^{*} -algebra A to $\mathcal{L}(E)$, the C^{*}-algebra of all adjointable operators on a Hilbert C^{*}-module over B. Moreover, we prove that Suen's construction is unique up to unitary equivalence.

Hilbert C^{*}-modules are generalizations of Hilbert spaces by allowing the inner-product to take values in a C^{*}-algebra rather than in the field of complex numbers.

Definition 1.3. *A pre-Hilbert A-module is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle\cdot,\cdot\rangle : E \times E \rightarrow A$ which is $\mathbb{C}$ -and A-linear in its second variable and satisfies the following relations:*

1. $\langle\xi,\eta\rangle^{*} = \langle\eta,\xi\rangle$ *for every* $\xi, \eta \in E$;
2. $\langle\xi,\xi\rangle \geq 0$ *for every* $\xi \in E$;
3. $\langle\xi,\xi\rangle = 0$ *if and only if* $\xi = 0$.

We say that E is a Hilbert A-module if E is complete with respect to the topology determined by the norm $\|\cdot\|$ given by $\|\xi\| = \sqrt{\|\langle\xi,\xi\rangle\|}$.

A C^{*}-algebra A is a Hilbert C^{*}-module over A with the inner-product defined by $\langle a, b\rangle = a^{*}b$ for a and b in A.

Given two Hilbert A -modules E and F, the Banach space of all bounded module homomorphisms from E to F is denoted by $\mathcal{B}(E, F)$. The subset of $\mathcal{B}(E, F)$ consisting of all adjointable module homomorphisms from E to F (that is, $T \in \mathcal{B}(E, F)$ such that there is $T^{*} \in \mathcal{B}(F, E)$ satisfying $\langle\eta, T\xi\rangle = \langle T^{*}\eta, \xi\rangle$ for all $\xi \in E$ and for all $\eta \in F$) is denoted by $\mathcal{L}(E, F)$. We will write $\mathcal{B}(E)$ for $\mathcal{B}(E, E)$ and $\mathcal{L}(E)$ for $\mathcal{L}(E, E)$.

For a C^{*} -algebra A, the C^{*} -algebra $\mathcal{L}(A)$ is isomorphic with $M(A)$, the multiplier algebra of A, and so A can be identified with a C^{*} -subalgebra of $\mathcal{L}(A)$.

In general, $\mathcal{L}(E, F) \neq \mathcal{B}(E, F)$. So the theory of Hilbert C^{*}-modules is different from the theory of Hilbert spaces.

The Banach space $E^{\#}$ of all bounded module homomorphisms from E to A becomes a right A -module with the action of A on $E^{\#}$ defined by $(aT)(\xi) = a^{*}(T\xi)$ for $a \in A$, $T \in E^{\#}$ and $\xi \in E$. We say that E is self-dual if $E^{\#} = E$ as right A -modules.

If E and F are self-dual, then $\mathcal{B}(E, F) = \mathcal{L}(E, F)$ [**5**, Proposition 3.4].

Suppose that A is a W^{*}-algebra. Then the A -valued inner- product on E extends to an A -valued inner-product on $E^{\#}$ and in this way $E^{\#}$ becomes a self-dual Hilbert A -module [**5**, Theorem 3.2]. Moreover, any bounded module homomorphism T from E to F extends uniquely to a bounded homomorphism $\widetilde{T}$ from $E^{\#}$ to $F^{\#}$ [**5**, Proposition 3.6].

A representation of a C^{*}-algebra A on a Hilbert C^{*}-module E over a C^{*}-algebra B is a $*$- morphism Φ from A to $\mathcal{L}(E)$.

Construction 1.4. *([**4**, **5**, **9**]) Let E be a Hilbert C^*-module over a C^*-algebra B. The algebraic tensor product $E \otimes_{alg} B^{**}$, where B^{**} is the enveloping W^*-algebra of B, becomes a right B^{**}-module if we define $(\xi \otimes b)\, c = \xi \otimes bc$, for $\xi \in E$, and $b, c \in B^{**}$.*

The map $[\cdot,\cdot] : (E \otimes_{alg} B^{**}) \times (E \otimes_{alg} B^{**}) \to B^{**}$ defined by

$$\left[\sum_{i=1}^{n} \xi_i \otimes b_i, \sum_{j=1}^{m} \eta_j \otimes c_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{m} b_i^* \left\langle \xi_i, \eta_j \right\rangle c_j$$

is a B^{**}-valued inner-product on $E \otimes_{alg} B^{**}$ and the quotient module $E \otimes_{alg} B^{**}/N_E$, where $N_E = \{\zeta \in E \otimes_{alg} B^{**}; [\zeta,\zeta] = 0\}$, becomes a pre-Hilbert B^{**}-module. The Hilbert C^*-module $\overline{E \otimes_{alg} B^{**}/N_E}$ obtained by the completion of $E \otimes_{alg} B^{**}/N_E$ with respect to the norm induced by the inner product $[\cdot,\cdot]$ is called the extension of E by the C^*-algebra B^{**}. Moreover, E can be regarded as a B-submodule of $\overline{E \otimes_{alg} B^{**}/N_E}$, since the map $\xi \mapsto \xi \otimes 1_{B^{**}} + N_E$ from E to $\overline{E \otimes_{alg} B^{**}/N_E}$ is an isometric inclusion.

The self-dual Hilbert B^{**}-module $\left(\overline{E \otimes_{alg} B^{**}/N_E}\right)^{\#}$ is denoted by $\widetilde{E}$, and we can consider E as embedded in $\widetilde{E}$ without making distinction.

Let $T \in \mathcal{B}_B(E,F)$. For $b_1, ..., b_m \in B^{**}$ and $\xi_1, ..., \xi_m$ in E we denote by b the element in $(B^{**})^m$ whose components are $b_1, ..., b_m$, by X the matrix in $M_n(B^{**})$ whose (i,j) -entry is $\left\langle \xi_i, \xi_j \right\rangle$ and by X_T the matrix in $M_n(B^{**})$ whose (i,j) -entry is $\left\langle T\xi_i, T\xi_j \right\rangle$. By Lemma 4.2 in [**3**], $0 \le X_T \le \|T\|\, X$. Identifying $M_n(B^{**})$ with $\mathcal{L}((B^{**})^n)$, we have

$$\begin{aligned}\left[\sum_{i=1}^{m} T\xi_i \otimes b_i, \sum_{i=1}^{m} T\xi_i \otimes b_i\right] &= \sum_{i,j=1}^{m} b_i^* \left\langle T\xi_i, T\xi_j \right\rangle b_j = \langle b, X_T b\rangle \\ &\le \|T\| \langle b, Xb\rangle = \|T\| \left[\sum_{i=1}^{m} \xi_i \otimes b_i, \sum_{i=1}^{m} \xi_i \otimes b_i\right].\end{aligned}$$

Therefore T extends uniquely to a bounded module homomorphism $\widehat{T}$ from $\overline{E \otimes_{\text{alg}} B^{**}/N_E}$ to $\overline{F \otimes_{\text{alg}} B^{**}/N_F}$ such that

$$\widehat{T}\left(\sum_{i=1}^{m} \xi_i \otimes b_i\right) = \sum_{i=1}^{m} T\xi_i \otimes b_i$$

and by Proposition 3.6 in [**5**], this extends uniquely to a bounded module homomorphism $\widetilde{T}$ from $\widetilde{E}$ to $\widetilde{F}$ such that $\|T\| = \left\|\widetilde{T}\right\|$.

Remark 1.5. *Any element $T \in \mathcal{B}(E,F)$ extends uniquely to an element $\widetilde{T} \in \mathcal{B}(\widetilde{E},\widetilde{F})$ such that $\|T\| = \left\|\widetilde{T}\right\|$. Moreover, $\widetilde{TS} = \widetilde{T}\widetilde{S}$ for all $T \in \mathcal{B}(E,F)$ and $S \in \mathcal{B}(F,E)$, and if $T \in \mathcal{L}(E,F)$, then $\widetilde{T^*} = \widetilde{T}^*$.*

Remark 1.6. *A representation Φ of a C^*-algebra A on a Hilbert C^*-module E over a C^*-algebra B induces a representation $\widetilde{\Phi}$ of A on $\widetilde{E}$ defined by $\widetilde{\Phi}(a) = \widetilde{\Phi(a)}$ for all $a \in A$.*

2. Representations associated with completely n -positive linear maps

Let A be a unital C^* -algebra, let E be a Hilbert C^* -module over B and let $\rho : A \to \mathcal{L}(E)$ be a unital completely positive linear map. It is well known that there is a representation Φ_ρ of A on a Hilbert B -module E_ρ and an isometry $V_\rho : E \to E_\rho$ such that

$$\rho\left(a\right) = V_\rho^* \Phi_\rho\left(a\right) V_\rho$$

for all $a \in A$ and $\{ \Phi_\rho\left(a\right) V_\rho \xi ; a \in A, \xi \in E\}$ spans a dense subspace of E_ρ [**3**, Theorem 5.6]. The triple $(\Phi_\rho, E_\rho, V_\rho)$ is called the KSGNS (Kasparov, Stinespring, Gel'fand, Naimark, Segal) construction associated with ρ. Moreover, the KSGNS construction associated with ρ is unique up to a unitary equivalence. Any completely positive linear map ρ from A to B induces a representation $\widetilde{\Phi_\rho}$ of A on a self -dual Hilbert B^{**}-module $\widetilde{E_\rho}$.

Let $C\left(\rho\right)$ be the C^* -subalgebra of $\mathcal{L}\left(\widetilde{E_\rho}\right)$ generated by $\{T \in \mathcal{L}\left(\widetilde{E_\rho}\right); T\widetilde{\Phi_\rho}(a) = \widetilde{\Phi_\rho}(a)T, \widetilde{V_\rho}^* T\widetilde{\Phi_\rho}(a)\widetilde{V_\rho}|_E \in \mathcal{L}(E)$ for all $a \in A\}$. For any positive element $T \in C\left(\rho\right)$, the map $\rho_T : A \to \mathcal{L}(E)$ defined by

$$\rho_T\left(a\right) = \widetilde{V_\rho}^* T \widetilde{\Phi_\rho}(a)\widetilde{V_\rho}|_E$$

is completely positive [**2**, Lemma 2.3]. Moreover, the map $T \to \rho_T$ is an affine order isomorphism of partially ordered set of operators $\{T \in C\left(\rho\right); 0 \le T \le I_{\mathcal{L}\left(\widetilde{E_\rho}\right)}\}$ onto $[0, \rho] = \{\theta \in CP(A, E); \theta \le \rho\}$, where $CP(A, E)$ is the set of completely positive linear maps from A to $\mathcal{L}(E)$ and $\theta \le \rho$ if $\rho - \theta$ is an element in $CP(A, E)$ [**2**, Theorem 2.6].

Let $\rho : M_2\left(A\right) \longrightarrow M_2\left(\mathcal{L}\left(E\right)\right)$ be a unital completely positive linear map defined by

$$\rho\left(\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\right) = \left[\begin{array}{cc} \rho_{11}(a) & \rho_{12}(b) \\ \rho_{21}(c) & \rho_{22}(d) \end{array}\right].$$

Clearly, ρ_{11} and ρ_{22} are unital completely positive linear maps from A to B and $\rho_{12}^* = \rho_{21}$ (that is $\rho_{21}^*\left(a\right) = \rho_{12}\left(a\right)^*$ for all $a \in A$). Let $\varphi = \frac{1}{2}(\rho_{11} + \rho_{22})$. Clearly, φ is a unital completely positive linear map from A to B.

Lemma 2.1. *Let A be a unital C^*-algebra, let E be a Hilbert C^*-module over B and let $\rho : M_2\left(A\right) \longrightarrow M_2\left(\mathcal{L}\left(E\right)\right)$ be a unital completely positive linear map defined by*

$$\rho\left(\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\right) = \left[\begin{array}{cc} \rho_{11}(a) & \rho_{12}(b) \\ \rho_{21}(c) & \rho_{22}(d) \end{array}\right].$$

Then there is a unique element T in $C\left(\varphi\right)$ such that $\rho_{12}(a) = \widetilde{V_\varphi}^* T\widetilde{\Phi_\varphi}(a)\widetilde{V_\varphi}|_E$, where $\varphi = \frac{1}{2}(\rho_{11} + \rho_{22})$, for all $a \in A$.

Proof. Let λ be a complex number with $|\lambda| = 1$. The map $\theta : A \to \mathcal{L}(E)$ defined by

$$\theta\left(a\right) = \left[\begin{array}{cc} 1 & \overline{\lambda} \end{array}\right] \rho\left(\left[\begin{array}{cc} a & a \\ a & a \end{array}\right]\right) \left[\begin{array}{c} 1 \\ \lambda \end{array}\right]$$

is completely positive, since compositions of completely positive linear maps are completely positive. From

$$\begin{aligned}\left[\begin{array}{cc} 1 & \overline{\lambda} \end{array}\right] \rho\left(\left[\begin{array}{cc} a & a \\ a & a \end{array}\right]\right)\left[\begin{array}{c} 1 \\ \lambda \end{array}\right] &= \rho_{11}(a)+\lambda\rho_{12}(a)+\overline{\lambda}\rho_{21}(a)+\rho_{22}(a) \\ &= 2\left(\varphi+\operatorname{Re}(\lambda\rho_{12})\right)(a)\end{aligned}$$

for all $a \in A$, we deduce that the linear map $\varphi+\operatorname{Re}(\lambda\rho_{12}) : A \to \mathcal{L}(E)$ is completely positive. In particular, $\varphi \pm \operatorname{Re}\rho_{12}$ and $\varphi \pm \operatorname{Im}\rho_{12}$ are completely positive. Moreover, since $\varphi - \frac{1}{2}(\varphi+\operatorname{Re}\rho_{12}) = \frac{1}{2}(\varphi-\operatorname{Re}\rho_{12})$ and $\varphi - \frac{1}{2}(\varphi+\operatorname{Im}\rho_{12}) = \frac{1}{2}(\varphi-\operatorname{Im}\rho_{12})$, we have $\varphi \geqslant \frac{1}{2}(\varphi+\operatorname{Re}\rho_{12})$ and $\varphi \geqslant \frac{1}{2}(\varphi+\operatorname{Im}\rho_{12})$.

Let $(\Phi_\varphi, E_\varphi, V_\varphi)$ be the KSGNS construction associated with φ. By [**2,** Theorem 2.6] there is a unique positive $S \in C(\varphi)$, $S \leqslant I_{\mathcal{L}(\widetilde{E_\varphi})}$ such that

$$\frac{1}{2}\left(\varphi(a)+(\operatorname{Re}\rho_{12})(a)\right) = \widetilde{V_\varphi}^{*} S \widetilde{\Phi_\varphi}(a)\widetilde{V_\varphi}|_E$$

for all $a \in A$ and, consequently,

$$(\operatorname{Re}\rho_{12})(a) = \widetilde{V_\varphi}^{*}(2S - I_{L(\widetilde{E_\varphi})})\widetilde{\Phi_\varphi}(a)\widetilde{V_\varphi}|_E$$

for all $a \in A$. Setting $S_1 = 2S - I_{\mathcal{L}(\widetilde{E_\varphi})}$, we have $S_1 = S_1^*$ and

$$(\operatorname{Re}\rho_{12})(a) = \widetilde{V_\varphi}^{*} S_1 \widetilde{\Phi_\varphi}(a)\widetilde{V_\varphi}|_E$$

for all $a \in A$. Similarly, there is $S_2 \in C(\varphi)$ such that $S_2 = S_2^*$ and

$$(\operatorname{Im}\rho_{12})(a) = \widetilde{V_\varphi}^{*} S_2 \widetilde{\Phi_\varphi}(a)\widetilde{V_\varphi}|_E$$

for all $a \in A$. Setting $T = S_1 + iS_2$, we have $T \in C(\varphi)$ and

$$\rho_{12}(a) = \widetilde{V_\varphi}^{*} T \widetilde{\Phi_\varphi}(a)\widetilde{V_\varphi}|_E$$

for all $a \in A$.

To show that T is unique, let $T_0 \in C(\varphi)$ such that

$$\rho_{12}(a) = \widetilde{V_\varphi}^{*} T_0 \widetilde{\Phi_\varphi}(a)\widetilde{V_\varphi}|_E$$

for all $a \in A$. Then

$$(\operatorname{Re}\rho_{12})(a) = \widetilde{V_\varphi}^{*} \frac{1}{2}(T_0+T_0^*)\widetilde{\Phi_\varphi}(a)\widetilde{V_\varphi}|_E$$

for all $a \in A$ and so

$$\frac{1}{2}\left(\varphi(a)+(\operatorname{Re}\rho_{12})(a)\right) = \widetilde{V_\varphi}^{*} \frac{1}{2}\left(\frac{1}{2}(T_0+T_0^*)+I_{\mathcal{L}(\widetilde{E_\varphi})}\right)\widetilde{\Phi_\varphi}(a)\widetilde{V_\varphi}|_E$$

for all $a \in A$. By [**2,** Theorem 2.6], $\frac{1}{2}\left(\frac{1}{2}(T_0+T_0^*)+I_{L(\widetilde{E_\varphi})}\right) = S$ and then

$$\frac{1}{2}(T_0+T_0^*) = 2S - I_{\mathcal{L}(\widetilde{E_\varphi})} = \frac{1}{2}(T+T^*).$$

Hence $T_0+T_0^* = T+T^*$. In the same way we obtain $T_0-T_0^* = T-T^*$. These imply that $T_0 = T$. ▮

The following theorem is a generalization of Proposition 2.7 [**8**].

Theorem 2.2. *Let A be a unital C^*-algebra, let E be a Hilbert C^* -module over B, let $[\rho_{ij}]_{i,j=1}^{n}$ be a unital completely positive linear map from A to $\mathcal{L}(E)$.*

1. *Then there is a representation* Φ *of* A *on a Hilbert* C^**-module* F *over* B*, an isometry* $V : E \to F$*, and* $[T_{ij}]_{i,j=1}^n \in M_n(\widetilde{\Phi}(A)')$ *such that*
   - (a) $\widetilde{V}^* T_{ij} \widetilde{\Phi}(a) \widetilde{V}\,|_E \in \mathcal{L}(E)$ *for all* $a \in A$ *and for all* $i,j = 1,2,...n$*,* $\left[\widetilde{V}^* T_{ij} \widetilde{V}\,|_E\right]_{i,j=1}^n$ *is a positive element in* $M_n\left(\mathcal{L}(E)\right)$*, and* $\sum\limits_{k=1}^n T_{kk} = nI_{\mathcal{L}(F)}$;
   - (b) $\{\Phi(a)V\xi; a \in A, \xi \in E\}$ *is dense in* F *;*
   - (c) $\rho_{ij}(a) = \widetilde{V}^* T_{ij} \widetilde{\Phi}(a) \widetilde{V}\,|_E$ *for all* $a \in A$ *and* $i,j = 1,...,n$*.*
2. *If* Ψ *is another representation of* A *on a Hilbert* C^**-module* G *over* B*,* $W : E \to G$ *is an isometry and* $[S_{ij}]_{i,j=1}^n \in M_n(\widetilde{\Psi}(A)')$ *such that*
   - (a) $\widetilde{W}^* S_{ij} \widetilde{\Psi}(a) \widetilde{W}\,|_E \in \mathcal{L}(E)$ *for all* $a \in A$ *and for all* $i,j = 1,2,...n$*,* $\left[\widetilde{W}^* S_{ij} \widetilde{W}\,|_E\right]_{i,j=1}^n$ *is a positive element in* $M_n\left(\mathcal{L}(E)\right)$*, and* $\sum\limits_{k=1}^n S_{kk} = nI_{\mathcal{L}(G)}$;
   - (b) $\{\Psi(a)W\xi; a \in A, \xi \in E\}$ *is dense in* G *;*
   - (c) $\rho_{ij}(a) = \widetilde{W}^* S_{ij} \widetilde{\Psi}(a) \widetilde{W}\,|_E$ *for all* $a \in A$ *and* $i,j = 1,2,...,n$*,* *then there is a unitary operator* $U : F \to G$ *such that*
     - (i) $U\Phi(a) = \Psi(a)\,U$ *for all* $a \in A$;
     - (ii) $UV = W$;
     - (iii) $\widetilde{U}T_{ij} = S_{ij}\widetilde{U}$ *for all* $i,j = 1,2,...n$*.*

Proof. 1. It is easy to verify that the map $\begin{bmatrix} \rho_{ii} & \rho_{ij} \\ \rho_{ji} & \rho_{jj} \end{bmatrix} : M_2(A) \longrightarrow M_2(\mathcal{L}(E))$ is completely positive for each $i,j = 1,...,n$ with $i \neq j$. Then by the proof of the above lemma, we have $\frac{1}{2}(\rho_{ii} + \rho_{jj}) + \operatorname{Re}(\lambda\rho_{ij})$ is completely positive for all $i,j = 1,2,...,n$, with $i \neq j$ and for all complex numbers λ with $|\lambda| = 1$, and so $\frac{1}{2}(\sum_{k=1}^n \rho_{kk}) + \operatorname{Re}(\lambda\rho_{ij})$ is completely positive for $i,j = 1,.2,...n$ with $i \neq j$. In particular, $\frac{1}{n}(\sum_{k=1}^n \rho_{kk}) \pm \frac{2}{n}\operatorname{Re}\rho_{ij}$ and $\frac{1}{n}(\sum_{k=1}^n \rho_{kk}) \pm \frac{2}{n}\operatorname{Im}\rho_{ij}$ are completely positive for each $i,j = 1,2,...n$, with $i \neq j$.

Setting $\rho = \frac{1}{n}\sum_{k=1}^n \rho_{kk}$, the map ρ is completely positive, because it is a sum of completely positive linear maps. By the KSGNS construction associated with ρ, there is a Hilbert C^*-module F over B, a representation Φ of A on F, and an isometry $V : E \to F$ such that the vector subspace $\{\Phi(a)V\xi; a \in A, \xi \in E\}$ is dense in F. For each $i,j = 1,2,...,n$ with $i \neq j$, by the proof of Lemma 2.1, since $\rho - \frac{1}{2}\left(\rho + \frac{2}{n}\operatorname{Re}\rho_{ij}\right) = \frac{1}{2}\left(\rho - \frac{2}{n}\operatorname{Re}\rho_{ij}\right)$ and $\rho - \frac{1}{2}\left(\rho + \frac{2}{n}\operatorname{Im}\rho_{ij}\right) = \frac{1}{2}\left(\rho - \frac{2}{n}\operatorname{Im}\rho_{ij}\right)$, there is a unique element T_{ij} in $C(\rho)$ (that is, $T_{ij} \in \widetilde{\Phi}(A)'$ and $\widetilde{V}^* T_{ij} \widetilde{\Phi}(a) \widetilde{V}\,|_E \in \mathcal{L}(E)$ for all $a \in A$) such that

$$\rho_{ij}(a) = \widetilde{V}^* T_{ij} \widetilde{\Phi}(a) \widetilde{V}\,|_E$$

for all $a \in A$.

For each $i = 1,2,...,n$, clearly $\frac{1}{n}\rho_{ii} \leq \rho$, and by [**2,** Theorem 2.6] there is a unique element $T_{ii}^0 \in C(\rho)$ such that

$$\frac{1}{n}\rho_{ii}(a) = \widetilde{V}^* T_{ii}^0 \widetilde{\Phi}(a) \widetilde{V}\,|_E$$

for all $a \in A$. Let $T_{ii} = nT_{ii}^{0}$. Then $T_{ii} \in C(\rho)$ and $\rho_{ii}(a) = \widetilde{V}^{*} T_{ii} \widetilde{\Phi}(a) \widetilde{V}\,|_{E}$ for all $a \in A$. From

$$\rho\left(a\right) = \frac{1}{n}\sum_{k=1}^{n}\rho_{kk}\left(a\right) = \frac{1}{n}\sum_{k=1}^{n}\widetilde{V}^{*}T_{kk}\widetilde{\Phi}(a)\widetilde{V}\,|_{E} = \widetilde{V}^{*}\sum_{k=1}^{n}\frac{1}{n}T_{kk}\widetilde{\Phi}(a)\widetilde{V}\,|_{E}$$

for all $a \in A$,and [**2,** Theorem 2.6] we deduce that $\sum_{k=1}^{n}\frac{1}{n}T_{kk} = I_{\mathcal{L}(F)}$.

To show that $\left[\widetilde{V}^{*}T_{ij}\widetilde{V}\,|_{E}\right]_{i,j=1}^{n}$ is a positive element in $M_{n}\left(\mathcal{L}(E)\right)$, let $a_{k} \in A$ and $\xi_{k} \in E$, $k = 1, 2, ..., n$. We have

$$\begin{aligned}
\sum_{i,j=1}^{n}\left[T_{ij}\left(\Phi(a_{i})V\xi_{i}\right), \Phi(a_{j})V\xi_{j}\right] &= \sum_{i,j=1}^{n}\left[\widetilde{V}^{*}T_{ij}\widetilde{\Phi}(a_{j}^{*}a_{i})\widetilde{V}\xi_{i}, \xi_{j}\right] \\
&= \sum_{i,j=1}^{n}\left\langle\widetilde{V}^{*}T_{ij}\widetilde{\Phi}(a_{j}^{*}a_{i})\widetilde{V}\xi_{i}, \xi_{j}\right\rangle \\
&= \sum_{i,j=1}^{n}\left\langle\rho_{ij}\left(a_{j}^{*}a_{i}\right)\xi_{i}, \xi_{j}\right\rangle \geq 0
\end{aligned}$$

since $[\rho_{ij}]_{i,j=1}^{n}$ is completely positive. From this fact and taking into account that the vector subspace generated by $\{\Phi(a)V\xi; a \in A, \xi \in E\}$ is dense in F, we deduce that

$$\sum_{i,j=1}^{n}\left[T_{ij}\left(\eta_{i}\right), \eta_{j}\right] \geq 0$$

for all $\eta_{1}, ..., \eta_{n} \in E$ and then

$$\sum_{i,j=1}^{n}\left\langle\widetilde{V}^{*}T_{ij}\widetilde{V}\xi_{i}, \xi_{j}\right\rangle = \sum_{i,j=1}^{n}\left[\widetilde{V}^{*}T_{ij}\widetilde{V}\xi_{i}, \xi_{j}\right] = \sum_{i,j=1}^{n}\left[T_{ij}\widetilde{V}\xi_{i}, \widetilde{V}\xi_{j}\right] \geq 0$$

for all $\xi_{1}, ..., \xi_{n} \in E$. Thus we showed that $\left[\widetilde{V}^{*}T_{ij}\widetilde{V}\,|_{E}\right]_{i,j=1}^{n}$ is a positive element in $M_{n}\left(\mathcal{L}(E)\right)$.

2. Consider the linear map U :$\mathrm{Sp}\{\Phi(a)V\xi; a \in A, \xi \in E\} \rightarrow \mathrm{Sp}\{\Psi(a)W\xi; a \in A, \xi \in E\}$ defined by

$$U\left(\Phi(a)V\xi\right) = \Psi(a)W\xi.$$

From

$$\begin{aligned}
\left\langle U\left(\Phi(a)V\xi\right), U\left(\Phi(a)V\xi\right)\right\rangle &= \left\langle\Psi(a)W\xi, \Psi(a)W\xi\right\rangle \\
&= \left\langle W^{*}\Psi\left(a^{*}a\right)W\xi, \xi\right\rangle = \left\langle\widetilde{W}^{*}I_{\mathcal{L}(\widetilde{G})}\widetilde{\Psi}(a^{*}a)\widetilde{W}\xi, \xi\right\rangle \\
&= \sum_{k=1}^{n}\frac{1}{n}\left\langle\widetilde{W}^{*}S_{kk}\widetilde{\Psi}(a^{*}a)\widetilde{W}\xi, \xi\right\rangle = \sum_{k=1}^{n}\frac{1}{n}\left\langle\rho_{kk}(a^{*}a)\xi, \xi\right\rangle \\
&= \sum_{k=1}^{n}\frac{1}{n}\left\langle\widetilde{V}^{*}T_{kk}\widetilde{\Phi}(a^{*}a)\widetilde{V}\xi, \xi\right\rangle \\
&= \left\langle\widetilde{V}^{*}I_{\mathcal{L}(\widetilde{F})}\widetilde{\Phi}(a^{*}a)\widetilde{V}\xi, \xi\right\rangle \\
&= \left\langle\Phi(a)V\xi, \Phi(a)V\xi\right\rangle
\end{aligned}$$

for all $a \in A$ and for all $\xi \in E$ and taking into account that $\mathrm{Sp}\{\Phi(a)V\xi; a \in A, \xi \in E\}$ is dense in F and $\mathrm{Sp}\{\Psi(a)W\xi; a \in A, \xi \in E\}$ is dense in G, we deduce that U extends to a unitary operator from F to G. It is not difficult to check that $U\Phi(a) = \Psi(a)U$ for all $a \in A$ and $UV = W$.

Clearly, $\widetilde{U}^* S_{ij} \widetilde{U} \in C(\rho)$ for all $i, j = 1, 2, ..., n$. From

$$\begin{aligned} \rho_{ij}(a) &= \widetilde{W}^* S_{ij} \widetilde{\Psi}(a) \widetilde{W} \mid_E = \widetilde{V}^* \widetilde{U}^* S_{ij} \widetilde{\Psi}(a) \widetilde{U} \widetilde{V} \mid_E \\ &= \widetilde{V}^* (\widetilde{U}^* S_{ij} \widetilde{U}) \widetilde{\Phi}(a) \widetilde{V} \mid_E \end{aligned}$$

for all $a \in A$ and for all $i, j = 1, 2, ..., n$, and the uniqueness of the operators $T_{ij} \in C(\rho)$ such that $\rho_{ij}(a) = \widetilde{V}^* T_{ij} \widetilde{\Phi}(a) \widetilde{V} \mid_E$ for all $a \in A$, we deduce that $T_{ij} = \widetilde{U}^* S_{ij} \widetilde{U}$ for all $i, j = 1, 2, ..., n$, and the theorem is proved. ▮

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