Hilbert modules over locally C*-algebras

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Contents

Preface

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than the field of complex numbers. The notion of Hilbert module over a commutative C^*-algebra first appeared in the work of Kaplansky [**25**], who used it to prove that derivations of type I AW^*-algebras are inner. The general theory of Hilbert C^*-modules has appeared 32 years ago in the basic papers of Paschke [**36**] and Rieffel [**41, 42**]. This theory has prove to be a convenient tool in the theory of operator algebras, allowing to study C^*-algebras by studying Hilbert C^*-modules over them. Thus, the theory of Hilbert C^*-modules is an important tool for studying Morita equivalence of C^*-algebras and its application to group representation theory and crossed product C^*-algebras, K-theory and KK -theory of operator algebras, completely positive maps between C^*-algebras, unbounded operators and quantum groups, vector bundles, non-commutative geometry, mathematical and theoretical physics. Beside these, theory of Hilbert C^*-modules is very interesting on it's own.

The finitely generated modules equipped with inner products over some topological $*$ -algebras and the standard Hilbert module H_A over a locally C^*-algebra A were first considered by Mallios [**32**], who used them to construct the index theory for elliptic operators over a locally C^*-algebra. Locally C^*-algebras are generalizations of C^*-algebras. Instead of being given by a single C^*-norm, the topology of a locally C^*-algebra is defined by a directed family of C^*-seminorms. Such many concepts as Hilbert C^*-module, adjointable operator, compact operator, (induced) representation, strong Morita equivalence can be defined with obvious modifications in the framework of locally C^*-algebras. Most of the basic properties of Hilbert C^*-modules are still valid for Hilbert modules over locally C^*-algebras, but the proofs are not always straightforward. Many important results have been obtained about module homorphisms of Hilbert modules over locally C^*-algebras [**16**], [**18**], [**20**], [**23**], [**24**], [**38**], [**45**], representations of locally C^* -algebras [**15**], [**17**], [**22**], frames and bases in Hilbert modules over locally

C^*-algebras [**28**] and Finsler modules over locally C^*-algebras [**27**].

This book is an introduction in theory of Hilbert modules over locally C^*-algebras. We did not purpose to discuss here all aspects of Hilbert modules over locally C^*-algebras, but we have tried to explain the basic notions and theorems of this theory, a number important of examples and some results about representations of locally C^* -algebras. A significant part of the results presented here was obtained by the author [**13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24**].

The detailed bibliography of the theory of Hilbert C^*-modules can be found in *Hilbert C*-Modules Homepage*

(http://www.imn.htwk-leipzig.de/~mfrank/hilmod.html).

Reader's Guide

Chapter 1 begins with some results about locally C^*-algebras, which will be necessary for us further on. The notion of Hilbert module over a locally C^*-algebra is discussed in Section 1.2 and there are presented some examples of Hilbert modules over locally C^*-algebras. In Section 1.3 it is shown that a Hilbert module E over a locally C^*-algebra A induces a structure of Hilbert C^*-module on the set $b(E)$ of bounded elements in E, i.e. of those elements for which any admissible seminorm applied to them takes a finite value, and the connection between $b(H_E)$ and $H_{b(E)}$ is discussed.

Chapter 2 is about module homomorphisms.

In Section 2.1 it is shown that the set of bounded module homomorphisms on a Hilbert module can be equipped with a structure of complete locally m -convex algebra. Moreover, this algebra can be identified with an inverse limit of Banach algebras. In Section 2.1, by analogy with the case of Hilbert C^*-modules, it is proved that a module homomorphism which has an adjoint is bounded. Also it is proved that the set of adjointable operators on a Hilbert module is a locally

C^*-algebra. Section 2.3 is devoted to "compact " operators on Hilbert modules. It is shown that the set $K_A(E)$ of compact operators on the Hilbert A -module E is a locally C^* -algebra, the algebra $ML(K_A(E))$ of left multipliers of $K_A(E)$ is isomorphic with the locally m -convex algebra $B_A(E)$ of bounded module homomorphisms on E and the algebra $M(K_A(E))$ of multipliers of $K_A(E)$ is isomorphic with the locally C^* -algebra $L_A(E)$ of adjointable module homomorphisms on E. In Section 2.4, it is introduced the notion of strongly bounded module homomorphism, that is a module homomorphism for which any admissible seminorm applied to it takes a finite value. It is established an isometric isomorphism between the Banach space $b(B_A(E,F))$ of all strongly bounded module homomorphisms from E to F and the Banach space $B_{b(A)}(b(E),B(F))$ of all bounded $b(A)$-module homomorphisms the bounded part $b(E)$ of E to the bounded part $b(F)$ of F, where $b(A)$ denotes the bounded part of A. In particular, $b(B_A(E))$ is a Banach algebra isometrically isomorphic to $B_{b(A)}(b(E))$, and $b(L_A(E))$ is a C^*-algebra isometrically isomorphic to $L_{b(A)}(b(E))$. Most remarkably, the respective sets of "compact" operators $b(K_A(E,F))$ and $K_{b(A)}(b(E),b(F))$ are not isomorphic, in general, as shown by an example. Another class of operators on Hilbert modules, unitary operators, is discussed in Section 2.5. Also in Section 2.5 a result of Brown, Lin, Lance and Frank [**10**] concerning isomorphisms of Hilbert C^*-modules and $*$-isomorphisms of related operator C^*-algebras is extended in the context of locally C^*-algebras. Thus it is proved that Hilbert A-modules $(E,\langle\cdot,\cdot\rangle_1)$ and $(E,\langle\cdot,\cdot\rangle_2)$ are isomorphic as Hilbert modules if and only if the locally C^*-algebras of adjointable operators on $(E,\langle\cdot,\cdot\rangle_1)$ and $(E,\langle\cdot,\cdot\rangle_2)$ are isomorphic if and only if the locally C^* -algebras of compact operators on $(E,\langle\cdot,\cdot\rangle_1)$ and $(E,\langle\cdot,\cdot\rangle_2)$ are isomorphic if and only if the C^* -algebras of compact operators on $(b(E),\langle\cdot,\cdot\rangle_1)$ and $(b(E),\langle\cdot,\cdot\rangle_2)$ are isomorphic.

Chapter 3 concerns projections on Hilbert modules over locally C^*-algebras, orthogonally complemented submodules and the polar decomposition of an adjointable operator on Hilbert modules. It is proved that a closed submodule E_0 of a Hilbert module E is complemented if and only if $b(E_0)$ is complemented if and only if E_0 is the range of an adjointable operator on E. Also it is proved a necessary and sufficient condition for an adjointable operator to admit a polar decomposition.

Chapter 4 is about tensor products of Hilbert modules over locally C^*-algebras and operators on tensor products. By analogy with the case of Hilbert C^*-modules, in Section 4.1 it is defined the notion of exterior tensor product of Hilbert modules over locally C^*-algebras and it is shown that the exterior tensor product $E \otimes F$ of E and F is in fact an inverse limit of exterior tensor products of Hilbert C^*-modules. In Section 4.2 it defined the notion of inner tensor product of Hilbert modules over locally C^*-algebras and the connection between the bounded part of the Hilbert A -module $E \otimes_\Phi F$ and the Hilbert C^*-module $b(E) \otimes_{\Phi|_{b(A)}} b(F)$ over $b(A)$ it is discussed. In Section 4.3 it is constructed an injective $*$-morphism j from $L_A(E) \otimes L_B(F)$ to $L_{A\otimes B}(E \otimes F)$ and it is shown that the locally C^*-algebras $K_A(E) \otimes K_B(F)$ and $K_{A\otimes B}(E \otimes F)$ are isomorphic. Also it is constructed a $*$-morphism Φ_* from $L_A(E)$ to $L_B(E \otimes_\Phi F)$, which is injective if Φ is an injective $*$-morphism .

Chapter 5 concerns full Hilbert modules, countably generated Hilbert modules and strong Morita equivalence of locally C^*-algebras. In Section 5.1, the full Hilbert modules over locally C^*-algebras are characterized. It is clear that E is a Hilbert module over a locally C^*-algebra A such that $b(E)$ is full, then E is full. The converse of this statement is not true in general. We present some example in this sense. Section 5.2 is devoted to famous Kasparov stabilization theorem [**26**]. It is showed that the stabilization theorem is still valid for countably generated Hilbert modules over arbitrary locally C^*-algebras and it is proved a necessary and sufficient condition that a Hilbert module over a Fréchet locally C^*-algebra to be countably generated. Also it is extended to context of locally C^*-algebras a result of Mingo and Phillips [**35**], which states that if E is a full countably generated Hilbert C^*-module over a σ-unital C^*-algebra A then the Hilbert C^*-modules H_A and H_E are unitarily equivalent. In Section 5.3, it is extended the notion of strong Morita equivalence in the context of locally C^*-

algebras. It is shown that strong Morita equivalence is an equivalence relation on the set of locally C^*-algebras and it is proved that two Fréchet locally C^*-algebras are strongly Morita equivalent if and only if they are stably isomorphic (this extends a well-known theorem of Brown, Green and Rieffel [**5**]).

Chapter 6 is devoted to representations of locally C^*-algebras on Hilbert modules. The non-degenerate representations of a locally C^*-algebra A on a Hilbert module E over a locally C^*-algebra B are characterized in Section 6.1. Also in this section it is shown that any locally C^*-algebra A admits a non-degenerate representation on H_B, where B is an arbitrary locally C^*-algebra. Section 6.2 is about completely positive linear maps between locally C^*-algebras. In Section 6.3 is obtained a construction of type KSGNS (Kasparov, Stinespring, Gel'fand, Naimark, Segal) for continuous strict completely positive linear maps between locally C^*-algebras.

Chapter 7 is about induced representations of locally C^*-algebras.

By analogy with the case of C^*-algebras, in Section 7.2 it is extended the notion of induced representation in the context of locally C^*-algebras and it is shown that theorem on induction in stage [Theorem 5.9, **41**] is still valid. Also in Section 7.3 it is proved an imprimitivity theorem for representations of locally C^*-algebras.

Chapter 1

Hilbert modules over locally C^*-algebras

1.1 Locally C^*-algebras

The basic information about locally C^*-algebras can be found in the works [**7**, **8**, **9**, **11**, **33**, **38, 39**]. We will present some results on locally C^*-algebras, which will be necessary for us further on.

Recall that a C^*-seminorm on a topological $*$-algebra A is a seminorm p such that $p(ab) \leq p(a)p(b)$ and $p(aa^*) = p(a)^2$ for all a and b in A.

Definition 1.1.1 *A locally C^*-algebra is a Hausdorff complete complex topological $*$ -algebra A whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_i\}_{i\in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for all continuous C^*-seminorm p on A.*

A pre-locally C^*-algebra is Hausdorff complex topological $*$ -algebra whose the topology is determined by a directed family of C^*-seminorms.

These objects are called pro-C^*-algebras in [**2, 3**, **38, 39**]. If the topology is determined by only countably many of C^*-seminorms, then we have σ-C^*-algebras in [**38, 39**] and Fréchet locally C^*-algebras in [**7**, **8**, **9**, **11**, **33**].

Example 1.1.2 *Any C^*-algebra is a locally C^*-algebra.*

Example 1.1.3 *Any closed subalgebra of a locally C^*-algebra is a locally C^*-algebra.*

Example 1.1.4 *If $\{A_\lambda; \pi_{\lambda\mu}\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ is an inverse system of C^*-algebras, then $\lim\limits_{\overleftarrow{\lambda}} A_\lambda$ is a locally C^*-algebras with the topology defined by the family of C^*-seminorms $\{p_\lambda\}_{\lambda\in\Lambda}$ defined by $p_\lambda\left((a_\mu)_\mu\right) = \|a_\lambda\|_\lambda$, where $\|\cdot\|_\lambda$ denotes the C^* -norm on A_λ, $\lambda \in \Lambda$.*

Example 1.1.5 *If X is a compactly generated space, then $C(X)$, the set of all continuous complex valued functions on X with the topology of uniform convergence on compact subsets is a locally C^*-algebra.*

Definition 1.1.6 *Let A and B be two locally C^*-algebras. A $*$ -morphism from A to B is a linear map $\Phi : A \to B$ such that $\Phi(ab) = \Phi(a)\Phi(b)$ and $\Phi(a^*) = \Phi(a)^*$ for all a and b in A.*

A morphism of locally C^*-algebras from A to B is a continuous $*$ -morphism from A to B.

An isomorphism of locally C^*-algebras from A to B is a bijective morphism of locally C^*-algebras $\Phi : A \to B$ such that $\Phi^{-1} : B \to A$ is a morphism of locally C^*-algebras.

Two locally C^*-algebras A and B are isomorphic if there is an isomorphism of locally C^*-algebras from A to B.

Remark 1.1.7 *([8]). If A is a locally C^*-algebra and B is a Fréchet locally C^*-algebra, then any $*$ -morphism from A to B is a morphism of locally C^*-algebras.*

Let A be a locally C^* -algebra. We denote by $S(A)$ the set of all continuous C^*-seminorms on A. For $p \in S(A)$, we let $\ker p$ be the set $\{a \in A; p(a) = 0\}$, which is a closed $*$ -ideal in A. We also let A_p be the quotient $*$ -algebra $A/\ker p$. The canonical map from A to A_p is denoted by π_p. It is not difficult to check that A_p is a pre-C^*-algebra with respect to the norm $\|\cdot\|_p$ induced by p (that is, $\|a+\ker p\|_p = p(a)$ for all a in A). We will prove that A_p is a C^*-algebra (Theorem 1.1.16). The set $S(A)$ is directed with the order $p \geq q$ if $p\,(a) \geq q\,(a)$ for all a in A. Then for p and q in $S(A)$ with $p \geq q$, there is a canonical surjective $*$ -morphism $\pi_{pq} : A_p \to A_q$ such that $\pi_{pq}(\pi_p\,(a)) = \pi_q(a)$ for all a in A. For each $p \in S(A)$ we denote by $\widetilde{A_p}$ the completion of the pre-C^*-algebra A_p. If p, $q \in S(A)$ and $p \geq q$, then clearly the surjective $*$ -morphism $\pi_{pq} : A_p \to A_q$ extends to a surjective $*$ -morphism $\widetilde{\pi_{pq}} : \widetilde{A_p} \to \widetilde{A_q}$.

Theorem 1.1.8 *([**1**, **9**, **11**, **33**]). Let A be a locally C^*-algebra. Then:*

1. $\{\widetilde{A_p}; \widetilde{\pi_{pq}}\}_{p\geq q,p,q\in S(A)}$ *is an inverse system of C^*-algebras.*
2. *The map $\Phi : A \to \lim\limits_{\leftarrow p} \widetilde{A_p}$ defined by*

$$\Phi\,(a) = (\pi_p(a))_p$$

is an isomorphism of locally C^*-algebras.

Proof. 1. It is a simple verification.

2. The topology on $\lim\limits_{\leftarrow p} \widetilde{A_p}$ is defined by the family of C^*-seminorms $\{\widetilde{p}\}_{p\in S(A)}$, where $\widetilde{p}\,((a_q)_q) = \|a_p\|_p$ for all $(a_q)_q \in \lim\limits_{\leftarrow p} \widetilde{A_p}$. It is not difficult to check that Φ is a $*$ -morphism from A to $\lim\limits_{\leftarrow p} \widetilde{A_p}$. Since

$$\widetilde{p}(\Phi\,(a)) = p(a)$$

for all $p \in S(A)$ and for all $a \in A$, Φ is an injective morphism of locally C^*-algebras with closed range. Then, by Lemma III 3.2, [**33**], we have

$$\Phi(A) = \lim_{\leftarrow p} \overline{\chi_p\,(\Phi(A))} = \lim_{\leftarrow p} \overline{\pi_p(A)} = \lim_{\leftarrow p} \widetilde{A_p}$$

where $\chi_p, p \in S(A)$ are the canonical maps from $\lim\limits_{\leftarrow p} \widetilde{A_p}$ to $\widetilde{A_p}$, and $\overline{\chi_p(\Phi(A))}$ denotes the closure of the vector subspace $\chi_p\,(\Phi(A))$ in $\widetilde{A_p}$ for each $p \in S(A)$.

From these facts we conclude that Φ is a bijective morphism of locally C^*-algebras and Φ^{-1} is continuous. Therefore Φ is an isomorphism of locally C^*-algebras. ■

If A is a unital locally C^*-algebra, then the pre-C^*-algebras A_p, $p \in S(A)$ are unital. Moreover, if 1 is the unity of A, then $\pi_p(1)$ is the unity of A_p for all $p \in S(A)$.

An element a in A is invertible if there is an element in A, denoted by a^{-1}, such that $aa^{-1} = a^{-1}a = 1$.

Remark 1.1.9 *Let A be a unital locally C^* -algebra and let $a \in A$. Then a is invertible if and only if $\pi_p(a)$ is invertible in A_p for all $p \in S(A)$. Moreover, $\pi_p(a)^{-1} = \pi_p(a^{-1})$ for all $p \in S(A)$.*

Let A be a locally C^*-algebra without unity, and let $A^+ = A \oplus \mathbb{C}$. Then A^+ under the multiplication

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$$

and the involution

$$(a, \lambda)^* = \left(a^*, \overline{\lambda}\right)$$

is an algebra with involution. Any continuous C^*-seminorm p can be extended up to a C^*-seminorm p^+ on A^+ and thus A^+ with the topology determined by the family of C^*-seminorms $\{p^+, p \in S(A)\}$ is a locally C^*-algebra. Moreover, A^+ can be identified with $\lim\limits_{\substack{\leftarrow \\ p}} \widetilde{A_p}^+$, where $\widetilde{A_p}^+$ is the unitization of $\widetilde{A_p}$, $p \in S(A)$.

Definition 1.1.10 *Let A be a unital locally C^*-algebra and let a be an element in A. The spectrum of A, $Sp(a)$ is the set of all complex numbers λ such that $a - \lambda 1$ is not invertible in A. If A is not unital, $Sp(a)$ is the set of all complex numbers λ such that $a - \lambda 1$ is not invertible in A^+.*

Remark 1.1.11 $Sp(a) = \bigcup\limits_{p} Sp(\pi_p(a))$.

An element a in A is self-adjoint if $a^* = a$.

Remark 1.1.12 *Let A be a locally C^* -algebra and $a \in A$. Then the following assertions are equivalent:*

(**a**) *a is self-adjoint;*

(**b**) *$Sp(a) \subseteq \mathbb{R}$;*

(**c**) *$\pi_p(a)$ is self-adjoint for all $p \in S(A)$.*

An element a in A is called positive and we write $a \geq 0$, if there is an element b in A such that $a = b^*b$. In particular, for a and b in A we will write $a \geq b$ if $a - b \geq 0$.

Remark 1.1.13 *Let A be a locally C^*-algebra and $a \in A$. Then the following assertions are equivalent:*

(**a**) *$a \geq 0$;*

(**b**) *$\pi_p(a) \geq 0$ for all $p \in S(A)$;*

(**c**) *$a = h^2$ for some h in A;*

(**d**) *$Sp(a) \subseteq [0, \infty)$*

The set of all positive elements in A is denoted by $P(A)$ and it is a closed convex con in A such that $P(A) \cap \{-P(A)\} = \{0\}$.

Proposition 1.1.14 *([**11, 38**]) Let A be a locally C^*-algebra, and let $a \in A$ be a normal element (that is $aa^* = a^*a$). There is a unique morphism of locally C^*-algebras from the locally C^*-algebra of all continuous functions $f : Sp(a) \to \mathbb{C}$ such that $f(0) = 0$ to A which sends the identity function to a. If A is unital, then this map extends uniquely to a morphism from the locally C^*-algebra of all continuous functions $f : Sp\ (a) \to \mathbb{C}$ to A which sends 1 to 1.*

Proof. Let $f \in \{h \in C(\text{ Sp } (a));\ h(0) = 0\}$, and let $p, q \in S(A)$ with $p \geq q$. Then

$$\pi_{pq}\left(f(\pi_p(a))\right) = f(\pi_{pq}\left(\pi_p(a)\right)) = f(\pi_q(a)).$$

Thus we can define a map $\Phi : \{h \in C(\text{ Sp } (a)); h(0) = 0\} \to A$ by $\Phi(f) = (f(\pi_p(a)))_p$. It is not difficult to check that the required map is Φ. ■

Definition 1.1.15 *Let A be a locally C^*-algebra. An element a in A is bounded if*

$$\sup\{p(a); p \in A\} < \infty.$$

The set of all bounded elements in A is denoted by $b(A)$.

Theorem 1.1.16 *Let A be a locally C^*-algebra. Then:*

1. *The map $\|\cdot\|_\infty : b(A) \to [0, \infty)$ defined by*

$$\|a\|_\infty = \sup\{p(a); p \in A\}$$

   is a C^*-norm on A.

2. *$b(A)$ equipped with the C^*-norm $\|\cdot\|_\infty$ is a C^*-algebra.*

3. *$b(A)$ is dense in A;*

4. *For each $p \in S(A)$, A_p is a C^*-algebra.*

Proof. 1. It is a simple verification.

2. Let $\{a_n\}_n$ be a Cauchy sequence in $b(A)$. Then there is a positive number M such that $\|a_n\|_\infty \leq M$ for all n and $\{a_n\}_n$ is a Cauchy sequence in A and so it converges in A to an element a.

To show that a is bounded, let $p \in S(A)$. Then

$$p(a) \leq p(a - a_n) + p(a_n) \leq p(a - a_n) + M$$

for all n. This implies that $p(a) \leq M$. Therefore $a \in b(A)$.

Let $\varepsilon > 0$. Since $\{a_n\}_n$ is a Cauchy sequence in $b(A)$, there is a positive integer n_0 such that

$$\|a_n - a_m\|_\infty < \varepsilon$$

for all $m \geq n_0$ and for all $n \geq n_0$. Then

$$p(a - a_n) = \lim_m p(a_m - a_n) \leq \lim_m \|a_n - a_m\|_\infty < \varepsilon$$

for all $p \in S(A)$ and for all $n \geq n_0$. This implies that the sequence $\{a_n\}_n$ is convergent in $b(A)$, and the assertion is proved.

3. Let $a \in A$. For each positive integer n, the element $1 + \frac{1}{n}a^*a$ is invertible in A^+. Let $a_n = a\left(1 + \frac{1}{n}a^*a\right)^{-1}$. By functional calculus, we have

$$p(a_n) = \sqrt{p\left(a_n^* a_n\right)} = \sqrt{p\left(a^*a\left(1 + \frac{1}{n}a^*a\right)^{-2}\right)} \leq \sqrt{n}$$

and

$$\begin{aligned} p(a - a_n) &= p\left(\frac{1}{n}aa^*a\left(1 + \frac{1}{n}a^*a\right)^{-1}\right) \leq \frac{p(a)^2}{n}p\left(a\left(1 + \frac{1}{n}a^*a\right)^{-1}\right) \\ &\leq \frac{p(a)^2}{\sqrt{n}} \end{aligned}$$

for all $p \in S(A)$. From these facts, we conclude that $\{a_n\}_n$ is a sequence in $b(A)$ which converges in A to a. This shows that $b(A)$ is dense in A.

4. Let $p \in S(A)$ and let $N_p = b(A) \cap \ker p$. Clearly, N_p is a closed $*$ -ideal of $b(A)$, and then $b(A)/N_p$ is a C^*-algebra with respect to the topology determined by the norm $\|\cdot\|$ definrd by $\|a + N_p\| = \inf\{\|a + b\|_\infty ; b \in N_p\}$, $a \in b(A)$. Since

$$\|a + \ker p\|_p = p(a) \leq p(a + b) \leq \|a + b\|_\infty$$

for all $b \in N_p$, we can define a map $\varphi : b(A)/N_p \to \widetilde{A_p}$, by $\varphi\left(a + N_p\right) = a+ \ker p$. Clearly, φ is an injective morphism of C^*- algebras. Moreover, $\varphi\left(b(A)/N_p\right) \subseteq A_p$.

On the other hand, we seen that for $a \in A$ there is a sequence $\{a_n\}_n$ in $b(A)$ which converges to a. Then the sequence $\{a_n + \ker p\}_n$ converges to $a + \ker p$ in $\widetilde{A_p}$. From thise facts and taking into account that $\varphi\left(a_n + N_p\right) = a_n + \ker p$ and φ has closed range we conclude that $A_p \subseteq \varphi\left(b(A)/N_p\right)$. Therefore $\varphi\left(b(A)/N_p\right) = A_p$. Thus, we showed that $\varphi : b(A)/N_p \to A_p$ is an isomorphism and so A_p is a C^*-algebras. ∎

Definition 1.1.17 *Let A be a locally C^*-algebra. An approximate unit for A is an increasing net $\{e_i\}_{i\in I}$ of positive elements in A such that $p(e_i) \leq 1$ for all $p \in S(A)$ and for all $i \in I$, and, for all $a \in A$ we have $p(a - ae_i) \to 0$ and $p(a - e_i a) \to 0$ for all $p \in S(A)$.*

Remark 1.1.18 *From Theorem 1.1.16 and Definition 1.1.17, we conclude that any locally C^*-algebra has an approximate unit.*

A locally C^*-algebra A is strongly spectrally bounded if $b(A) = A$ as set.

Let A be a locally C^*-algebra. A left multiplier of A is a linear map $l : A \to A$ such that $l(ab) = l(a)b$ for all a and b in A, and a right multiplier of A is a linear map $r : A \to A$ such that $r(ab) = ar(b)$ for all a and b in A. The set $LM(A)$ of all left multipliers of A is an algebra. For each $p \in S(A)$, the map $p_{LM(A)} : LM(A) \to [0, \infty)$ defined by $p_{LM(A)}(l) = \sup\{p(l(a)); a \in A, p(a) \leq 1\}$ is a seminorm on $LM(A)$ such that $p_{LM(A)}(l_1 l_2) \leq p_{LM(A)}(l_1) p_{LM(A)}(l_2)$ for all $l_1, l_2 \in LM(A)$.

A multiplier of A is a pair (l, r), where l is a left multiplier and r is a right multiplier, such that $al(b) = r(a)b$ for all a and b in A. The set $M(A)$ of all multipliers of A is an algebra with involution; addition is defined as usual, multiplication is $(l_1, r_1)(l_2, r_2) = (l_1 l_2, r_1 r_2)$ and involution is $(l, r)^* = (r^*, l^*)$, where $r^*(a) = r(a^*)^*$ and $l^*(a) = l(a^*)^*$ for all $a \in A$. For each $p \in S(A)$, the map $p_{M(A)} : M(A) \to [0, \infty)$ defined by $p_{M(A)}(l, r) = \sup\{p(l(a)); a \in A, p(a) \leq 1\}$ is a C^*-seminorm on $M(A)$.

Exactly as in the case of C^* -algebras, any left (right) multiplier of a locally C^*-algebra A is automatically continuous. Moreover, if $l \in LM(A)$, then for each $p \in S(A)$, there is a unique $l_p \in LM(A_p)$ such that $\pi_p \circ l = l_p \circ \pi_p$ (see [**24**]), and if $(l, r) \in M(A)$, then for each $p \in S(A)$, there is a unique $(l_p, r_p) \in M(A_p)$ such that $\pi_p \circ l = l_p \circ \pi_p$ and $\pi_p \circ r = r_p \circ \pi_p$ (see, for example, [**38**]).

Theorem 1.1.19 *Let A be a locally C^*-algebra. Then:*

1. *$LM(A)$ equipped with the topology determined by the family of seminorms $\{p_{LM(A)}\}_{p\in S(A)}$ is a complete locally m -convex algebra.*

2. *$M(A)$ is a locally C^*-algebra with respect to the topology determined by the family of C^*-seminorms $\{p_{M(A)}\}_{p\in S(A)}$.*

Proof. Let $p, q \in S(A)$ with $p \geq q$. Since π_{pq} is surjective, it extends uniquely to a morphism of W^*-algebras $\pi_{pq}^{''}$ from $A_p^{''}$ to $A_q^{''}$, where $A_p^{''}$ is the enveloping W^*-algebra of A_p. Morover, $\pi_{pq}^{''}(LM(A_p)) \subseteq LM(A_q)$ and $\pi_{pq}^{''}(M(A_p)) \subseteq M(A_q)$. It is not difficult to check that $\{LM(A_p); \pi_{pq}^{''}|_{LM(A_p)}\}_{p\geq q,p,q\in S(A)}$ is an inverse system of Banach algebras and $\{M(A_p); \pi_{pq}^{''}|_{M(A_p)}\}_{p\geq q,p,q\in S(A)}$ is an inverse system of C^*-algebras.

1. Since $\lim\limits_{\overleftarrow{p}} LM(A_p)$ is a complete locally m -convex algebras, to show that $LM(A)$ is a complete locally m-convex algebra it is sufficient to prove that $LM(A)$ is isomorphic to $\lim\limits_{\overleftarrow{p}} LM(A_p)$. Let $l \in LM(A)$. It is not difficult to check that $(l_p)_p$, where $\pi_p \circ l = l_p \circ \pi_p$ for all $p \in S(A)$, is a coherent sequence in $LM(A_p)$. Then we can define a map $\Psi : LM(A) \to \lim\limits_{\overleftarrow{p}} LM(A_p)$ by $\Psi(l) = (l_p)_p$. A simple calculus shows that Ψ is linear and $\Psi(l_1 l_2) = \Psi(l_1)\Psi(l_2)$ for all $l_1, l_2 \in LM(A)$. We denote by $\{\widetilde{p}\}_{p\in S(A)}$ the family of seminorms which defines the topology on $\lim\limits_{\overleftarrow{p}} LM(A_p)$. Clearly, $\widetilde{p}\left((l_q)_q\right) = \|l_p\|_{LM(A_p)}$. Then

$$\begin{aligned}\widetilde{p}(\Psi(l)) &= \|l_p\|_{LM(A_p)} = \sup\{\|l_p(\pi_p(a))\|_{A_p}; \|\pi_p(a)\|_{A_p} \leq 1\} \\ &= \sup\{\|\pi_p(l(a))\|_{A_p}; \|\pi_p(a)\|_{A_p} \leq 1\} \\ &= \sup\{p(l(a)); p(a) \leq 1\} = p_{LM(A)}(l)\end{aligned}$$

for all $p \in S(A)$ and for all $l \in LM(A)$. From this fact we conclude that Ψ is an injective morphism of locally m -convex algebras with closed range. To show that Ψ is surjective, let $(l_p)_p$ be a coherent sequence in $LM(A_p)$. Define a map $l : A \to A$ by $l(a) = (l_p(\pi_p(a)))_p$. Since

$$\pi_{pq}(l_p(\pi_p(a))) = \pi_{pq}^{''}(l_p(\pi_p(a)) = \pi_{pq}^{''}(l_p)\pi_{pq}^{''}(\pi_p(a)) = l_q(\pi_q(a))$$

for all $a \in A$ and for all $p, q \in S(A)$ with $p \geq q$, l is well defined. Moreover, $l \in LM(A)$, since

$$l(ab) = (l_p(\pi_p(ab)))_p = (l_p(\pi_p(a))\pi_p(b))_p = l(a)b$$

for all $a, b \in A$. Clearly, $\Psi(l) = (l_p)_p$. Therefore Ψ is surjective. Thus we showed that Ψ is a bijective continuous morphism from $LM(A)$ onto $\lim\limits_{\leftarrow p} LM(A_p)$ and since $\widetilde{p}(\Psi(l)) = p_{LM(A)}(l)$ for all $p \in S(A)$ and for all $l \in LM(A)$, Ψ is an isomorphism of locally m -convex algebras.

2. In the same manner as in the proof of the assertion 1, we show that the map $\Phi : M(A) \rightarrow \lim\limits_{\leftarrow p} M(A_p)$ defined by $\Phi(l, r) = ((l_p, r_p))_p$ is a bijective morphism from $M(A)$ onto $\lim\limits_{\leftarrow p} M(A_p)$ and Φ^{-1} is a continuous morphism. From these facts and taking into account that $\lim\limits_{\leftarrow p} M(A_p)$ is a locally C^*-algebra and $M(A)$ is a pre-locally C^*-algebra, we conclude that $M(A)$ is a locally C^*-algebra. ■

Corollary 1.1.20 *Let $\{A_\lambda; \pi_{\lambda\mu}\}_{\lambda \geq \mu, \lambda, \mu \in \Lambda}$ be an inverse system of C^*-algebras such that the canonical maps χ_λ from $\lim\limits_{\leftarrow \lambda} A_\lambda$ to A_λ are all surjective. If $A = \lim\limits_{\leftarrow \lambda} A_\lambda$, then the complete locally m -convex algebras $LM(A)$ and $\lim\limits_{\leftarrow \lambda} LM(A_\lambda)$ as well as the locally C^*-algebras $M(A)$ and $\lim\limits_{\leftarrow \lambda} M(A_\lambda)$ are isomorphic.*

Proof. The topology on A is defined by the family of C^* -seminorms $\{p_\lambda\}_{\lambda \in \Lambda}$, where $p_\lambda((a_\mu)_\mu) = \|a_\lambda\|_{A_\lambda}$. Since the canonical maps χ_λ from $\lim\limits_{\leftarrow \lambda} A_\lambda$ to A_λ are all surjective, it is not difficult to check that the C^*-algebras A_λ and A_{p_λ} are isomorphic for each $\lambda \in \Lambda$. Then to prove the corollary we apply Theorem 1.1.19 for the locally C^*-algebra A. ■

Let A be a locally C^*-algebra and let H be a Hilbert space. A representation of A on H is a continuous $*$-morphism φ from A to $L(H)$. If φ is a representation of A, then there is $p \in S(A)$ such that $\|\varphi(a)\| \leq p(a)$ for all $a \in A$ and so there is a unique representation φ_p of A_p such that $\varphi_p \circ \pi_p = \pi_p \circ \varphi$. Clearly, if φ_p is a representation of the C^*-algebra A_p, then $\varphi_p \circ \pi_p$ is a representation of A.

Let $R(A) = \{\varphi;\ \varphi$ is a representation of $A\}$ and for each $p \in S(A)$, let $R_p(A) = \{\varphi \in R(A); \|\varphi(a)\| \leq p(a)$ for all $a \in A\}$. Clearly, $R(A) = \bigcup\limits_p R_p(A)$.

Let A and B be two locally C^*-algebras. For $p \in S(A)$ and $q \in S(B)$ define

$t_{(p,q)} : A \otimes_{\text{alg}} B \to [0, \infty)$ by

$$t_{(p,q)}(c) = \sup\{\|(\varphi \otimes \psi)(c)\| \,; \varphi \in R_p(A), \psi \in R_q(B)\}.$$

Clearly, $t_{(p,q)}$ is a C^*-seminorm on $A \otimes_{\text{alg}} B$. The minimal or injective tensor product of the locally C^* -algebras A and B, denoted by $A \otimes B$, is the completion of the algebraic tensor product $A \otimes_{\text{alg}} B$ with respect to the topology determined by the family of C^*-seminorms $\{t_{(p,q)}\}_{(p,q) \in S(A) \times S(B)}$.

Proposition 1.1.21 *Let A and B be two locally C^*-algebras. Then the C^*-algebras $(A \otimes B)_{(p,q)}$ and $A_p \otimes B_q$ are isomorphic for all $p \in S(A)$ and for all $q \in S(B)$.*

Proof. Let $p \in S(A)$ and let $q \in S(B)$. Since

$$\begin{aligned} t_{(p,q)}\left(\sum_{i=1}^{n} a_i \otimes b_i\right) &= \sup\left\{\left\|(\varphi \otimes \psi)\left(\sum_{i=1}^{n} a_i \otimes b_i\right)\right\| ; \varphi \in R_p(A), \psi \in R_q(B)\right\} \\ &= \sup\left\{\left\|(\varphi_p \otimes \psi_p)\left(\sum_{i=1}^{n} \pi_p(a_i) \otimes \pi_q(b_i)\right)\right\| ; \varphi_p \in R(A_p), \psi_q \in R(B_q)\right\} \\ &= \left\|\sum_{i=1}^{n} \pi_p(a_i) \otimes \pi_q(b_i)\right\|_{A_p \otimes B_q} \end{aligned}$$

for all $\sum_{i=1}^{n} a_i \otimes b_i \in A \otimes_{\text{alg}} B$, we can define a linear map $\varphi_{(p,q)} : A \otimes_{\text{alg}} B / \ker\left(t_{(p,q)}\right) \to A_p \otimes B_q$ by

$$\varphi_{(p,q)}\left(\sum_{i=1}^{n} a_i \otimes b_i + \ker\left(t_{(p,q)}\right)\right) = \sum_{i=1}^{n} \pi_p(a_i) \otimes \pi_q(b_i).$$

It is not difficult to check that $\varphi_{(p,q)}$ is an isometric $*$ -morphism from $A \otimes_{\text{alg}} B / \ker\left(t_{(p,q)}\right)$ to $A_p \otimes B_q$. Moreover, $\varphi_{(p,q)}\left(A \otimes_{\text{alg}} B / \ker\left(t_{(p,q)}\right)\right) = A_p \otimes_{\text{alg}} B_q$. From these facts and taking into account that $A_p \otimes_{\text{alg}} B_q$ is dense in $A_p \otimes B_q$ and $A \otimes_{\text{alg}} B / \ker\left(t_{(p,q)}\right)$ is dense in $(A \otimes B)_{(p,q)}$ (see, for example, [**38**]), we conclude that $\varphi_{(p,q)}$ extends to an isomorphism from $(A \otimes B)_{(p,q)}$ onto $A_p \otimes B_q$. ∎

Corollary 1.1.22 *Let A and B be two locally C^*-algebras. Then the locally C^*-algebras $A \otimes B$ and $\varprojlim_{(p,q)} A_p \otimes B_q$ are isomorphic.*

Corollary 1.1.23 *Let A be a locally C^*-algebra. Then the locally C^*-algebras $A \otimes A$ and $\lim\limits_{\substack{\leftarrow \\ p}} A_p \otimes A_p$ are isomorphic as well as $A \otimes \mathbb{C}$ and A.*

1.2 Definitions, notation and examples of Hilbert modules

In this Section we introduce the notion of Hilbert modules over a locally C^*-algebra and we present some examples of Hilbert modules.

Let A be a locally C^*-algebra.

Definition 1.2.1 *A pre-Hilbert A-module is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ which is $\mathbb{C}$- and A-linear in its second variable and satisfies the following relations:*

*(**a**) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$;*

*(**b**) $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$;*

*(**c**) $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.*

Proposition 1.2.2 *(Cauchy-Schwarz Inequality, [**14**, **45**]). Let E be a right A - module equipped with an A -valued inner-product $\langle \cdot, \cdot \rangle$ which is $\mathbb{C}$ - and A -linear in its second variable and satisfies the conditions (a) and (b) from Definition 1.2.1. Then for each $p \in S(A)$ and for all $\xi, \eta \in E$, we have*

$$p(\langle \xi, \eta \rangle)^2 \leq p(\langle \xi, \xi \rangle)\, p(\langle \eta, \eta \rangle).$$

Proof. When $\xi, \eta \in E$ and $a \in A$,

$$0 \leq \langle \xi a - \eta, \xi a - \eta \rangle = a^* \langle \xi, \xi \rangle a - a^* \langle \xi, \eta \rangle - \langle \eta, \xi \rangle a + \langle \eta, \eta \rangle.$$

By taking $a = \lambda \langle \xi, \eta \rangle$, where λ is a positive number, we obtain

$$0 \leq 2\lambda \langle \xi, \eta \rangle^* \langle \xi, \eta \rangle \leq \lambda^2 \langle \xi, \eta \rangle^* \langle \xi, \xi \rangle \langle \xi, \eta \rangle + \langle \eta, \eta \rangle.$$

From this relation and Corollary 2.3 in [**11**], we conclude that

$$0 \leq 2\lambda p\left(\langle \xi, \eta \rangle\right)^2 \leq \lambda^2 p\left(\langle \xi, \xi \rangle\right) p\left(\langle \xi, \eta \rangle\right)^2 + p\left(\langle \eta, \eta \rangle\right).$$

If $p\left(\langle \xi, \xi \rangle\right) \neq 0$, for $\lambda = p\left(\langle \xi, \xi \rangle\right)^{-1}$, we obtain

$$2p\left(\langle \xi, \xi \rangle\right)^{-1} p\left(\langle \xi, \eta \rangle\right)^2 \leq p\left(\langle \xi, \xi \rangle\right)^{-1} p\left(\langle \xi, \eta \rangle\right)^2 + p\left(\langle \eta, \eta \rangle\right),$$

whence

$$p\left(\langle \xi, \eta \rangle\right)^2 \leq p\left(\langle \xi, \xi \rangle\right) p\left(\langle \eta, \eta \rangle\right).$$

If $p\left(\langle \xi, \xi \rangle\right) = 0$, we have $2\lambda p\left(\langle \xi, \eta \rangle\right)^2 \leq p\left(\langle \eta, \eta \rangle\right)$, whence, since λ is an arbitrary positive number, we conclude that $p\left(\langle \xi, \eta \rangle\right) = 0$. Therefore

$$p\left(\langle \xi, \eta \rangle\right)^2 = p\left(\langle \xi, \xi \rangle\right) p\left(\langle \eta, \eta \rangle\right) = 0$$

and so the inequality is true in this case too. ■

Corollary 1.2.3 *([45]). Let E be a pre-Hilbert A-module. Then for each $p \in S(A)$ the map $\overline{p} : E \to [0, \infty)$ defined by*

$$\overline{p}(\xi) = \sqrt{p\left(\langle \xi, \xi \rangle\right)}, \quad \xi \in E$$

is a seminorm on E. Moreover, the following relations hold:

i. $\overline{p}(\xi a) \leq \overline{p}(\xi) p(a)$ *for all* $\xi \in E$ *and for all* $a \in A$;

ii. $\overline{p}(\xi) = \sup\left\{p\left(\langle \xi, \eta \rangle\right); \overline{p}(\eta) \leq 1\right\}$;

iii. $\overline{p}(\xi) = 0$ *for all* $p \in S(A)$ *implies* $\xi = 0$.

Remark 1.2.4 *If E is a pre-Hilbert A -module, then E equipped with the topology determined by the family of seminorms $\{\overline{p}\}_{p \in S(A)}$ is a separable locally convex space.*

Definition 1.2.5 *A Hilbert A-module is a pre-Hilbert A -module E which is complete with respect to the topology determined by the family of seminorms $\{\overline{p}\}_{p \in S(A)}$.*

We will use the notation $\overline{p}_E$ in place $\overline{p}$, when we are dealing with more than one Hilbert module over the same locally C^*-algebra.

Definition 1.2.6 *Two Hilbert A -modules E and F are isomorphic if there is a surjective module homomorphism Φ from E onto F such that*

$$\langle \Phi(\xi), \Phi(\eta) \rangle = \langle \xi, \eta \rangle$$

for all $\xi, \eta \in E$.

Remark 1.2.7 *Let E be a right A -module equipped with an A -valued inner-product $\langle \cdot, \cdot \rangle$ which is $\mathbb{C}$ - and A -linear in its second variable and satisfies the conditions (a) and (b) of Definition 1.2.1. From Proposition 1.2.2, we conclude that*

$$N = \{\xi \in E; \langle \xi, \xi \rangle = 0\}$$

is a closed A -submodule of E. On the quotient A -module E/N, we define an A -valued inner-product $\langle \cdot, \cdot \rangle_0$ by

$$\langle \xi + N, \eta + N \rangle_0 = \langle \xi, \eta \rangle, \xi, \eta \in E.$$

According to Proposition 1.2.2, this inner-product is well -defined. Moreover, E/N equipped with this inner-product becomes a pre-Hilbert A -module.

Remark 1.2.8 *If E_0 is a pre -Hilbert A -module and E is its completion with respect to the topology induced by the inner-product, then E is a Hilbert A -module.*

Proof. Indeed, for ξ and η in E there are the nets $\{\xi_i\}_{i \in I}$ and $\{\eta_j\}_{j \in J}$ in E_0 such that $\xi = \lim_i \xi_i$ and $\eta = \lim_j \eta_j$. Then for each $p \in S(A)$, the nets of real numbers $\{\overline{p}(\xi_i)\}_{i \in I}$ and $\{\overline{p}(\eta_j)\}_{j \in J}$ are convergent and so they are bounded.

Let $p \in S(A)$, $M_p > 0$ such that $\overline{p}(\xi_i) < M_p$ for all $i \in I$ and $\overline{p}(\eta_j) < M_p$ for all $j \in J$ and $\varepsilon > 0$. Then there is $i_0 \in I$ and $j_0 \in J$ such that

$$\overline{p}(\xi_{i_1} - \xi_{i_2}) < M_p \varepsilon / 2 \text{ and } \overline{p}(\eta_{j_1} - \eta_{j_2}) < M_p \varepsilon / 2$$

for all i_1 and i_2 in I with $i_1 \geq i_0$ and $i_2 \geq i_0$ and for all j_1 and j_2 in J with $j_1 \geq j_0$ and $j_2 \geq j_0$. Moreover,

$$\begin{aligned} p(\langle \xi_{i_1}, \eta_{j_1} \rangle - \langle \xi_{i_2}, \eta_{j_2} \rangle) &\leq p(\langle \xi_{i_1} - \xi_{i_2}, \eta_{j_1} \rangle) + p(\langle \xi_{i_2}, \eta_{j_1} - \eta_{j_2} \rangle) \\ &\leq \overline{p}(\xi_{i_1} - \xi_{i_2}) \overline{p}(\eta_{j_1}) + \overline{p}(\xi_{i_2}) \overline{p}(\eta_{j_1} - \eta_{j_2}) < \varepsilon \end{aligned}$$

for all (i_1, j_1) and (i_2, j_2) in $I \times J$ with $(i_1, j_1) \geq (i_0, j_0)$ and $(i_2, j_2) \geq (i_0, j_0)$. So the net $\{\langle \xi_i, \eta_j \rangle\}_{(i,j) \in I \times J}$ is convergent in A.

Let $\{\widetilde{\xi}_{i'}\}_{i' \in I'}$ and $\{\widetilde{\eta}_{j'}\}_{j' \in J'}$ be another nets in E_0 such that $\xi = \lim\limits_{i'} \widetilde{\xi}_{i'}$ and $\eta = \lim\limits_{j'} \widetilde{\eta}_{j'}$ and $a \in A$. From

$$\overline{p}(\xi_i a - \widetilde{\xi}_{i'} a) \leq (\overline{p}(\xi_i - \xi) + \overline{p}(\xi - \widetilde{\xi}_{i'}))p(a)$$

for all $i \in I$ and $i' \in I'$ and

$$\begin{aligned} p(\langle \xi_i, \eta_j \rangle - \left\langle \widetilde{\xi}_{i'}, \widetilde{\eta}_{j'} \right\rangle) \ &\leq \ (\overline{p}(\xi_i - \xi) + \overline{p}(\xi - \widetilde{\xi}_{i'}))\overline{p}(\eta_j) \\ &\quad + (\overline{p}(\eta_j - \eta) + \overline{p}(\widetilde{\eta}_{j'} - \eta))\overline{p}(\widetilde{\xi}_{i'}) \end{aligned}$$

for all $(i, j) \in I \times J$ and for all $(i', j') \in I' \times J'$, we conclude that $\lim\limits_{i} \xi_i a = \lim\limits_{i'} \widetilde{\xi}_{i'} a$ and $\lim\limits_{(i,j)} \langle \xi_i, \eta_j \rangle = \lim\limits_{(i',j')} \left\langle \widetilde{\xi}_{i'}, \widetilde{\eta}_{j'} \right\rangle$. Thus the module action of A on E_0 extends to a module action of A on E by

$$\xi a = \lim_i \xi_i a$$

and the inner-product on E_0 extends to an inner-product on E by

$$\langle \xi, \eta \rangle = \lim_{(i,j)} \langle \xi_i, \eta_j \rangle .$$

In this way, E becomes a Hilbert A -module. ■

Remark 1.2.9 *Let A_0 be a pre-locally C^*-algebra and let A be its completion. Suppose that E is a Hilbert A_0-module, that is, E is a right A_0 -module equipped with an A_0 -valued inner-product which is $\mathbb{C}$ - and A_0 -linear in its second variable and verifies the conditions from Definition 1.2.1, and is complete with respect to the topology induced by the inner-product.*

Let a in A and ξ in E. Then there is a net $\{a_i\}_{i \in I}$ in A_0 such that $a = \lim\limits_i a_i$. From Corollary 1.2.3 (i), we deduce that $\{\xi a_i\}_{i \in I}$ is a fundamental net in E and so it is convergent. If $\{b_j\}_{j \in J}$ is another net in A_0 which converges to a, then from Corollary 1.2.3 (i), we conclude that the nets $\{\xi a_i\}_{i \in I}$ and $\{\xi b_j\}_{j \in J}$ have the same limit. Thus we can extend the module action of A_0 on E by continuity to a module action of A on E and thus E becomes a Hilbert A -module.

Remark 1.2.10 *Let A_0 be a pre-locally C^*-algebra and let E_0 be a pre-Hilbert A_0-module. If A is the completion of A_0 and E is the completion of E_0, by Remarks 1.2.8 and 1.2.9, E becomes a Hilbert A -module.*

Remark 1.2.11 *Let A and B be two isomorphic locally C^*-algebras. If E is a Hilbert A -module, then E becomes a Hilbert B -module with the module action of B on E defined by*

$$\xi b = \xi \Phi^{-1}(b), \xi \in E, b \in B$$

and the B -valued inner-product defined by

$$\langle \xi, \eta \rangle_B = \Phi\left(\langle \xi, \eta \rangle_A\right)$$

where Φ is an isomorphism from A onto B and $\langle \cdot, \cdot \rangle_A$ denotes the A -valued inner-product on E.

Let E be a Hilbert A -module. Then the closed span of the set $\{\langle \xi, \eta \rangle ; \xi, \eta \in E\}$ is a two-sided $*$ -ideal in A. We denote it by $\langle E, E \rangle$.

Proposition 1.2.12 *If E is a Hilbert A -module, then $E\langle E, E \rangle$ is dense in E.*

Proof. Let $\{u_i\}_{i \in I}$ be an approximate unit for $\langle E, E \rangle$, $\xi \in E$ and $p \in S(A)$. Then

$$\begin{aligned} \overline{p}\left(\xi u_i - \xi\right)^2 &= p\left(\langle \xi u_i - \xi, \xi u_i - \xi \rangle\right) \\ &= p\left(\langle \xi, \xi \rangle - \langle \xi, \xi \rangle u_i - u_i \langle \xi, \xi \rangle + u_i \langle \xi, \xi \rangle u_i\right) \\ &\leq 2p\left(\langle \xi, \xi \rangle u_i - \langle \xi, \xi \rangle\right) \to 0. \end{aligned}$$

This shows that $\{\xi u_i\}_{i \in I}$ converges to ξ and so $E\langle E, E \rangle$ is dense in E. ■

Corollary 1.2.13 *If E is a Hilbert A -module, then EA is dense in E.*

Remark 1.2.14 *If E is a Hilbert A -module and A is unital, then $\xi 1 = \xi$ for all $\xi \in E$.*

If A is not unital and A^+ is the unitization of A, then E becomes a Hilbert A^+ -module if we define $\xi 1 = \xi$ for all $\xi \in E$.

Example 1.2.15 *Any locally C^*-algebra A is a Hilbert A -module with the inner-product defined by $\langle a,b\rangle = a^*b,\ a,b\in A$.*

Example 1.2.16 *Any closed right ideal I of a locally C^*-algebra A equipped with the inner product $\langle a,b\rangle = a^*b,\ a,b\in I$ is a Hilbert A -module.*

Example 1.2.17 *If $\{E_i\}_{i=1}^n$ is a finite set of Hilbert A -modules, then the direct sum $\bigoplus\limits_{i=1}^{n} E_i$ is a right A -module in the obvious way and it becomes a Hilbert A -module if we define the inner-product by*

$$\langle (\xi_i)_{i=1}^n , (\eta_i)_{i=1}^n \rangle = \sum_{i=1}^{n} \langle \xi_i, \eta_i \rangle .$$

Example 1.2.18 *Let $\{E_n\}_n$ be a countable set of Hilbert A -modules. We denote by $\bigoplus\limits_{n} E_n$ the set of all sequences $(\xi_n)_n$ with ξ_n in E_n such that $\sum\limits_n \langle \xi_n, \xi_n\rangle$ converges in A. Then $\bigoplus\limits_{n} E_n$ becomes a Hilbert A -module with the action of A on $\bigoplus\limits_{n} E_n$ defined by $(\xi_n)_n a = (\xi_n a)_n$ and the inner product defined by*

$$\langle (\xi_n)_n , (\eta_n)_n \rangle = \sum_{n} \langle \xi_n, \eta_n \rangle .$$

To show that the module action of A on $\bigoplus\limits_{n} E_n$ and the A -valued inner-product on $\bigoplus\limits_{n} E_n$ are well-defined, let $a\in A$, $(\xi_n)_n$, $(\eta_n)_n \in \bigoplus\limits_{n} E_n$, $p\in S(A)$ and $\varepsilon > 0$. Then there is n_0 such that

$$p(\sum_{k=n}^{m} \langle \xi_k, \xi_k\rangle) < \varepsilon \text{ and } p(\sum_{k=n}^{m} \langle \eta_k, \eta_k\rangle) < \varepsilon$$

for all positive integers n and m with $m\geq n\geq n_0$ and so

$$p(\sum_{k=n}^{m} \langle \xi_k a, \xi_k a\rangle) = p(a^* \sum_{k=n}^{m} \langle \xi_k, \xi_k\rangle\, a) \leq p(a)^2 \varepsilon$$

and

$$p(\sum_{k=n}^{m} \langle \xi_k, \eta_k\rangle)^2 \ = \ p\left(\langle (\xi_k)_{k=n}^m , (\eta_k)_{k=n}^m \rangle\right)^2$$

(*cf. Cauchy-Schwarz Inequality*)

$$\leq \; p\left(\langle(\xi_k)_{k=n}^m, (\xi_k)_{k=n}^m\rangle\right) p\left(\langle(\eta_k)_{k=n}^m, (\eta_k)_{k=n}^m\rangle\right)$$

$$= \; p(\sum_{k=n}^m \langle \xi_k, \xi_k\rangle)p(\sum_{k=n}^m \langle \eta_k, \eta_k\rangle) < \varepsilon^2$$

for all positive integers n *and* m *with* $m \geq n \geq n_0$. *These relations show that the module action of* A *on* $\bigoplus_n E_n$ *and the* A *-valued inner-product on* $\bigoplus_n E_n$ *are well-defined. It is not difficult to check that* $\bigoplus_n E_n$ *becomes a pre-Hilbert* A *-module.*

To show that $\bigoplus_n E_n$ *is complete with respect to the topology induced by the inner-product, let* $\{(\xi_n^i)_n\}_{i\in I}$ *be a fundamental net in* $\bigoplus_n E_n$, $\varepsilon > 0$ *and* $p \in S(A)$. *Since* $\{(\xi_n^i)_n\}_{i\in I}$ *is a fundamental net in* $\bigoplus_n E_n$, *there is* i_0 *in* I *such that*

$$p\left(\sum_n \left\langle \xi_n^{i_1} - \xi_n^{i_2}, \xi_n^{i_1} - \xi_n^{i_2}\right\rangle\right) \leq \varepsilon/8$$

for all $i_1, i_2 \in I$ *with* $i_1 \geq i_0$ *and* $i_2 \geq i_0$. *From this inequality, we conclude that*

$$p\left(\left\langle \xi_n^{i_1} - \xi_n^{i_2}, \xi_n^{i_1} - \xi_n^{i_2}\right\rangle\right) \leq \varepsilon/8$$

for all $i_1, i_2 \in I$ *with* $i_1 \geq i_0$ *and* $i_2 \geq i_0$ *and for all positive integer* n. *Therefore, for any positive integer* n, $\{\xi_n^i\}_{i\in I}$ *is a fundamental net in* E_n, *and so it converges to an element* ξ_n *in* E_n.

We show that $(\xi_n)_n$ *is an element in* $\bigoplus_n E_n$. *Let* $i_3 \in I$ *such that* $i_3 \geq i_0$. *Since* $(\xi_n^{i_3})_n$ *is an element in* $\bigoplus_n E_n$ *there is a positive integer* n_0 *such that*

$$p\left(\sum_{k=n_0}^n \left\langle \xi_k^{i_3}, \xi_k^{i^3}\right\rangle\right) \leq \varepsilon/8$$

for all positive integer n *with* $n \geq n_0$. *From*

$$p\left(\sum_{k=n_0}^n \left\langle \xi_k^i, \xi_k^i\right\rangle\right) \; \leq \; p\left(\sum_n \left\langle \xi_n^i - \xi_n^{i_3}, \xi_n^i - \xi_n^{i_3}\right\rangle\right) + p\left(\sum_{k=n_0}^n \left\langle \xi_k^{i_3}, \xi_k^{i_3}\right\rangle\right)$$

$$
\begin{aligned}
&\quad +2p\left(\sum_{k=n_0}^{n}\left\langle \xi_k^i-\xi_k^{i_3},\xi_k^i\right\rangle\right)\\
&\leq\ \varepsilon/8+\varepsilon/8\\
&\quad +2p\left(\sum_{k=n_0}^{n}\left\langle \xi_k^i-\xi_k^{i_3},\xi_k^i-\xi_k^{i_3}\right\rangle\right)^{1/2} p\left(\sum_{k=n_0}^{n}\left\langle \xi_k^i,\xi_k^i\right\rangle\right)^{1/2}\\
&\leq\ \varepsilon/4+\sqrt{\varepsilon/2}\,p\left(\sum_{k=n_0}^{n}\left\langle \xi_k^i,\xi_k^i\right\rangle\right)^{1/2}
\end{aligned}
$$

for all positive integer n *with* $n\geq n_0$ *and for all* $i\in I$ *with* $i\geq i_3$, *we obtain*

$$
p\left(\sum_{k=n_0}^{n}\left\langle \xi_k^i,\xi_k^i\right\rangle\right)\leq\varepsilon
$$

for all positive integer n *with* $n\geq n_0$ *and for all* $i\in I$ *with* $i\geq i_3$.

Therefore

$$
p\left(\sum_{k=n_0}^{n}\left\langle \xi_k,\xi_k\right\rangle\right)=\lim_i p\left(\sum_{k=n_0}^{n}\left\langle \xi_k^i,\xi_k^i\right\rangle\right)\leq\varepsilon
$$

for all positive integer n *with* $n\geq n_0$. *This shows that* $(\xi_n)_n$ *is an element in* $\bigoplus\limits_n E_n$.

From

$$
\begin{aligned}
p\left(\sum_{k=1}^{n}\left\langle \xi_k-\xi_k^{i_1},\xi_k-\xi_k^{i_1}\right\rangle\right)\ &=\ \lim_i p\left(\sum_{k=1}^{n}\left\langle \xi_k^i-\xi_k^{i_1},\xi_k^i-\xi_k^{i_1}\right\rangle\right)\\
&\leq\ \lim_i p\left(\sum_{n}\left\langle \xi_k^i-\xi_k^{i_1},\xi_k^i-\xi_k^{i_1}\right\rangle\right)\leq\varepsilon/8
\end{aligned}
$$

for all positive integer n *and for all* $i_1\in I$ *with* $i_1\geq i_0$, *we conclude that the net* $\{(\xi_n^i)_n\}_{i\in I}$ *converges to* $(\xi_n)_n$.

The direct sum of a countable number of copies of a Hilbert A *-module* E *will be denoted by* H_E.

Let A be a locally C^*-algebra. Let $M_n(A)$ denote the $*$ -algebra of all $n \times n$ matrices bover A, with the usual algebraic operations and the topology obtained by regarding it as a direc sum of n^2 copies of A. Thus $M_n(A)$ is a locally C^*-algebra and moreover, $M_n(A)$ is isomorphic with $\lim\limits_{\underset{p}{\leftarrow}} M_n(A_p)$.

Example 1.2.19 *Let E be a Hilbert A -module and let n be a positive integer. It is not difficult to check that $\bigoplus\limits_{i=1}^{n} E$ is a right $M_n(A)$ -module with the action of $M_n(A)$ on $\bigoplus\limits_{i=1}^{n} E$ defined by $(\xi_i)_{i=1}^n [a_{ij}]_{i,j=1}^n = \left(\sum\limits_{i=1}^{n} \xi_i a_{ij}\right)_{j=1}^n$. The map $\langle \cdot, \cdot \rangle_{M_n(A)}$ from $\left(\bigoplus\limits_{i=1}^{n} E\right) \times \left(\bigoplus\limits_{i=1}^{n} E\right)$ to $M_n(A)$ defined by*

$$\langle (\xi_i)_{i=1}^n, (\eta_i)_{i=1}^n \rangle_{M_n(A)} = \left[\langle \xi_i, \eta_j \rangle\right]_{i,j=1}^n$$

is a $M_n(A)$ -valued inner-product on $\bigoplus\limits_{i=1}^{n} E$ which is $\mathbb{C}$ - and $M_n(A)$ -linear in its second variable and verifies the condition (a) and (c) from Definition 1.2.1. Let $(\xi_i)_{i=1}^n \in \bigoplus\limits_{i=1}^{n} E$. Since

$$\sum_{i,j=1}^{n} a_i^* \langle \xi_i, \xi_j \rangle a_j = \sum_{i,j=1}^{n} \langle \xi_i a_i, \xi_j a_j \rangle = \left\langle \sum_{i=1}^{n} \xi_i a_i, \sum_{j=1}^{n} \xi_j a_j \right\rangle \geq 0$$

for all $a_1, ..., a_n \in A$, $\left[\langle \xi_i, \xi_j \rangle\right]_{i,j=1}^n$ is positive in $M_n(A)$ and so $\langle (\xi_i)_{i=1}^n, (\xi_i)_{i=1}^n \rangle_{M_n(A)} \geq 0$. Therefore $\left(\bigoplus\limits_{i=1}^{n} E, \langle \cdot, \cdot \rangle_{M_n(A)}\right)$ is a pre-Hilbert $M_n(A)$ -module. It is not difficult to check that $\bigoplus\limits_{i=1}^{n} E$ is complete with respect to the topology induced by the inner-product $\langle \cdot, \cdot \rangle_{M_n(A)}$. Hence $\bigoplus\limits_{i=1}^{n} E$ is a Hilbert $M_n(A)$ -module.

Definition 1.2.20 *Let $\{A_\lambda; \pi_{\lambda\mu}\}_{\lambda \geq \mu, \lambda, \mu \in \Lambda}$ be an inverse system of C^*-algebras and let $\{E_\lambda; \sigma_{\lambda\mu}\}_{\lambda \geq \mu, \lambda, \mu \in \Lambda}$ be an inverse system of vector spaces. We say that $\{E_\lambda; \sigma_{\lambda\mu}; A_\lambda; \pi_{\lambda\mu}\}_{\lambda \geq \mu, \lambda, \mu \in \Lambda}$ is an inverse system of Hilbert C^*-modules if for each $\lambda \in \Lambda$, E_λ is a Hilbert C^*-module over A_λ, and the following conditions are satisfied:*

(**a**) $\sigma_{\lambda\mu}(\xi_\lambda a_\lambda) = \sigma_{\lambda\mu}(\xi_\lambda)\pi_{\lambda\mu}(a_\lambda)$ *for all* $\xi_\lambda \in E_\lambda$ *and* $a_\lambda \in A_\lambda$;

(**b**) $\langle \sigma_{\lambda\mu}(\xi_\lambda), \sigma_{\lambda\mu}(\eta_\lambda)\rangle = \pi_{\lambda\mu}(\langle \xi_\lambda, \eta_\lambda\rangle)$ *for all* $\xi_\lambda, \eta_\lambda \in E_\lambda$.

Proposition 1.2.21 *Let* $\{E_\lambda; \sigma_{\lambda\mu}; A_\lambda; \pi_{\lambda\mu}\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ *be an inverse system of Hilbert* C^**-modules. Then* $\lim\limits_{\overleftarrow{\lambda}} E_\lambda$ *is a Hilbert* $\lim\limits_{\overleftarrow{\lambda}} A_\lambda$ *-module with the action of* $\lim\limits_{\overleftarrow{\lambda}} A_\lambda$ *on* $\lim\limits_{\overleftarrow{\lambda}} E_\lambda$ *defined by*

$$(\xi_\lambda)_\lambda (a_\lambda)_\lambda = (\xi_\lambda a_\lambda)_\lambda$$

and the inner-product defined by

$$\langle (\xi_\lambda)_\lambda, (\eta_\lambda)_\lambda \rangle = (\langle \xi_\lambda, \eta_\lambda \rangle)_\lambda .$$

Proof. Let $(\xi_\lambda)_\lambda, (\eta_\lambda)_\lambda \in \lim\limits_{\overleftarrow{\lambda}} E_\lambda$ and $(a_\lambda)_\lambda \in \lim\limits_{\overleftarrow{\lambda}} A_\lambda$. Since

$$\sigma_{\lambda\mu}(\xi_\lambda a_\lambda) = \sigma_{\lambda\mu}(\xi_\lambda)\pi_{\lambda\mu}(a_\lambda) = \xi_\mu a_\mu$$

and

$$\pi_{\lambda\mu}(\langle \xi_\lambda, \eta_\lambda\rangle) = \langle \sigma_{\lambda\mu}(\xi_\lambda), \sigma_{\lambda\mu}(\eta_\lambda)\rangle = \langle \xi_\mu, \eta_\mu \rangle$$

for all $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, the module action of $\lim\limits_{\overleftarrow{\lambda}} A_\lambda$ on $\lim\limits_{\overleftarrow{\lambda}} E_\lambda$ and the inner-product on $\lim\limits_{\overleftarrow{\lambda}} E_\lambda$ are well-defined. It is not hard to check that in this way $\lim\limits_{\overleftarrow{\lambda}} E_\lambda$ becomes a pre-Hilbert $\lim\limits_{\overleftarrow{\lambda}} A_\lambda$ -module.

To show that $\lim\limits_{\overleftarrow{\lambda}} E_\lambda$ is a Hilbert $\lim\limits_{\overleftarrow{\lambda}} A_\lambda$-module, let $\{(\xi_\lambda^i)_\lambda\}_{i\in I}$ be a fundamental net in $\lim\limits_{\overleftarrow{\lambda}} E_\lambda$. Then, for any $\varepsilon > 0$ and for any $\lambda \in \Lambda$, there is $i_0 \in I$ such that

$$\left\| \xi_\lambda^{i_1} - \xi_\lambda^{i_2} \right\|_\lambda = \overline{p}_\lambda((\xi_\mu^{i_1})_\mu - (\xi_\mu^{i_2})_\mu) < \varepsilon$$

for all i_1, i_2 in I with $i_1 \geq i_0$ and $i_2 \geq i_0$, where $\|\cdot\|_\lambda$ means the norm on E_λ induced by the inner-product. Therefore for any $\lambda \in \Lambda$, the net $\{\xi_\lambda^i\}_{i\in I}$ converges in E_λ to an element ξ_λ. Since

$$\sigma_{\lambda\mu}(\xi_\lambda) = \lim_i \sigma_{\lambda\mu}(\xi_\lambda^i) = \lim_i \xi_\mu^i = \xi_\mu$$

for all $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, $(\xi_\lambda)_\lambda \in \lim\limits_{\overleftarrow{\lambda}} E_\lambda$.

Let $\lambda \in \Lambda$. Then

$$\overline{p}_\lambda((\xi^i_\mu)_\mu - (\xi_\mu)_\mu) = \left\|\xi^i_\lambda - \xi_\lambda\right\|_\lambda \to 0.$$

This shows that $\lim\limits_{\overleftarrow{\lambda}} E_\lambda$ is complete and the proposition is proved. ■

References for Section 1.2: **[14]**, **[29]**, **[38]**, **[45]**.

1.3 Bounded elements

In this Section we consider the set $b(E)$ of all bounded elements in a Hilbert module E over a locally C^*-algebra A, and we show that E induces on $b(E)$ a structure of Hilbert C^*-module over $b(A)$. Also we prove that a Hilbert module E over A can be identified with $\lim\limits_{\overleftarrow{p}} E_p$ up to an isomorphism of Hilbert modules and we study the connection between $b(H_E)$ and $H_{b(E)}$.

Let A be a locally C^*-algebra and let E be a Hilbert A-module.

Definition 1.3.1 *An element ξ in E is said to be bounded if*

$$\|\xi\|_\infty = \sup\{\overline{p}(\xi)\,; p \in S(A)\} < \infty.$$

The set of all bounded elements of E is denoted by $b(E)$.

Theorem 1.3.2 *([**18**, **38**, **45**]). Let E be a Hilbert A-module. Then:*

1. *$b(E)$ is a Hilbert $b(A)$-module;*

2. *$b(E)$ is dense in E.*

Proof. 1. First we will show that the restriction of the inner product $\langle\cdot,\cdot\rangle$ on $b(E)$ is a $b(A)$-valued inner-product on $b(E)$. Let $\xi, \eta \in E$. Then, by Cauchy-Schwarz Inequality, we have

$$p(\langle\xi,\eta\rangle) \leq \sqrt{p(\langle\xi,\xi\rangle)\,p(\langle\eta,\eta\rangle)} = \overline{p}(\xi)\,\overline{p}(\eta) \leq \|\xi\|_\infty \|\eta\|_\infty$$

for all $p \in S(A)$. This means that $\langle \xi, \eta \rangle \in b(A)$, and so $b(E)$ is a pre-Hilbert $b(A)$-module. We remark that

$$\|\xi\|_\infty = \sqrt{\|\langle \xi, \xi \rangle\|_\infty}$$

for all $\xi \in b(E)$. Hence $\|\cdot\|_\infty$ is the norm on $b(E)$ induced by the inner-product.

To show the completeness of $b(E)$ with respect to the norm $\|\cdot\|_\infty$ induced by the inner-product, let $\{\xi_n\}_n$ be a fundamental sequence in $b(E)$. Since

$$\overline{p}(\xi_n - \xi_m) \leq \|\xi_n - \xi_m\|_\infty$$

for all positive integers n and m and for all $p \in S(A)$, the sequence $\{\xi_n\}_n$ is a fundamental sequence in E, and since E is complete, it converges to an element ξ in E. From

$$\|\xi_n\|_\infty - \|\xi_m\|_\infty \leq \|\xi_n - \xi_m\|_\infty$$

for all positive integers m and n, we conclude that the sequence of positive numbers $\{\|\xi_n\|_\infty\}_n$ is a fundamental sequence and so it is bounded. Let $M > 0$ such that $\|\xi_n\|_\infty \leq M$ for all positive integer n. Then

$$\overline{p}(\xi) \leq \overline{p}(\xi_n - \xi) + \overline{p}(\xi_n) \leq \overline{p}(\xi_n - \xi) + M$$

for all positive integer n and for all $p \in S(A)$. This implies that $\overline{p}(\xi) \leq M$ for all $p \in S(A)$, and so $\xi \in b(E)$.

To show that $\{\xi_n\}_n$ converges to ξ with respect to the norm $\|\cdot\|_\infty$, let $\varepsilon > 0$. Since $\{\xi_n\}_n$ is a fundamental sequence in $b(E)$, there is a positive integer n_ε such that

$$\|\xi_n - \xi_m\|_\infty \leq \varepsilon$$

for all positive integers n and m with $n \geq n_\varepsilon$ and $m \geq n_\varepsilon$. Then

$$\overline{p}(\xi_n - \xi) = \lim_m \overline{p}(\xi_n - \xi_m) \leq \lim_m \|\xi_n - \xi_m\|_\infty \leq \varepsilon$$

for all positive integer n with $n \geq n_\varepsilon$ and for all $p \in S(A)$. This shows that the sequence $\{\xi_n\}_n$ converges to ξ with respect to the norm $\|\cdot\|_\infty$, and the assertion 1. is proved.

2. According to Remark 1.2.14 we can suppose that A has a unit 1. Let $\xi \in E$. For any positive integer n, the element $1 + \frac{1}{n}\langle \xi, \xi \rangle$ is an invertible element in A, [**11**]. Consider the sequence $\{\xi_n\}_n$ of elements in E, where $\xi_n = \xi \left(1 + \frac{1}{n}\langle \xi, \xi \rangle\right)^{-1}$. By functional calculus

$$\overline{p}(\xi_n) = p\left(\langle \xi, \xi \rangle^{\frac{1}{2}} \left(1 + \frac{1}{n}\langle \xi, \xi \rangle\right)^{-1}\right) \leq \sqrt{n}$$

for all $p \in S(A)$. This shows that $\{\xi_n\}_n$ is a sequence of elements in $b(E)$, and since

$$\begin{aligned}
\overline{p}(\xi_n - \xi) &= p\left(\langle \xi, \xi \rangle^{\frac{1}{2}} - \langle \xi, \xi \rangle^{\frac{1}{2}} \left(1 + \frac{1}{n}\langle \xi, \xi \rangle\right)^{-1}\right) \\
&= \frac{1}{n} p\left(\langle \xi, \xi \rangle^{\frac{3}{2}} \left(1 + \frac{1}{n}\langle \xi, \xi \rangle\right)^{-1}\right) \\
&\leq \frac{\overline{p}(\xi)^2}{n} p\left(\langle \xi, \xi \rangle^{\frac{1}{2}} \left(1 + \frac{1}{n}\langle \xi, \xi \rangle\right)^{-1}\right) \\
&\leq \frac{\overline{p}(\xi)^2}{\sqrt{n}}
\end{aligned}$$

for all $p \in S(A)$, $\{\xi_n\}_n$ converges to ξ. Therefore $b(E)$ is dense in E. ■

Remark 1.3.3 *If A is a strongly spectral bounded locally C^*-algebra and E is a Hilbert A -module, then $b(E) = E$ as set.*

Remark 1.3.4 *Let $\{E_n\}_n$ be a countable set of Hilbert A -modules. If $\xi = (\xi_n)_n$ is an element in $\bigoplus_n b(E_n)$, then $\sum_n \langle \xi_n, \xi_n \rangle$ converges in $b(A)$, and so it converges with respect to the topology determined by the family $S(A)$ of C^* -seminorms to an element in $b(A)$. This shows that ξ is an element in $b(\bigoplus_n E_n)$. Therefore $\bigoplus_n b(E_n)$ is a subset of $b(\bigoplus_n E_n)$, and moreover, since the restriction of the $b(A)$-valued inner-product from $b(\bigoplus_n E_n)$ to $\bigoplus_n b(E_n)$ coincides with the $b(A)$-valued inner-product on $\bigoplus_n b(E_n)$, $\bigoplus_n b(E_n)$ is a closed submodule of $b(\bigoplus_n E_n)$.*

In general, $\bigoplus_n b(E_n)$ does not coincide with $b(\bigoplus_n E_n)$.

Example 1.3.5 *Let $A = C(\mathbb{Z}^+)$, the locally C^* -algebra of all $\mathbb{C}$-valued functions on $\mathbb{Z}^+$ endowed with the topology pointwise convergence. Then $H_{b(A)} \subsetneqq b(H_A)$.*

Indeed, for any positive integer n, we consider the function f_n from $\mathbb{Z}^+$ to $\mathbb{C}$ defined by

$$f_n(m) = \begin{cases} 1 & \text{if } m = n \\ \\ 0 & \text{if } m \neq n \end{cases}.$$

It is easy to check that $\sum\limits_n |f_n|^2$ converges in A to the function f from $\mathbb{Z}^+$ to $\mathbb{C}$ defined by $f(m) = 1$ for all positive integer m. Hence $(f_n)_n$ is an element in $b(H_A)$. Since

$$\sup\{|\sum_{n \geq n_0} |f_n|^2 (m)|; m \in \mathbb{Z}^+\} = 1$$

for any positive integer n_0, $\sum\limits_n |f_n|^2$ is not convergent in $b(A)$, and so $(f_n)_n \notin H_{b(A)}$.

Example 1.3.6 *Let $A = C_{cc}[0,1]$, the locally C^*-algebra of all $\mathbb{C}$-valued continuous functions on $[0,1]$ endowed with the topology of uniform convergence on the countable compact subsets of $[0,1]$, [**8**] . In this case, $H_{b(A)} = b(H_A)$.*

Indeed, if $(f_n)_n \in b(H_A)$, then $\sum\limits_n |f_n|^2$ converges in A, and by Dini's Theorem it converges in $b(A)$. Therefore $(f_n)_n \in H_{b(A)}$.

Remark 1.3.7 *If $\{E_i\}_{i=1}^n$ is a finite set of Hilbert A -modules, then $b(\bigoplus\limits_{i=1}^n E_i)$ coincides with $\bigoplus\limits_{i=1}^n b(E_i)$.*

Indeed, we seen that $\bigoplus\limits_{i=1}^n b(E_i)$ is a closed submodule of $b(\bigoplus\limits_{i=1}^n E_i)$. Let $\xi = (\xi_i)_{i=1}^n$ be an element in $b(\bigoplus\limits_{i=1}^n E_i)$. Then $\sum\limits_{i=1}^n \langle \xi_i, \xi_i \rangle$ is a positive element in $b(A)$, and since

$$0 \leq \langle \xi_k, \xi_k \rangle \leq \sum_{i=1}^n \langle \xi_i, \xi_i \rangle$$

for all $k = 1, ..., n$, ξ_k is an element in $b(E_k)$ for all $k = 1, ..., n$. Hence $\xi \in \bigoplus\limits_{i=1}^n b(E_i)$.

Lemma 1.3.8 *Let $\{E_n\}_n$ be a countable set of Hilbert A -modules. Then $\bigoplus_n b(E_n)$ is dense in $\bigoplus_n E_n$.*

Proof. Since $b(\bigoplus_n E_n)$ is dense in $\bigoplus_n E_n$, it is enough to show that $b(\bigoplus_n E_n)$ is contained in the closure of $\bigoplus_n b(E_n)$ with respect to the topology induced by the inner-product on $\bigoplus_n E_n$.

Let $\xi = (\xi_n)_n \in b(\bigoplus_n E_n)$. For each positive integer m, we denote by η_m the element in $\bigoplus_n E_n$ which has all the components zero except at the first m components which are $\xi_1, ..., \xi_m$. Since $\xi_n \in b(E_n)$ for all positive integer n, and since $\langle \eta_m, \eta_m \rangle = \sum_{n=1}^{m} \langle \xi_n, \xi_n \rangle$, η_m is an element $\bigoplus_n b(E_n)$.

Let $\varepsilon > 0$ and $p \in S(A)$. Since $\sum_n \langle \xi_n, \xi_n \rangle$ converges in A, there is a positive integer n_ε such that

$$p(\sum_{n \geq n_\varepsilon} \langle \xi_n, \xi_n \rangle) \leq \varepsilon.$$

Then

$$\overline{p}\,(\xi - \eta_m)^2 = p\,(\langle \xi - \eta_m, \xi - \eta_m \rangle) = p(\sum_{n \geq m} \langle \xi_n, \xi_n \rangle) \leq \varepsilon$$

for all positive integer m with $m \geq n_\varepsilon$. This shows that $\{\eta_m\}_m$ converges to ξ with respect to the topology induced by the inner product on $\bigoplus_n E_n$ and the lemma is proved. ■

Let E be a Hilbert A -module and let $p \in S(A)$. Using Cauchy-Schwarz Inequality, it is easy to check that $N_p = \{\xi \in E; p\,(\langle \xi, \xi \rangle) = 0\}$ is a closed A -submodule of E and $\mathcal{N}_p = \{\xi \in b(E); p\,(\langle \xi, \xi \rangle) = 0\}$ is a closed $b(A)$ -submodule of $b(E)$. The quotient vector space E/N_p is denoted by E_p.

Theorem 1.3.9 *([38, 45]). Let E be a Hilbert A -module and $p \in S(A)$. Then E_p is a Hilbert A_p-module.*

Proof. We define an action of A_p on E_p by

$$(\xi + N_p)\,\pi_p(a) = \xi a + N_p, \;\; \xi \in E, a \in A$$

and an A_p -valued inner-product on E_p by

$$\langle \xi + N_p, \eta + N_p \rangle = \pi_p \left(\langle \xi, \eta \rangle \right), \quad \xi, \eta \in E.$$

It is not difficult to check E_p with the action of A_p on E_p and the A_p -valued inner-product defined above becomes a pre-Hilbert A_p -module. It remains to show that E_p is complete with respect to the norm induced by the inner-product. The norm on E_p induced by the inner- product is denoted by $\|\cdot\|_{\overline{p}}$.

We know that the vector space $b(E)/\mathcal{N}_p$ is a Banach space with respect to the topology determined by that norm

$$\|\xi + \mathcal{N}_p\| = \inf \left\{ \|\xi + \eta\|_\infty ; \eta \in \mathcal{N}_p \right\}, \ \xi \in b(E).$$

To show that E_p is complete, it is enough to show that the map $\Phi : b(E)/\mathcal{N}_p \to E_p$ defined by

$$\Phi \left(\xi + \mathcal{N}_p \right) = \xi + N_p, \ \xi \in b(E).$$

is a linear isometry with dense range. Clearly, Φ is well-defined and it is linear.

Let $\xi \in b(E)$. Then

$$\begin{aligned} \|\Phi \left(\xi + \mathcal{N}_p \right)\|_{\overline{p}} &= \|\xi + N_p\|_{\overline{p}} = \sqrt{\|\langle \xi + N_p, \xi + N_p \rangle\|_p} \\ &= \sqrt{p \left(\langle \xi, \xi \rangle \right)} = \overline{p} \left(\xi \right). \end{aligned}$$

Thus, to show that Φ is an isometry, we must show that $\overline{p}(\xi) = \|\xi + \mathcal{N}_p\|$ for all $\xi \in b(E)$. Let $\eta \in \mathcal{N}_p$. Then

$$\overline{p}(\xi) \leq \overline{p}(\xi + \eta) + \overline{p}(\eta) = \overline{p}(\xi + \eta) \leq \|\xi + \eta\|_\infty .$$

From this relation we conclude that $\overline{p}(\xi) \leq \|\xi + \mathcal{N}_p\|$.

On the other hand, if $\{e_i\}_{i \in I}$ is an approximate unit for $\ker p \cap b(A)$, then

$$\begin{aligned} \|\xi + \mathcal{N}_p\| &\leq \lim_i \|\xi - \xi e_i\|_\infty = \lim_i \sqrt{\|\langle \xi - \xi e_i, \xi - \xi e_i \rangle\|_\infty} \\ &= \lim_i \left\| \langle \xi, \xi \rangle^{\frac{1}{2}} - \langle \xi, \xi \rangle^{\frac{1}{2}} e_i \right\|_\infty = \left\| \pi_p \left(\langle \xi, \xi \rangle^{\frac{1}{2}} \right) \right\|_p \\ &= \sqrt{p \left(\langle \xi, \xi \rangle \right)} = \overline{p} \left(\xi \right) \end{aligned}$$

(see, for example, [**37**], 1.5.4). Hence Φ is a linear isometry.

Let $\xi \in E$. Since $b(E)$ is dense in E, there is a sequence $\{\xi_n\}_n$ in $b(E)$ such that $\overline{p}\,(\xi - \xi_n) \to 0$ for all $p \in S(A)$. Then

$$\|\Phi\,(\xi_n + \mathcal{N}_p) - (\xi + N_p)\|_{\overline{p}} = \|\xi_n - \xi + N_p\|_{\overline{p}} = \overline{p}\,(\xi_n - \xi) \to 0.$$

This implies that Φ has dense range and the theorem is proved. ■

Let E be a Hilbert A -module. The canonical maps from E onto E_p, $p \in S(A)$ are denoted by σ_p^E, $p \in S(A)$ and the image of ξ under σ_p^E by ξ_p.

Let $p, q \in S(A)$ with $p \geq q$. Since $N_p \subseteq N_q$, there is a unique canonical map σ_{pq}^E from E_p onto E_q such that $\sigma_{pq}^E \circ \sigma_p^E = \sigma_q^E$. It is easy to see that $\{E_p; \sigma_{pq}^E; A_p; \pi_{pq}\}_{p,q\in S(A), p\geq q}$ is an inverse system of Hilbert C^*-modules in the sense of Definition 1.2.19.

Proposition 1.3.10 *Let E be a Hilbert A -module. Then the Hilbert A -modules E and $\lim\limits_{\leftarrow p} E_p$ are isomorphic.*

Proof. Define $\Phi : E \to \lim\limits_{\leftarrow p} E_p$ by

$$\Phi\,(\xi) = \left(\sigma_p^E\,(\xi)\right)_p, \quad \xi \in E.$$

Clearly Φ is linear. From

$$\begin{aligned} \langle \Phi\,(\xi)\,, \Phi\,(\eta)\rangle &= \left(\langle \sigma_p^E\,(\xi)\,, \sigma_p^E\,(\eta)\rangle\right)_p = \left(\pi_p\,(\langle \xi, \eta\rangle)\right)_p \\ &= \langle \xi, \eta\rangle \end{aligned}$$

for all $\xi, \eta \in E$, we conclude that $\Phi\,(E)$ is a closed A -submodule of $\lim\limits_{\leftarrow p} E_p$. By [**4**],

$$\Phi\,(E) = \overline{\Phi\,(E)} = \lim_{\leftarrow p} \overline{\widetilde{\sigma}_p\,(\Phi\,(E))} = \lim_{\leftarrow p} \overline{\sigma_p\,(E)} = \lim_{\leftarrow p} E_p$$

where $\overline{X}$ means the closure of the vector space X with respect to the topology determine by the inner- product and $\widetilde{\sigma}_p$, $p \in S(A)$ are the canonical maps from $\lim\limits_{\leftarrow p} E_p$ to E_p. Thus we showed that Φ is a surjective linear map which preserves the inner product and the proposition is proved. ■

Corollary 1.3.11 *Let E be a Hilbert A -module and let α be a real number such that $0 < \alpha < \frac{1}{2}$. Then for each $\xi \in E$ there is $\eta \in E$ such that $\xi = \eta \langle \xi, \xi \rangle^{\alpha}$.*

Proof. For each positive integer n, we consider the function $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(\lambda) = \begin{cases} n^{\alpha/2} & \text{if } \lambda \leq 1/n \\ \lambda^{-\alpha/2} & \text{if } \lambda > 1/n \end{cases}.$$

Let $p \in S(A)$. Then the sequence $\{\sigma_p(\xi) f_n(\langle \sigma_p(\xi), \sigma_p(\xi) \rangle^{1/2}\}_n$ converges in E_p to an element η_p, and moreover, $\sigma_p(\xi) = \eta_p \langle \sigma_p(\xi), \sigma_p(\xi) \rangle^{\alpha}$ (see, for example, [**29**]). By functional calculus, the sequence $\{\xi f_n(\langle \xi, \xi \rangle^{1/2}\}_n$ converges in E to an element η. Moreover, $\sigma_p(\eta) = \eta_p$ for all $p \in S(A)$ and then $\xi = \eta \langle \xi, \xi \rangle^{\alpha}$. ■

Lemma 1.3.12 *Let $\{E_n\}_n$ be a countable set of Hilbert A -modules. Then, for each $p \in S(A)$, the Hilbert A_p-modules $\left(\bigoplus\limits_n E_n \right)_p$ and $\bigoplus\limits_n (E_n)_p$ are isomorphic.*

Proof. Let $(\xi_n)_n$ be an element in $\bigoplus\limits_n E_n$ and $p \in S(A)$. Then $\sum\limits_n \langle \xi_n, \xi_n \rangle$ converges in A, and since π_p is continuous, $\sum\limits_n \pi_p(\langle \xi_n, \xi_n \rangle)$ converges in A_p. From this fact, knowing that

$$\pi_p(\langle \xi_n, \xi_n \rangle) = \left\langle \sigma_p^n(\xi_n), \sigma_p^n(\xi_n) \right\rangle$$

for all positive integer n, where σ_p^n is the canonical map from E_n onto $(E_n)_p$, we conclude that $\sum\limits_n \left\langle \sigma_p^n(\xi_n), \sigma_p^n(\xi_n) \right\rangle$ converges in A_p, and so $\left(\sigma_p^n(\xi_n)\right)_n$ is an element in $\bigoplus\limits_n (E_n)_p$. Moreover,

$$\left\langle \sigma_p^n(\xi_n)_n, \sigma_p^n(\xi_n)_n \right\rangle = \pi_p(\langle (\xi_n)_n, (\xi_n)_n \rangle).$$

Thus we can define a map U_p from $\left(\bigoplus\limits_n E_n \right)_p$ to $\bigoplus\limits_n (E_n)_p$ by

$$U_p((\xi_n)_n + N_p) = \left(\sigma_p^n(\xi_n)\right)_n.$$

It is not difficult to check that U_p is $\mathbb{C}$ - and A - linear and it preserves the inner product.

To show that the Hilbert A_p -modules $\left(\bigoplus_n E_n\right)_p$ and $\bigoplus_n (E_n)_p$ are isomorphic it is enough to show that U_p has dense range. Let $\xi = (\xi_n^p)_n$ be an element in $\bigoplus_n (E_n)_p$. For each positive integer m, we consider the element $\widetilde{\xi}_m$ in $\bigoplus_n (E_n)_p$ which has all the components zero at except the first m components which are $\xi_1^p, ..., \xi_m^p$. A simple calculus shows that the sequence $\{\widetilde{\xi}_m\}_m$ converges to ξ. Since the maps σ_p^n, $n = 1, 2, ...$, are all surjective, for each positive integer m there is an element $\widetilde{\eta}_m$ in $\bigoplus_n E_n$ such that $U_p(\widetilde{\eta}_m + N_p) = \widetilde{\xi}_m$. Hence U_p has dense range and the lemma is proved. ■

Remark 1.3.13 *If E is a Hilbert A -module, then the Hilbert A_p -modules $(H_E)_p$ and H_{E_p} are isomorphic.*

References for Section 1.3: **[14], [18], [23], [36], [38], [45].**

Chapter 2

Operators on Hilbert modules

2.1 Bounded operators

In this Section we introduce the notion of bounded module morphism between two Hilbert modules, and we show that the set $B_A(E)$ of all bounded module morphisms on E is a locally m -convex algebra which can be identified with $\lim\limits_{\substack{\leftarrow \\ p}} B_{A_p}(E_p)$, where $B_{A_p}(E_p)$ is the Banach space of all bounded module morphisms on E_p for each $p \in S(A)$, up to a topological isomorphism.

Let A be a locally C^*-algebra and let E, F and G be Hilbert A -modules.

Definition 2.1.1 *A $\mathbb{C}$- and A- linear map T from E to F is said to be a bounded operator from E to F if for each $p \in S(A)$ there is $M_p > 0$ such that*

$$\overline{p}_F\,(T\xi) \leq M_p \overline{p}_E\,(\xi)$$

for all $\xi \in E$.

The set of all bounded operators from E to F is denoted by $B_A(E,F)$ and we write $B_A(E)$ for $B_A(E,E)$.

Remark 2.1.2 *Let $T, S \in B_A(E,F)$, $R \in B_A(F,G)$, $\lambda \in \mathbb{C}$, $p \in S(A)$ and $\xi \in E$. Then:*

1. $\overline{p}_F((T+S)(\xi)) \leq \overline{p}_F(T\xi) + \overline{p}_F(S\xi) \leq M_{p,T}\overline{p}_E(\xi) + M_{p,S}\overline{p}_E(\xi) = (M_{p,T} + M_{p,S})\overline{p}_E(\xi)$;

2. $\overline{p}_F((\lambda T)(\xi)) = |\lambda|\overline{p}_F(T\xi) \leq |\lambda| M_{p,T}\overline{p}_E(\xi)$;

3. $\overline{p}_G(RT\xi) \leq M_{p,R}\overline{p}_F(T\xi) \leq M_{p,R}M_{p,T}\overline{p}_E(\xi)$.

From these facts we conclude that $B_A(E,F)$ is a vector space and $B_A(E)$ is an algebra.

Lemma 2.1.3 *For each $p \in S(A)$, the map $\widetilde{p} : B_A(E,F) \to [0,\infty)$ defined by*

$$\widetilde{p}\,(T) = \sup\left\{\overline{p}_F\,(T\xi)\,;\overline{p}_E\,(\xi) \leq 1\right\}$$

is a seminorm on $B_A(E,F)$. Moreover, if $F = E$, then $\widetilde{p}$ is a submultiplicative seminorm on $B_A(E)$.

Proof. It is straightforward. ■

Theorem 2.1.4 *The set $B_A(E,F)$ is a Hausdorff complete locally convex space with respect to the topology determined by the family of seminorms $\{\widetilde{p}\}_{p\in S(A)}$.*

Proof. Let $T \in B_A(E,F)$ such that $\widetilde{p}(T) = 0$ for all $p \in S(A)$. Then $\overline{p}_F(T\xi) = 0$ for all $p \in S(A)$ and for all $\xi \in E$, and since F is separable, $T\xi = 0$ for all $\xi \in E$. Therefore $T = 0$. Thus we showed that $B_A(E,F)$ is a Hausdorff locally convex space.

To show the completeness of $B_A(E,F)$, let $\{T_i\}_{i\in I}$ be a fundamental net in $B_A(E,F)$. First, we show that the net $\{T_i\xi\}_{i\in I}$ converges in F for each $\xi \in E$. For this, let $p \in S(A)$, $\xi \in E$ and $\varepsilon > 0$.

If $\overline{p}_E(\xi) \neq 0$, then, since $\{T_i\}_{i\in I}$ is a fundamental net in $B_A(E,F)$, there is $i_0 \in I$ such that

$$\widetilde{p}(T_{i_1} - T_{i_2}) \leq \varepsilon/\overline{p}_E(\xi)$$

for all $i_1, i_2 \in I$ with $i_1 \geq i_0$ and $i_2 \geq i_0$, and so

$$\overline{p}_F(T_{i_1}\xi - T_{i_2}\xi) \leq \varepsilon$$

for all $i_1, i_2 \in I$ with $i_1 \geq i_0$ and $i_2 \geq i_0$.

If $\overline{p}_E(\xi) = 0$, then

$$\overline{p}_F(T_{i_1}\xi - T_{i_2}\xi) \leq \widetilde{p}(T_{i_1} - T_{i_2})\overline{p}_E(\xi) = 0$$

for all $i_1, i_2 \in I$. Hence $\{T_i\xi\}_{i\in I}$ is a fundamental net in F and so it is convergent. Define a map $T : E \to F$ by

$$T\xi = \lim_i T_i\xi.$$

It is not hard to check that T is $\mathbb{C}$ - and A - linear. Since $\{T_i\}_{i\in I}$ is a fundamental net in $B_A(E, F)$, for each $p \in S(A)$, $\{\widetilde{p}(T_i)\}_{i\in I}$ is a fundamental net of positive numbers and so it is bounded. Let $M_p > 0$ such that $\widetilde{p}(T_i) \leq M_p$ for all $i \in I$. Then we have

$$\overline{p}_F(T\xi) = \lim_i \overline{p}_F(T_i\xi) \leq \lim_i \widetilde{p}(T_i)\overline{p}_E(\xi) \leq M_p\overline{p}_E(\xi)$$

for all $\xi \in E$ and for all $p \in S(A)$. This means that $T \in B_A(E, F)$.

To show that $\{T_i\}_{i\in I}$ converges to T, let $\varepsilon > 0$, $p \in S(A)$ and $\xi \in E$ such that $\overline{p}_E(\xi) \leq 1$. Since $\{T_i\}_{i\in I}$ is a fundamental net in $B_A(E, F)$, there is $i_0 \in I$ such that

$$\widetilde{p}(T_{i_1} - T_{i_2}) \leq \varepsilon$$

for all $i_1, i_2 \in I$ with $i_1 \geq i_0$ and $i_2 \geq i_0$. Then

$$\begin{aligned}\overline{p}_F(T\xi - T_{i_1}\xi) &= \lim_i \overline{p}_F(T_i\xi - T_{i_1}\xi) \leq \lim_i \widetilde{p}(T_i - T_{i_1})\overline{p}_E(\xi)\\ &\leq \lim_i \widetilde{p}(T_i - T_{i_1}) \leq \varepsilon\end{aligned}$$

for all $i_1 \in I$ with $i_1 \geq i_0$. Therefore

$$\widetilde{p}(T - T_{i_1}) = \sup\{\overline{p}_F(T\eta - T_{i_1}\eta); \overline{p}_E(\eta) \leq 1\} \leq \varepsilon$$

for all $i_1 \in I$ with $i_1 \geq i_0$. This means that the net $\{T_i\}_{i\in I}$ converges to T and the theorem is proved. ■

Corollary 2.1.5 *The algebra $B_A(E)$ is a Hausdorff complete locally convex algebra with respect to the topology determined by the family of seminorms $\{\widetilde{p}\}_{p\in S(A)}$.*

Let $\{E_\lambda;\sigma_{\lambda\mu};A_\lambda;\pi_{\lambda\mu}\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ and $\{F_\lambda;\chi_{\lambda\mu};A_\lambda;\pi_{\lambda\mu}\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ be two inverse systems of Hilbert C^*-modules such that the canonical maps σ_λ from $\lim\limits_{\overleftarrow{\lambda}} E_\lambda$ to E_λ, χ_λ from $\lim\limits_{\overleftarrow{\lambda}} F_\lambda$ to F_λ and π_λ from $\lim\limits_{\overleftarrow{\lambda}} A_\lambda$ to A_λ, $\lambda\in\Lambda$ are all surjective. Then the connecting maps $\sigma_{\lambda\mu}$ from E_λ to E_μ, $\chi_{\lambda\mu}$ from F_λ to F_μ and $\pi_{\lambda\mu}$ from A_λ to A_μ, $\lambda\geq\mu$, $\lambda,\mu\in\Lambda$ are all surjective.

Let $\lambda,\mu\in\Lambda$ with $\lambda\geq\mu$, $T\in B_{A_\lambda}(E_\lambda,F_\lambda)$, $\zeta\in E_\mu$ and $\xi\in E_\lambda$ such that $\sigma_{\lambda\mu}(\xi)=\zeta$. Then

$$\begin{aligned}\left\|\chi_{\lambda\mu}(T\xi)\right\|_{F_\mu}^2 &= \left\|\left\langle\chi_{\lambda\mu}(T\xi),\chi_{\lambda\mu}(T\xi)\right\rangle\right\|_{A_\mu}=\left\|\pi_{\lambda\mu}(\langle T\xi,T\xi\rangle)\right\|_{A_\mu}\\ &\quad \text{(cf. [}\mathbf{36}\text{], 2.8)}\\ &\leq \|T\|_\lambda^2\left\|\pi_{\lambda\mu}(\langle\xi,\xi\rangle)\right\|_{A_\mu}=\|T\|_\lambda^2\left\|\langle\sigma_{\lambda\mu}(\xi),\sigma_{\lambda\mu}(\xi)\rangle\right\|_{A_\mu}\\ &= \|T\|_\lambda^2\,\|\zeta\|_{E_\mu}^2\end{aligned}$$

where $\|\cdot\|_\lambda$ is the norm on $B_{A_\lambda}(E_\lambda,F_\lambda)$. Therefore we can define a map $(\pi_{\lambda\mu})_*(T)$ from E_μ to F_μ by

$$(\pi_{\lambda\mu})_*(T)(\zeta)=\chi_{\lambda\mu}(T\xi)\ \text{ if } \sigma_{\lambda\mu}(\xi)=\zeta.$$

Moreover,

$$\left\|(\pi_{\lambda\mu})_*(T)(\zeta)\right\|_{F_\mu}\leq\|T\|_\lambda\,\|\zeta\|_{E_\mu}.$$

Let $\zeta_1,\zeta_2\in E_\mu$, $\xi_1,\xi_2\in E_\lambda$ such that $\sigma_{\lambda\mu}(\xi_1)=\zeta_1$ and $\sigma_{\lambda\mu}(\xi_2)=\zeta_2$, $a\in A_\mu$, $b\in A_\lambda$ such that $\pi_{\lambda\mu}(b)=a$ and the complex numbers α_1 and α_2. Then, since

$$\sigma_{\lambda\mu}(\alpha_1\xi_1+\alpha_2\xi_2)=\alpha_1\zeta_1+\alpha_2\zeta_2 \text{ and } \sigma_{\lambda\mu}(\xi_1 b)=\sigma_{\lambda\mu}(\xi_1)\pi_{\lambda\mu}(b)=\zeta_1 a,$$

we have:

(**a**) $(\pi_{\lambda\mu})_*(T)(\alpha_1\zeta_1+\alpha_2\zeta_2)=\chi_{\lambda\mu}(T(\alpha_1\xi_1+\alpha_2\xi_2))$
$=\alpha_1\chi_{\lambda\mu}(T\xi_1)+\alpha_2\chi_{\lambda\mu}(T\xi_2)=\alpha_1(\pi_{\lambda\mu})_*(T)(\zeta_1)+\alpha_2(\pi_{\lambda\mu})_*(T)(\zeta_2)$
and

(**b**) $(\pi_{\lambda\mu})_*(T)(\zeta_1 a)=\chi_{\lambda\mu}(T(\xi_1 b))=\chi_{\lambda\mu}(T(\xi_1)b)=\chi_{\lambda\mu}(T\xi_1)\pi_{\lambda\mu}(b)$
$=(\pi_{\lambda\mu})_*(T)(\zeta_1)a.$

Therefore $(\pi_{\lambda\mu})_*(T)$ is an element in $B_{A_\mu}(E_\mu, F_\mu)$. Thus we have obtained a map $(\pi_{\lambda\mu})_*$ from $B_{A_\lambda}(E_\lambda, F_\lambda)$ to $B_{A_\mu}(E_\mu, F_\mu)$. From

$$\left\|(\pi_{\lambda\mu})_*(T)\right\|_\mu = \sup\{\left\|(\pi_{\lambda\mu})_*(T)(\zeta)\right\|_{F_\mu}; \|\zeta\|_{E_\mu} \leq 1\} \leq \|T\|_\lambda$$

for all $T \in B_{A_\lambda}(E_\lambda, F_\lambda)$, we conclude that $(\pi_{\lambda\mu})_*$ is continuous. It is not hard to check that $(\pi_{\lambda\mu})_*$ is linear and $\{B_{A_\lambda}(E_\lambda, F_\lambda); (\pi_{\lambda\mu})_*\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ is an inverse system of Banach spaces.

Let $A = \lim\limits_{\overleftarrow{\lambda}} A_\lambda$, $E = \lim\limits_{\overleftarrow{\lambda}} E_\lambda$ and $F = \lim\limits_{\overleftarrow{\lambda}} F_\lambda$.

Theorem 2.1.6 *Let A, E and F be as above. Then the Hausdorff complete locally convex spaces $B_A(E, F)$ and $\lim\limits_{\overleftarrow{\lambda}} B_{A_\lambda}(E_\lambda, F_\lambda)$ are isomorphic.*

Proof. Let $\lambda \in \Lambda$, $T \in B_A(E, F)$, $\xi_\lambda \in E_\lambda$ and $\xi \in E$ such that $\sigma_\lambda(\xi) = \xi_\lambda$. Then

$$\begin{aligned}
\left\|\chi_\lambda(T\xi)\right\|_{F_\lambda}^2 &= \left\|\langle\chi_\lambda(T\xi), \chi_\lambda(T\xi)\rangle\right\|_{A_\lambda} \\
&= \|\pi_\lambda(\langle T\xi, T\xi\rangle)\|_{A_\lambda} \\
&= p_\lambda(\langle T\xi, T\xi\rangle) = \overline{p}_{\lambda,F}(T\xi)^2 \\
&\leq \widetilde{p}_\lambda(T)^2 \overline{p}_{\lambda,E}(\xi)^2 \\
&= \widetilde{p}_\lambda(T)^2 p_\lambda(\langle\xi, \xi\rangle) \\
&= \widetilde{p}_\lambda(T)^2 \|\pi_\lambda(\langle\xi, \xi\rangle)\|_{A_\lambda} \\
&= \widetilde{p}_\lambda(T)^2 \left\|\xi_\lambda\right\|_{E_\lambda}^2.
\end{aligned}$$

This implies that the map $(\pi_\lambda)_*(T)$ from E_λ to F_λ defined by

$$(\pi_\lambda)_*(T)(\xi_\lambda) = \chi_\lambda(T\xi) \text{ if } \sigma_\lambda(\xi) = \xi_\lambda$$

is well-defined, and moreover,

$$\left\|(\pi_\lambda)_*(T)(\xi_\lambda)\right\|_{F_\lambda} \leq \widetilde{p}_\lambda(T) \left\|\xi_\lambda\right\|_{E_\lambda}.$$

It is not difficult to check that $(\pi_\lambda)_*(T)$ is an element in $B_{A_\lambda}(E_\lambda, F_\lambda)$. In this way we have defined a map $(\pi_\lambda)_*$ from $B_A(E,F)$ to $B_{A_\lambda}(E_\lambda, F_\lambda)$. Also it is not difficult to check that $(\pi_\lambda)_*$ is linear. Moreover, since

$$\|(\pi_\lambda)_*(T)\|_\lambda = \sup\{\left\|(\pi_\lambda)_*(T)(\xi_\lambda)\right\|_{F_\lambda}; \left\|\xi_\lambda\right\|_{E_\lambda} \leq 1\} = \widetilde{p}_\lambda(T)$$

for all $T \in B_A(E,F)$, $(\pi_\lambda)_*$ is continuous.

Define Ψ from $B_A(E,F)$ to $\lim\limits_{\substack{\leftarrow \\ \lambda}} B_{A_\lambda}(E_\lambda, F_\lambda)$ by

$$\Psi(T) = \left((\pi_\lambda)_*(T)\right)_\lambda.$$

Let $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, $T \in B_A(E,F)$, $\zeta \in E_\mu$ and $\xi \in E_\lambda$ such that $\sigma_{\lambda\mu}(\xi) = \zeta$ and $\eta \in E$ such that $\sigma_\lambda(\eta) = \xi$. Then, since $\sigma_\mu(\eta) = \sigma_{\lambda\mu}(\sigma_\lambda(\eta)) = \sigma_{\lambda\mu}(\xi) = \zeta$, we have

$$\begin{aligned}\left((\pi_{\lambda\mu})_*\left((\pi_\lambda)_*(T)\right)\right)(\zeta) &= \chi_{\lambda\mu}\left((\pi_\lambda)_*(T)(\xi)\right) = \chi_{\lambda\mu}(\chi_\lambda(T\eta)) \\ &= \chi_\mu(T\eta) = \left(\pi_\mu\right)_*(T)(\zeta).\end{aligned}$$

Therefore Ψ is well-defined. It is not hard to check that Ψ is linear.

To show that Ψ is surjective, let $(T_\lambda)_\lambda \in \lim\limits_{\substack{\leftarrow \\ \lambda}} B_{A_\lambda}(E_\lambda, F_\lambda)$ and $\xi = \left(\xi_\lambda\right)_\lambda \in \lim\limits_{\substack{\leftarrow \\ \lambda}} E_\lambda$. Define T from E to F by $T\xi = \left(T_\lambda \xi_\lambda\right)_\lambda$. Let $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$. Then

$$\chi_{\lambda\mu}(T_\lambda \xi_\lambda) = (\pi_{\lambda\mu})_*(T_\lambda)(\sigma_{\lambda\mu}(\xi_\lambda)) = T_\mu \xi_\mu.$$

This shows that T is well-defined. It is easy to check that T is $\mathbb{C}$ -and A -linear, and since

$$\overline{p}_{\lambda,F}(T\xi) = \left\|T_\lambda \xi_\lambda\right\|_{F_\lambda} \leq \|T_\lambda\|_\lambda \left\|\xi_\lambda\right\|_{E_\lambda} = \|T_\lambda\|_\lambda \overline{p}_{\lambda,E}(\xi)$$

for all $\xi \in E$ and for all $\lambda \in \Lambda$, T is a bounded operator from E to F. Moreover, since

$$(\pi_\lambda)_*(T)(\sigma_\lambda(\xi)) = \chi_\lambda(T\xi) = T_\lambda(\sigma_\lambda(\xi))$$

for all $\xi \in E$ and for all $\lambda \in \Lambda$, $\Psi(T) = (T_\lambda)_\lambda$. Therefore Ψ is surjective.

Let $\{q_\lambda\}_{\lambda\in\Lambda}$ be the family of seminorms which define the topology on $\lim\limits_{\overleftarrow{\lambda}} B_{A_\lambda}$ (E_λ, F_λ) (that is, $q_\lambda((T_\mu)_\mu) = \|T_\lambda\|_\lambda$ From

$$q_\lambda(\Psi(T)) = \|(\pi_\lambda)_*(T)\|_\lambda = \widetilde{p}_\lambda(T)$$

for all $\lambda \in \Lambda$ and for all $T \in B_A(E, F)$, we conclude that Ψ is injective, and moreover, Ψ and its inverse are continuous. ■

Corollary 2.1.7 *Let A be a locally C^* -algebra and let E and F be Hilbert A -modules. Then $\{B_{A_p}(E_p, F_p); (\pi_{pq})_*\}_{p\geq q,p,q\in S(A)}$ is an inverse system of Banach spaces and the Hausdorff complete locally convex spaces $B_A(E, F)$ and $\lim\limits_{\overleftarrow{p}} B_{A_p}(E_p, F_p)$ are isomorphic.*

Let $\{E_\lambda; \sigma_{\lambda\mu}; A_\lambda; \pi_{\lambda\mu}\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ be an inverse system of Hilbert C^*-modules such that the canonical maps σ_λ from $\lim\limits_{\overleftarrow{\lambda}} E_\lambda$ to E_λ and π_λ from $\lim\limits_{\overleftarrow{\lambda}} A_\lambda$ to A_λ, $\lambda \in \Lambda$ are all surjective.

Let $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$. We seen that the map $(\pi_{\lambda\mu})_*$ from $B_{A_\lambda}(E_\lambda)$ to $B_{A_\mu}(E_\mu)$ defined by

$$(\pi_{\lambda\mu})_*(T)(\zeta) = \sigma_{\lambda\mu}(T\xi) \text{ if } \sigma_{\lambda\mu}(\xi) = \zeta$$

is a continuous morphism of Banach spaces. To show that $(\pi_{\lambda\mu})_*$ is a morphism of algebras, let $T, S \in B_{A_\lambda}(E_\lambda)$, $\zeta \in E_\mu$ and $\xi \in E_\lambda$ such that $\sigma_{\lambda\mu}(\xi) = \zeta$. Then

$$\begin{aligned}(\pi_{\lambda\mu})_*(TS)(\zeta) &= \sigma_{\lambda\mu}(TS\xi) = (\pi_{\lambda\mu})_*(T)(\sigma_{\lambda\mu}(S\xi)) \\ &= (\pi_{\lambda\mu})_*(T)\,(\pi_{\lambda\mu})_*(S)(\zeta).\end{aligned}$$

Therefore $(\pi_{\lambda\mu})_*$ is a continuous morphism of Banach algebras and $\{ B_{A_\lambda}(E_\lambda);$ $(\pi_{\lambda\mu})_*\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ is an inverse system of Banach algebras.

Let $A = \lim\limits_{\overleftarrow{\lambda}} A_\lambda$ and $E = \lim\limits_{\overleftarrow{\lambda}} E_\lambda$.

Proposition 2.1.8 *Let A and E be as above. Then the Hausdorff complete locally m -convex algebras $B_A(E)$ and $\lim\limits_{\overleftarrow{\lambda}} B_{A_\lambda}(E_\lambda)$ are isomorphic.*

Proof. By Theorem 2.1.6, to prove the proposition it remains to show that the map Ψ from $B_A(E)$ onto $\lim\limits_{\leftarrow \lambda} B_{A_\lambda}(E_\lambda)$ defined by $\Psi(T) = ((\pi_\lambda)_*(T))_\lambda$ is a morphism of algebras. Let $\lambda \in \Lambda$, $T, S \in B_A(E)$, $\xi \in E$ and $\zeta \in E_\lambda$ such that $\sigma_\lambda(\xi) = \zeta$. Then

$$\begin{aligned} (\pi_\lambda)_* (TS)(\zeta) &= \sigma_\lambda (TS\xi) = (\pi_\lambda)_* (T)(\sigma_\lambda (S\xi)) \\ &= (\pi_\lambda)_* (T) (\pi_\lambda)_* (S)(\zeta). \end{aligned}$$

Therefore $(\pi_\lambda)_*$ is a morphism of algebras for all $\lambda \in \Lambda$. This implies that Ψ is a morphism of algebras and the proposition is proved. ■

Corollary 2.1.9 *Let A be a locally C^* -algebra and let E be a Hilbert A -module. Then $\{B_{A_p}(E_p); (\pi_{pq})_*\}_{p \geq q, p, q \in S(A)}$ is an inverse system of Banach algebras and the Hausdorff complete locally m-convex algebras $B_A(E)$ and $\lim\limits_{\leftarrow p} B_{A_p}(E_p)$ are isomorphic.*

References for Section 2.1: **[12]**, **[14]**, **[24]**, **[29]**, **[36]**, **[38]**, **[45]**.

2.2 Operators admitting an adjoint

In this Section we consider the notion of adjointable module morphism between two Hilbert A-modules E and F, and we prove that the set $L_A(E)$ of all adjointable module morphisms on E is a locally C^*-algebra which can be identified with $\lim\limits_{\leftarrow p} L_{A_p}(E_p)$, where $L_{A_p}(E_p)$ is the C^*-algebra of all adjointable module morphisms on E_p for each $p \in S(A)$, up to an isomorphism of locally C^*-algebras.

Let A be a locally C^* -algebra and let E, F and G be Hilbert modules.

Definition 2.2.1 *A map T from E to F is said to be adjointable if there is a map T^* from F to E such that*

$$\langle \eta, T\xi \rangle = \langle T^*\eta, \xi \rangle$$

for all $\xi \in E$ and for all $\eta \in F$.

The set of all adjointable operators from E to F is denoted by $L_A(E,F)$. When $F = E$, for simplifying, we write $L_A(E)$.

Remark 2.2.2 *Let $T, S \in L_A(E,F)$, $R \in L_A(F,G)$, $\xi \in E$, $\eta \in F$, $\zeta \in G$ and the complex numbers α, β. Then:*

1. $\langle \eta, (\alpha T + \beta S)\xi \rangle = \alpha \langle \eta, T\xi \rangle + \beta \langle \eta, S\xi \rangle = \alpha \langle T^*\eta, \xi \rangle + \beta \langle S^*\eta, \xi \rangle$
$= \left\langle (\overline{\alpha} T^* + \overline{\beta} S^*)\eta, \xi \right\rangle;$

2. $\langle \zeta, RT\xi \rangle = \langle R^*\zeta, T\xi \rangle = \langle T^*R^*\zeta, \xi \rangle$ *and*

3. $\langle \xi, T^*\eta \rangle = (\langle T^*\eta, \xi \rangle)^* = (\langle \eta, T\xi \rangle)^* = \langle T\xi, \eta \rangle$.

Therefore $L_A(E,F)$ is a vector space, $L_A(E)$ is an algebra and moreover,

*(**a**)* $(\alpha T + \beta S)^* = \overline{\alpha} T^* + \overline{\beta} S^*$;

*(**b**)* $(RT)^* = T^*R^*$;

*(**c**)* $(T^*)^* = T$.

Lemma 2.2.3 *If T is an adjointable operator from E to F, then T and T^* are bounded operators, and moreover, $\widetilde{p}(T) = \widetilde{p}(T^*)$ for all $p \in S(A)$.*

Proof. Let $\xi, \zeta \in E$, $\eta \in F$, $a \in A$ and $\lambda \in \mathbb{C}$. Then:

1. $\langle \eta, T(\xi + \zeta) \rangle = \langle T^*\eta, \xi + \zeta \rangle = \langle T^*\eta, \xi \rangle + \langle T^*\eta, \zeta \rangle$
$= \langle \eta, T\xi \rangle + \langle \eta, T\zeta \rangle = \langle \eta, T\xi + T\zeta \rangle;$

2. $\langle \eta, T(\lambda\xi) \rangle = \langle T^*\eta, \lambda\xi \rangle = \lambda \langle T^*\eta, \xi \rangle = \lambda \langle \eta, T\zeta \rangle = \langle \eta, \lambda T\zeta \rangle$; and

3. $\langle \eta, T(\xi a) \rangle = \langle T^*\eta, \xi a \rangle = \langle T^*\eta, \xi \rangle a = \langle \eta, T\zeta \rangle a = \langle \eta, (T\zeta)a \rangle$.

From these relations we conclude that T is $\mathbb{C}$ - and A -linear. In the same way we show that T^* is a $\mathbb{C}$ - and A -linear map from F to E.

To show that T is a bounded operator from F to E, first we show that $T(N_p^E) \subseteq N_p^F$ for all $p \in S(A)$. Let $p \in S(A)$ and $\xi \in N_p^E$. Then

$$\begin{aligned}\overline{p}_F(T\xi)^2 &= p(\langle T\xi, T\xi\rangle) = p(\langle T^*T\xi, \xi\rangle) \\ &\quad \text{(cf. Cauchy-Schwarz Inequqlity)} \\ &\leq \overline{p}_F(T^*T\xi)\,\overline{p}_E(\xi) = 0.\end{aligned}$$

Hence $T(N_p^E) \subseteq N_p^F$ for all $p \in S(A)$. In the same way we show that $T^*(N_p^F) \subseteq N_p^E$ for all $p \in S(A)$. Then, for each $p \in S(A)$, we can define the maps T_p from E_p to F_p by

$$T_p\left(\sigma_p^E(\xi)\right) = \sigma_p^F(T\xi)$$

and T_p^* from F_p to E_p by

$$T_p^*\left(\sigma_p^F(\eta)\right) = \sigma_p^E(T^*\eta).$$

Moreover, we have

$$\begin{aligned}\left\langle \sigma_p^F(\eta), T_p\left(\sigma_p^E(\xi)\right)\right\rangle &= \left\langle \sigma_p^F(\eta), \sigma_p^F(T\xi)\right\rangle \\ &= \pi_p(\langle \eta, T\xi\rangle) = \pi_p(\langle T^*\eta, \xi\rangle) \\ &= \left\langle \sigma_p^E(T^*\eta), \sigma_p^E(\xi)\right\rangle = \left\langle T_p^*\left(\sigma_p^F(\eta)\right), \sigma_p^E(\xi)\right\rangle\end{aligned}$$

for all $\xi \in E$ and for all $\eta \in F$. This means that T_p is an adjointable operator from E_p to F_p and so T_p and T_p^* are bounded operators between Hilbert C^*-modules (see, for example, [**36**]).

Then

$$\begin{aligned}\overline{p}_F(T\xi)^2 &= p(\langle T\xi, T\xi\rangle) = \|\pi_p(\langle T\xi, T\xi\rangle)\|_p \\ &= \left\|\left\langle \sigma_p^F(T\xi), \sigma_p^F(T\xi)\right\rangle\right\|_p \\ &= \left\|\left\langle T_p\left(\sigma_p^E(\xi)\right), T_p\left(\sigma_p^E(\xi)\right)\right\rangle\right\|_p \\ &\leq \|T_p\|^2 \left\|\left\langle \sigma_p^E(\xi), \sigma_p^E(\xi)\right\rangle\right\|_p \\ &\quad \text{(cf. [\textbf{36}], 2.8)} \\ &= \|T_p\|^2 \|\pi_p(\langle \xi, \xi\rangle)\|_p \\ &= \|T_p\|^2\, \overline{p}_E(\xi)^2,\end{aligned}$$

for $p \in S(A)$ and for all $\xi \in E$. Hence T is bounded. Moreover,

$$\begin{aligned}\widetilde{p}(T) &= \sup\{\overline{p}_F\left(T\xi\right); \xi \in E,\ p_E(\xi) \le 1\} \\ &= \sup\{\left\|T_p\left(\sigma_p^E\left(\xi\right)\right)\right\|_{\overline{p}_F}; \xi \in E,\ \left\|\sigma_p^E\left(\xi\right)\right\|_{\overline{p}_E} \le 1\} \\ &= \left\|T_p\right\|.\end{aligned}$$

In the same way we show that T^* is a bounded operator from F to E and $\widetilde{p}(T^*) = \left\|T_p^*\right\|$. Moreover, since $\left\|T_p^*\right\| = \left\|T_p\right\|$ for all $p \in S(A)$, $\widetilde{p}(T) = \widetilde{p}(T^*)$ for all $p \in S(A)$. ■

Remark 2.2.4 *The map $T \to T^*$ defines an involution on $L_A(E)$.*

Proposition 2.2.5 *The set $L_A(E,F)$ of all adjointable operators from E to F is a closed subspace of $B_A(E,F)$.*

Proof. By Remark 2.2.2 and Lemma 2.2.3, $L_A(E,F)$ is a vector subspace of $B_A(E,F)$.

To show that $L_A(E,F)$ is closed, let $\{T_i\}_{i\in I}$ be a net in $L_A(E,F)$ which converges to an element T in $B_A(E,F)$. According to Lemma 2.2.3, $\{T_i^*\}_{i\in I}$ is a fundamental net in $B_A(E,F)$, and so it converges to an element S in $B_A(E,F)$. Then

$$\langle \eta, T\xi\rangle = \lim_i \langle \eta, T_i\xi\rangle = \lim_i \langle T_i^*\eta, \xi\rangle = \langle S\eta, \xi\rangle$$

for all $\xi \in E$ and for all $\eta \in F$. This means that T is adjointable, and the proposition is proved. ■

Let $p \in S(A)$ and $T \in L_A(E)$. We seen that $\widetilde{p}(T) = \widetilde{p}(T^*)$. Moreover,

$$\widetilde{p}(T^*T) \le \widetilde{p}(T^*)\widetilde{p}(T) = \widetilde{p}(T)^2.$$

On the other hand,

$$\begin{aligned}\widetilde{p}(T^*T) &= \sup\{\overline{p}_E\left(T^*T\xi\right); p_E(\xi) \le 1\} = \\ &= \sup\{\sup\{p(\langle T^*T\xi, \eta\rangle); \eta \in E, p_E(\eta) \le 1\}; \xi \in E, p_E(\xi) \le 1\} \\ &\ge \sup\{p(\langle T^*T\xi, \xi\rangle); \xi \in E, p_E(\xi) \le 1\} = \widetilde{p}(T)^2.\end{aligned}$$

Therefore $\widetilde{p}$ is a C^*-seminorm on $L_A(E)$.

Theorem 2.2.6 *The set $L_A(E)$ of all adjointable operators on E is a locally C^*-algebra with respect to the topology determined by the family of C^*-seminorms $\{\widetilde{p}\}_{p\in S(A)}$ and the involution defined by $T \to T^*$.*

Proof. By Proposition 2.2.5, $L_A(E)$ equipped with the topology determined by the family of seminorms $\{\widetilde{p}\}_{p\in S(A)}$ is a Hausdorff complete locally convex space, and since $L_A(E)$ is an algebra with involution and $\{\widetilde{p}\}_{p\in S(A)}$ is a family of C^*-seminorms, $L_A(E)$ is a locally C^*-algebra. ■

Let $\{E_\lambda; \sigma_{\lambda\mu}; A_\lambda; \pi_{\lambda\mu}\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ and $\{F_\lambda; \chi_{\lambda\mu}; A_\lambda; \pi_{\lambda\mu}\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ be two inverse systems of Hilbert C^* -modules such that the canonical maps σ_λ from $\lim\limits_{\leftarrow \lambda} E_\lambda$ to E_λ, χ_λ from $\lim\limits_{\leftarrow \lambda} F_\lambda$ to F_λ and π_λ from $\lim\limits_{\leftarrow \lambda} A_\lambda$ to A_λ, $\lambda \in \Lambda$ are all surjective. We seen (pp. 36-37) that $\{B_{A_\lambda}(E_\lambda, F_\lambda); (\pi_{\lambda\mu})_*\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ is an inverse system of Banach spaces.

Let $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$, $T \in L_{A_\lambda}(E_\lambda, F_\lambda)$, $\xi_\mu \in E_\mu$ and $\xi_\lambda \in E_\lambda$ such that $\sigma_{\lambda\mu}(\xi_\lambda) = \xi_\mu$, $\eta_\mu \in F_\mu$ and $\eta_\lambda \in F_\lambda$ such that $\chi_{\lambda\mu}(\eta_\lambda) = \eta_\mu$. Then

$$\begin{aligned} \left\langle \eta_\mu, (\pi_{\lambda\mu})_* (T)(\xi_\mu) \right\rangle &= \left\langle \chi_{\lambda\mu}(\eta_\lambda), \chi_{\lambda\mu}(T\xi_\lambda) \right\rangle = \pi_{\lambda\mu}(\langle \eta_\lambda, T\xi_\lambda \rangle) \\ &= \pi_{\lambda\mu}(\langle T^*\eta_\lambda, \xi_\lambda \rangle) = \left\langle (\widetilde{\pi_{\lambda\mu}})_* (T^*)(\eta_\mu), \xi_\mu \right\rangle, \end{aligned}$$

where $(\widetilde{\pi_{\lambda\mu}})_*$ is the canonical map from $B_{A_\lambda}(F_\lambda, E_\lambda)$ to $B_{A_\mu}(F_\mu, E_\mu)$. This shows that $(\pi_{\lambda\mu})_* (T) \in L_{A_\mu}(E_\mu, F_\mu)$ and moreover, $(\pi_{\lambda\mu})_* (T)^* = (\widetilde{\pi_{\lambda\mu}})_* (T^*)$. The restriction of $(\pi_{\lambda\mu})_*$ on $L_{A_\lambda}(E_\lambda, F_\lambda)$ is also denoted by $(\pi_{\lambda\mu})_*$. Therefore $\{L_{A_\lambda}(E_\lambda, F_\lambda);\ (\pi_{\lambda\mu})_*\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ is an inverse system of Banach spaces and $\{L_{A_\lambda}(E_\lambda); (\pi_{\lambda\mu})_*\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ is an inverse system of C^*-algebras.

Let $A = \lim\limits_{\leftarrow \lambda} A_\lambda$, $E = \lim\limits_{\leftarrow \lambda} E_\lambda$and $F = \lim\limits_{\leftarrow \lambda} F_\lambda$.

Proposition 2.2.7 *Let A, E and F be as above. Then:*

1. *The Hausdorff complete locally convex spaces $L_A(E, F)$ and $\lim\limits_{\leftarrow \lambda} L_{A_\lambda}(E_\lambda, F_\lambda)$ are isomorphic.*

2. *The locally C^*-algebras $L_A(E)$ and $\lim\limits_{\overleftarrow{\lambda}} L_{A_\lambda}(E_\lambda)$ are isomorphic.*

Proof. 1. Let $\lambda \in \Lambda$, $T \in L_A(E,F)$, $\xi_\lambda \in E_\lambda$ and $\xi \in E$ such that $\sigma_\lambda(\xi) = \xi_\lambda, \eta_\lambda \in F_\lambda$ and $\eta \in F$ such that $\chi_\lambda(\eta) = \eta_\lambda$. Then

$$\begin{aligned}\langle \eta_\lambda, (\pi_\lambda)_*(T)(\xi_\lambda)\rangle &= \langle \chi_\lambda(\eta), \chi_\lambda(T\xi)\rangle = \pi_\lambda(\langle \eta, T\xi\rangle) \\ &= \pi_\lambda(\langle T^*\eta, \xi\rangle) = \langle (\widetilde{\pi_\lambda})_*(T^*)(\eta_\lambda), \xi_\lambda\rangle,\end{aligned}$$

where $(\widetilde{\pi_\lambda})_*$ is the canonical map from $B_A(E,F)$ to $B_{A_\lambda}(E_\lambda, F_\lambda)$. This shows that $(\pi_\lambda)_*(T) \in L_{A_\lambda}(E_\lambda, F_\lambda)$ and $(\pi_\lambda)_*(T)^* = (\widetilde{\pi_\lambda})_*(T^*)$.The restriction of $(\pi_\lambda)_*$ on $L_A(E,F)$ is also denoted by $(\pi_\lambda)_*$. Thus, if Ψ is the isomorphism from $B_A(E,F)$ onto $\lim\limits_{\overleftarrow{\lambda}} B_{A_\lambda}(E_\lambda, F_\lambda)$ defined by $\Psi(T) = ((\pi_\lambda)_*(T))_\lambda$ (Theorem 2.1.6), then $\Psi(\ L_A(E,F)) \subseteq \lim\limits_{\overleftarrow{\lambda}} L_{A_\lambda}(E_\lambda, F_\lambda)$. To show that $L_A(E,F)$ is isomorphic with $\lim\limits_{\overleftarrow{\lambda}} L_{A_\lambda}\ (E_\lambda, F_\lambda)$ it remains to show that the restriction of Ψ on $L_A(E,F)$ is a surjective map from $L_A(E,F)$ to $\lim\limits_{\overleftarrow{\lambda}} L_{A_\lambda}(E_\lambda, F_\lambda)$.

Let $(T_\lambda)_\lambda \in \lim\limits_{\overleftarrow{\lambda}} L_{A_\lambda}(E_\lambda, F_\lambda)$. Then there is $T \in B_A(E,F)$ such that $\Psi(T) = (T_\lambda)_\lambda$. Moreover, $T\xi = (T_\lambda(\sigma_\lambda(\xi)))_\lambda$ for all $\xi \in E$. Let $S \in B_A(F,E)$ defined by $S\eta = (T_\lambda^*(\sigma_\lambda(\eta)))_\lambda$, $\xi = (\xi_\lambda)_\lambda \in E$ and $\eta = (\eta_\lambda)_\lambda \in F$. Then

$$\begin{aligned}\langle \eta, T\xi\rangle &= \langle (\eta_\lambda)_\lambda, (T_\lambda(\xi_\lambda))_\lambda\rangle = (\langle \eta_\lambda, T_\lambda(\xi_\lambda)\rangle)_\lambda \\ &= (\langle T_\lambda^*(\eta_\lambda), (\xi_\lambda)\rangle)_\lambda = \langle (T_\lambda^*(\eta_\lambda))_\lambda, (\xi_\lambda)_\lambda\rangle = \langle S\eta, \xi\rangle.\end{aligned}$$

Therefore, $T \in L_A(E,F)$ and the assertion 1. is proved.

2. By the first part of the proposition, the map Ψ from $L_A(E)$ to $\lim\limits_{\overleftarrow{\lambda}} L_{A_\lambda}(E_\lambda)$ defined by $\Psi(T) = ((\pi_\lambda)_*(T))_\lambda$ is an isomorphism of locally convex spaces. Moreover, from the proof of Proposition 2.1.8, Ψ is also morphism of algebras.

Let $T \in L_A(E)$, $\xi = (\xi_\lambda)_\lambda, \eta = (\eta_\lambda)_\lambda \in E$. Then

$$\begin{aligned}\langle \eta, \Psi(T)\xi\rangle &= \langle (\eta_\lambda)_\lambda, ((\pi_\lambda)_*(T)\xi_\lambda)_\lambda\rangle = (\langle \eta_\lambda, (\pi_\lambda)_*(T)\xi_\lambda\rangle)_\lambda \\ &= (\langle (\pi_\lambda)_*(T)^*\eta_\lambda, \xi_\lambda\rangle)_\lambda = (\langle (\pi_\lambda)_*(T^*)\eta_\lambda, \xi_\lambda\rangle)_\lambda \\ &= \langle ((\pi_\lambda)_*(T^*)\eta_\lambda)_\lambda, (\xi_\lambda)_\lambda\rangle = \langle \Psi(T^*)\eta, \xi\rangle.\end{aligned}$$

This shows that $\Psi(T)^* = \Psi(T^*)$ and so Ψ is an isomorphism of locally C^* -algebras. ■

Corollary 2.2.8 *Let A be a locally C^*-algebra and let E and F be two Hilbert A-modules. Then :*

1. *The Hausdorff complete locally convex spaces $L_A(E,F)$ and $\lim\limits_{\underset{p}{\leftarrow}} L_{A_p}(E_p,F_p)$ are isomorphic.*
2. *The locally C^*-algebras $L_A(E)$ and $\lim\limits_{\underset{p}{\leftarrow}} L_{A_p}(E_p)$ are isomorphic.*

Corollary 2.2.9 *Let A be a locally C^*-algebra, let E be Hilbert C^*-module and let n be a positive integer. Then the locally C^*-algebras $L_A(\bigoplus\limits_{i=1}^{n} E)$ and $M_n(L_A(E))$ are isomorphic.*

Proof. By Lemma 1.3.12 and Corollary 2.2.8, the locally C^*-algebras $L_A(\bigoplus\limits_{i=1}^{n} E)$ and $\lim\limits_{\underset{p}{\leftarrow}} L_{A_p}(\bigoplus\limits_{i=1}^{n} E_p)$ are isomorphic as well as $M_n(L_A(E))$ and $\lim\limits_{\underset{p}{\leftarrow}} M_n(L_{A_p}(E_p))$. From this fact, taking into account that the C^*-algebras $L_{A_p}(\bigoplus\limits_{i=1}^{n} E_p)$ and $M_n(L_{A_p}$ $(E_p))$ are isomorphic, we conclude that the locally C^*-algebras $L_A(\bigoplus\limits_{i=1}^{n} E)$ and $M_n(L_A(E))$ are isomorphic ■

References for Section 2.2: **[12], [14], 24], [29], [36], [39], [40], [45].**

2.3 Compact operators

In this Section we introduce the notion of "compact operators" on Hilbert modules on locally C^*-algebras, and we prove that the set $K_A(E)$ of all compact operators on is a closed two -sided $*$ -ideal of $L_A(E)$ can be identified with $\lim\limits_{\underset{p}{\leftarrow}} K_{A_p}(E_p)$, where $K_{A_p}(E_p)$ is the set of all compact operators on E_p for each $p \in S(A)$, up to an isomorphism of locally C^*-algebras. It is well -known that the left multiplier algebra of the C^*-algebra of compact operators on a Hilbert

C^*-module E is isomorphic with the Banach algebra of all bounded operators on E and the multiplier algebra of the C^*-algebra of compact operators on E is isomorphic with the C^*-algebra of all adjointable operators on E. We show that these properties of the compact operators on a Hilbert C^*-modules are still hold in the context of Hilbert modules over locally C^*-algebras.

Let A be a locally C^*-algebra, let E, F and G be Hilbert A -modules. For $\xi \in E$ and $\eta \in F$ define $\theta_{\eta,\xi}$ from E to F by

$$\theta_{\eta,\xi}\left(\zeta\right) = \eta \left\langle \xi, \zeta \right\rangle .$$

We have

$$\begin{aligned} \left\langle \eta_1, \theta_{\eta,\xi}\left(\zeta\right), \right\rangle &= \left\langle \eta_1, \eta \left\langle \xi, \zeta \right\rangle \right\rangle = \left\langle \eta_1, \eta \right\rangle \left\langle \xi, \zeta \right\rangle \\ &= \left\langle \xi \left\langle \eta, \eta_1 \right\rangle, \zeta \right\rangle = \left\langle \theta_{\xi,\eta}\left(\eta_1\right), \zeta \right\rangle \end{aligned}$$

for all $\zeta \in E$ and for all $\eta_1 \in F$. Therefore $\theta_{\eta,\xi} \in L_A(E,F)$ and moreover, $\theta^*_{\eta,\xi} = \theta_{\xi,\eta}$. We say that $\theta_{\eta,\xi}$ is an "one-rank" module homomorphism from E to F. The vector subspace of $L_A(E,F)$ generated by $\{\theta_{\eta,\xi}; \xi \in E, \eta \in F\}$ is denoted by $\Theta_A\left(E,F\right)$. When $F = E$, for simplifying we write $\Theta_A\left(E\right)$. We say that an element in $\Theta_A\left(E,F\right)$ is a finite-rank operator from E to F.

Remark 2.3.1 *It is easy to check that:*

1. $\theta_{\zeta,\eta_2}\theta_{\eta_1,\xi} = \theta_{\zeta\langle\eta_2,\eta_1\rangle,\xi} = \theta_{\zeta,\xi\langle\eta_1,\eta_2\rangle}$ *for all* $\xi \in E$, $\eta_1, \eta_2 \in F$ *and* $\zeta \in G$;

2. $T\ \theta_{\eta,\xi} = \theta_{T\eta,\xi}$ *for all* $\xi \in E$, $\eta \in F$ *and* $T \in B_A(F,G)$;

3. $\theta_{\eta,\xi}S = \theta_{\eta,S^*\xi}$ *for all* $\xi \in E$, $\eta \in F$ *and* $S \in L_A(G,E)$.

Remark 2.3.2 **1**. $L_A(F,G)\Theta_A\left(E,F\right) \subseteq \Theta_A\left(E,G\right)$;

2. $\Theta_A\left(E,F\right) L_A(G,E) \subseteq \Theta_A\left(G,E\right)$;

3. $\Theta_A\left(E\right)$ *is a two-sided * -ideal of* $L_A(E)$.

The closed subspace of $L_A(E,F)$ generated by $\Theta_A(E,F)$ is denoted by $K_A(E,F)$. We write $K_A(E)$ for $K_A(E,E)$. An element in $K_A(E,F)$ is said "-compact operator" from E to F.

Remark 2.3.3 **1**. $L_A(F,G)K_A(E,F) \subseteq K_A(E,G)$;

2. $K_A(E,F)\, L_A(G,E) \subseteq K_A(G,E)$;

3. $K_A(E)$ *is a two-sided * -ideal of* $L_A(E)$

Let $\{E_\lambda;\sigma_{\lambda\mu};A_\lambda;\pi_{\lambda\mu}\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ and $\{F_\lambda;\chi_{\lambda\mu};A_\lambda;\pi_{\lambda\mu}\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ be two inverse systems of Hilbert C^*-modules such that the canonical maps σ_λ from $\lim\limits_{\leftarrow\lambda} E_\lambda$ to E_λ, χ_λ from $\lim\limits_{\leftarrow\lambda} F_\lambda$ to F_λ and π_λ from $\lim\limits_{\leftarrow\lambda} A_\lambda$ to A_λ, $\lambda\in\Lambda$ are all surjective. We seen (pp. 44) that $\{L_{A_\lambda}(E_\lambda,F_\lambda);(\pi_{\lambda\mu})_*\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ is an inverse system of Banach spaces.

Let $\lambda,\mu\in\Lambda$ with $\lambda\geq\mu$, $\xi\in E_\lambda$, $\eta\in F_\lambda$, $\zeta\in E_\mu$ and $\zeta_1\in E_\lambda$ such that $\sigma_{\lambda\mu}(\zeta_1)=\zeta$. Then

$$\begin{aligned}(\pi_{\lambda\mu})_*(\theta_{\eta,\xi})(\zeta) &= \chi_{\lambda\mu}(\theta_{\eta,\xi}(\zeta_1)) = \chi_{\lambda\mu}(\eta\langle\xi,\zeta_1\rangle)\\ &= \chi_{\lambda\mu}(\eta)\langle\sigma_{\lambda\mu}(\xi),\sigma_{\lambda\mu}(\zeta_1)\rangle = \theta_{\chi_{\lambda\mu}(\eta),\sigma_{\lambda\mu}(\xi)}(\zeta).\end{aligned}$$

From this relation we conclude that $(\pi_{\lambda\mu})_*\left(K_{A_\lambda}(E_\lambda,F_\lambda)\right)\subseteq K_{A_\mu}(E_\mu,F_\mu)$. Moreover, since $\chi_{\lambda\mu}$ and $\sigma_{\lambda\mu}$ are surjective, the closure of $(\pi_{\lambda\mu})_*\left(K_{A_\lambda}(E_\lambda,F_\lambda)\right)$ in $L_{A_\lambda}(E_\lambda,F_\lambda)$ coincides with $K_{A_\mu}(E_\mu,F_\mu)$. The restriction of $(\pi_{\lambda\mu})_*$ on $K_{A_\lambda}(E_\lambda,F_\lambda)$ is also denoted by $(\pi_{\lambda\mu})_*$. Therefore $\{K_{A_\lambda}(E_\lambda,F_\lambda);(\pi_{\lambda\mu})_*\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ is an inverse system of Banach spaces and $\{K_{A_\lambda}(E_\lambda);(\pi_{\lambda\mu})_*\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ is an inverse system of C^*-algebras with the connecting maps $(\pi_{\lambda\mu})_*$, $\lambda\geq\mu$, $\lambda,\mu\in\Lambda$ all surjective.

Let $A=\lim\limits_{\leftarrow\lambda} A_\lambda$, $E=\lim\limits_{\leftarrow\lambda} E_\lambda$ and $F=\lim\limits_{\leftarrow\lambda} F_\lambda$.

Proposition 2.3.4 *Let* A, E *and* F *be as above. Then:*

1. *The Hausdorff complete locally convex spaces* $K_A(E,F)$ *and* $\lim\limits_{\leftarrow\lambda} K_{A_\lambda}(E_\lambda,F_\lambda)$ *are isomorphic.*

2. *The locally C^*-algebras $K_A(E)$ and $\lim\limits_{\stackrel{\leftarrow}{\lambda}} K_{A_\lambda}(E_\lambda)$ are isomorphic.*

Proof. 1. Let $\lambda \in \Lambda$, $\xi \in E$, $\eta \in F$, $\zeta_\lambda \in E_\lambda$ and $\zeta \in E$ such that $\sigma_\lambda(\zeta) = \zeta_\lambda$. Then

$$\begin{aligned} (\pi_\lambda)_* (\theta_{\eta,\xi})(\zeta_\lambda) &= \chi_\lambda((\theta_{\eta,\xi})(\zeta)) = \chi_\lambda(\eta \langle \xi, \zeta\rangle) = \chi_\lambda(\eta)\pi_\lambda(\langle \xi, \zeta\rangle) \\ &= \chi_\lambda(\eta) \langle \sigma_\lambda(\xi), \sigma_\lambda(\zeta)\rangle = \theta_{\chi_\lambda(\eta),\sigma_\lambda(\xi)}(\zeta_\lambda). \end{aligned}$$

Therefore $(\pi_\lambda)_* (K_A(E,F)) \subseteq K_{A_\lambda}(E_\lambda, F_\lambda)$. Moreover, since σ_λ and χ_λ are surjective, the closure of $(\pi_\lambda)_* (K_A(E,F))$ in $L_{A_\lambda}(E_\lambda, F_\lambda)$ coincides with $K_{A_\lambda}(E_\lambda, F_\lambda)$. The restriction of $(\pi_\lambda)_*$ on $K_A(E,F)$ is also denoted by $(\pi_\lambda)_*$. Thus, if Ψ is the isomorphism from $L_A(E,F)$ onto $\lim\limits_{\stackrel{\leftarrow}{\lambda}} L_{A_\lambda}(E_\lambda, F_\lambda)$ defined by $\Psi(T) = ((\pi_\lambda)_* (T))_\lambda$ (Proposition 2.2.7 (1)), then $\Psi(K_A(E,F))$ is a closed subspace of $\lim\limits_{\stackrel{\leftarrow}{\lambda}} K_{A_\lambda}(E_\lambda, F_\lambda)$. Thus to show that $K_A(E,F)$ is isomorphic with $\lim\limits_{\stackrel{\leftarrow}{\lambda}} K_{A_\lambda}(E_\lambda, F_\lambda)$ it remains to show that the restriction of Ψ on $K_A(E,F)$ is a surjective map from $K_A(E,F)$ to $\lim\limits_{\stackrel{\leftarrow}{\lambda}} K_{A_\lambda}(E_\lambda, F_\lambda)$.

By Lemma III 3.2, [**33**],

$$\begin{aligned} \Psi(K_A(E,F)) &= \overline{\Psi(K_A(E,F))} = \lim_{\stackrel{\leftarrow}{\lambda}} \overline{\widetilde{\pi_\lambda}(\Psi(K_A(E,F)))} \\ &= \lim_{\stackrel{\leftarrow}{\lambda}} \overline{(\pi_\lambda)_* (K_A(E,F))} = \lim_{\stackrel{\leftarrow}{\lambda}} K_{A_\lambda}(E_\lambda, F_\lambda), \end{aligned}$$

where $\widetilde{\pi_\lambda}$, $\lambda \in \Lambda$ are the canonical maps from $\lim\limits_{\stackrel{\leftarrow}{\lambda}} K_{A_\lambda}(E_\lambda, F_\lambda)$ to $K_{A_\lambda}(E_\lambda, F_\lambda)$.

2. By the first part of the proposition, the restriction of the isomorphism Ψ from $L_A(E)$ to $\lim\limits_{\stackrel{\leftarrow}{\lambda}} L_{A_\lambda}(E_\lambda)$ on $K_A(E)$ is an isomorphism of locally convex spaces from $K_A(E)$ to $\lim\limits_{\stackrel{\leftarrow}{\lambda}} K_{A_\lambda}(E_\lambda)$, and since Ψ is also morphism of $*$ -algebras, the assertion is proved. ■

Corollary 2.3.5 *Let A be a locally C^*-algebra and let E and F be two Hilbert A-modules. Then :*

1. *The Hausdorff complete locally convex spaces $K_A(E,F)$ and $\lim\limits_{\leftarrow p} K_{A_p}(E_p,F_p)$ are isomorphic.*

2. *The locally C^*-algebras $K_A(E)$ and $\lim\limits_{\leftarrow p} K_{A_p}(E_p)$ are isomorphic.*

Corollary 2.3.6 *Let A be a locally C^*-algebra. Then $K_A(A)$ is isomorphic with A.*

Proof. Let $p \in S(A)$. We know that the map φ_p from $K_{A_p}(A_p)$ to A_p defined by $\varphi_p(\theta_{a,b}) = ab^*$ is an isomorphism of C^*-algebras (see, for example, [**26**, **34**]).

Let $p, q \in S(A)$ with $p \geq q$. Since

$$\left(\pi_{pq} \circ \varphi_p\right)(\theta_{a,b}) = \pi_{pq}(ab^*)$$

and

$$\left(\varphi_q \circ (\pi_{pq})_*\right)(\theta_{a,b}) = \varphi_q\left(\theta_{\pi_{pq}(a),\pi_{pq}(b)}\right) = \pi_{pq}(a)\pi_{pq}(b)^* = \pi_{pq}(ab^*)$$

for all $a, b \in A_p$, $\pi_{pq} \circ \varphi_p = \varphi_q \circ (\pi_{pq})_*$. Therefore $(\varphi_p)_p$ is an inverse system of C^*-isomorphisms. Let $\varphi = \lim\limits_{\leftarrow p} \varphi_p$. Then φ is an isomorphism of locally C^*-algebras from $\lim\limits_{\leftarrow p} K_{A_p}(E_p)$ to $\lim\limits_{\leftarrow p} A_p$. Therefore $K_A(A)$ is isomorphic with A. ■

Let A be a locally C^* -algebra and let E be a Hilbert A -module. For $\xi \in E$, consider the map T_ξ from E to A defined by $T_\xi(\eta) = \langle \xi, \eta \rangle$. It is easy to check that T_ξ is a module morphism from E to A and $\widetilde{p}(T_\xi) = \overline{p}_E(\xi)$ for all $p \in S(A)$. Exactly as in the case of Hilbert C^*-modules, we show that any element in $K_A(E, A)$ is of the form T_ξ, $\xi \in E$. This is a version of the Riesz-Fréchet theorem for Hilbert modules over locally C^*-algebras.

Corollary 2.3.7 *Let A be a locally C^* -algebra and let E be a Hilbert A -module. Then any element in $K_A(E, A)$ is of the form T_ξ, $\xi \in E$.*

Proof. For $p \in S(A)$, the map ψ_p from E_p to $K_{A_p}(E_p, A_p)$ defined by $\psi_p(\xi) = T_\xi$ is an isometric isomorphism of Banach spaces.

Let $p, q \in S(A)$ with $p \geq q$. Since

$$\left(\psi_q \circ \sigma_{pq}\right)(\zeta) = T_{\sigma_{pq}(\zeta)}$$

and

$$\left((\pi_{pq})_* \circ \psi_p\right)(\zeta) = (\pi_{pq})_* (T_\zeta) = T_{\sigma_{pq}(\zeta)}$$

for all $\xi \in A_p$, $\psi_q \circ \sigma_{pq} = (\pi_{pq})_* \circ \psi_p$. Therefore $(\psi_p)_p$ is an inverse system of isometric isomorphisms of Banach spaces. Let $\psi = \lim\limits_{\overleftarrow{p}} \psi_p$. Then ψ is an isomorphism of locally convex spaces from $\lim\limits_{\overleftarrow{p}} E_p$ to $\lim\limits_{\overleftarrow{p}} K_{A_p}(E_p, A_p)$. Therefore E is isomorphic with $K_A(E, A)$. Moreover, $\psi(\xi) = \left(\psi_p\left(\sigma_p\left(\xi\right)\right)\right)_p = \left(T_{\sigma_p(\xi)}\right)_p = T_\xi$ for all $\xi \in E$. ■

Remark 2.3.8 *Let A be a unital locally C^* -algebra and let E be a Hilbert A -module. If $T \in L_A(E, A)$, then $T = T_{T^*(1)}$, where 1 is the unity of A and so $T \in K_A(E, A)$. Therefore $K_A(E, A) = L_A(E, A)$.*

Proposition 2.3.9 *Let A be a locally C^* -algebra and let E be a Hilbert A -module. Then:*

1. *$LM(K_A(E))$ is isomorphic to $B_A(E)$ as locally convex algebras.*

2. *$M(K_A(E))$ is isomorphic to $L_A(E)$ as locally C^* -algebras.*

Proof. Since $\{K_{A_p}(E_p); (\pi_{pq})_*\}_{p \geq q, p, q \in S(A)}$ is an inverse system of C^* -algebras and the canonical maps $(\pi_p)_*, p \in S(A)$ are all dense range, by Corollary 1.1.20, $LM(K_A(E))$ is isomorphic with $\lim\limits_{\overleftarrow{p}} LM(K_{A_p}(E_p))$ and $M(K_A(E))$ is isomorphic with $\lim\limits_{\overleftarrow{p}} M(K_{A_p}(E_p))$.

On the other hand, by Corollary 2.1.9, $B_A(E)$ is isomorphic with $\lim\limits_{\overleftarrow{p}} B_{A_p}(E_p)$, and by Corollary 2.2.8 (2), $L_A(E)$ is isomorphic with $\lim\limits_{\overleftarrow{p}} L_{A_p}(E_p)$. Thus to prove the proposition it is sufficient to prove that the locally convex algebras

$\lim\limits_{\leftarrow p} B_{A_p}(E_p)$ and $\lim\limits_{\leftarrow p} LM(K_{A_p}(E_p))$ are isomorphic likewise the locally C^* -algebras $M(K_A(E))$ and $\lim\limits_{\leftarrow p} M(K_{A_p}(E_p))$.

1. For each $p \in S(A)$, the map φ_p from $B_{A_p}(E_p)$ to $LM(K_{A_p}(E_p))$ defined by

$$\varphi_p(T)(S) = TS$$

is an isometric isomorphism of Banach algebras (see, for example, [**31**], Theorem 1.5).

Let $p, q \in S(A)$ with $p \geq q$, $T \in B_{A_p}(E_p)$, $S \in K_{A_q}(E_q)$ and $\widetilde{S} \in K_{A_p}(E_p)$ such that $(\widetilde{\pi_{pq}})_* (\widetilde{S}) = S$. Then

$$\begin{aligned}\left(\left((\widetilde{\pi_{pq}})_* \circ \varphi_p\right)(T)\right)(S) &= \left((\widetilde{\pi_{pq}})_* \left(\varphi_p(T)\right)\right)(\widetilde{\pi_{pq}})_* (\widetilde{S}) \\ &= (\widetilde{\pi_{pq}})_* (T\widetilde{S})\end{aligned}$$

and

$$\left(\left(\varphi_q \circ (\widetilde{\pi_{pq}})_*\right)(T)\right)(S) = (\widetilde{\pi_{pq}})_* (T)\,(\widetilde{\pi_{pq}})_* (\widetilde{S}) = (\widetilde{\pi_{pq}})_* (T\widetilde{S}).$$

Therefore $\left(\varphi_p\right)_p$ is an inverse system of isometric isomorphisms of Banach algebras. Let $\varphi = \lim\limits_{\leftarrow p} \varphi_p$. Then φ is an isomorphism of locally m -convex algebras from $\lim\limits_{\leftarrow p} B_{A_p}(E_p)$ onto $\lim\limits_{\leftarrow p} LM(K_{A_p}(E_p))$.

2. For each $p \in S(A)$, the map ψ_p from $L_{A_p}(E_p)$ to $M(K_{A_p}(E_p))$ defined by

$$\varphi_p(T)(S) = (TS, ST)$$

is isomorphism of C^* -algebras (see, for example, [**26**], Theorem 1).

It is not hard to check that $\left(\psi_p\right)_p$ is an inverse system of isomorphisms of C^* -algebras. Let $\psi = \lim\limits_{\leftarrow p} \psi_p$. Then ψ is an isomorphism of locally C^* -algebras from $\lim\limits_{\leftarrow p} B_{A_p}(E_p)$ onto $\lim\limits_{\leftarrow p} LM(K_{A_p}(E_p))$. ■

Corollary 2.3.10 *If A is a locally C^*-algebra, then $M(A)$ is isomorphic to $L_A(A)$.*

References for Section 2.3: **[12], [24], [26], [31], [38].**

2.4 Strongly bounded operators

In this Section we introduce the notion of strongly bounded module morphism between Hilbert modules and we study the connection between the set $b(B_A(E,F))$ of all strongly bounded module morphisms from E to F (respectively the set $b(L_A(E,F))$ of all strongly adjointable module morphisms from E to F, respectively the set $b(K_A(E,F))$ of all strongly compact module morphisms from E to F) and the Banach space $B_{b(A)}(b(E),b(F))$ of all bounded module morphisms from $b(E)$ to $b(F)$ (respectively, the Banach space $L_{b(A)}(b(E),b(F))$ of all adjointable module morphisms from $b(E)$ to $b(F)$, respectively the Banach space $K_{b(A)}(b(E),b(F))$ of all compact module morphisms from $b(E)$ to $b(F)$).

Let A be a locally C^*-algebra and let E and F be Hilbert A -modules.

Definition 2.4.1 *A bounded operator T from E to F is strongly bounded if*

$$\sup\{\widetilde{p}(T); p \in S(A)\} < \infty.$$

The set of all strongly bounded operators from E to F is denoted by $b(B_A(E,F))$ and we write $b(B_A(E))$ for $b(B_A(E,E))$.

It is not difficult to check that $b(B_A(E,F))$ is a vector subspace of $B_A(E,F)$ and the map $T \to \|T\|_\infty$, where

$$\|T\|_\infty = \sup\{\widetilde{p}(T); p \in S(A)\}$$

defines a norm on $b(B_A(E,F))$. Also it is not difficult to check that $b(B_A(E))$ is a subalgebra of $B_A(E)$ and $\|\cdot\|_\infty$ is a submultiplicative norm on $b(B_A(E))$.

The connection between $b(B_A(E,F))$ and $B_{b(A)}(b(E),b(F))$, the set of all bounded operators from $b(E)$ to $b(F)$, is given by the following theorem.

Theorem 2.4.2 *Let A be a locally C^*-algebra and let E and F be Hilbert A -modules. Then:*

1. *The vector space $b(B_A(E,F))$ equipped with the norm $\|\cdot\|_\infty$ is a Banach space which is isometrically isomorphic to $B_{b(A)}(b(E),b(F))$.*

2. *The set $b(L_A(E,F))$ of all strongly bounded adjointable operators from E to F is a closed subspace of $b(B_A(E,F))$. Moreover, $b(L_A(E,F))$ is isometrically isomorphic with $L_{b(A)}(b(E),b(F))$.*

3. *The set $b(K_A(E,F))$ of all strongly bounded compact operators from E to F is a closed subspace of $b(B_A(E,F))$.*

Proof. Let $\{T_n\}_n$ be a fundamental sequence in $b(B_A(E,F))$. Since

$$\widetilde{p}(T_n - T_m) \leq \sup\{\widetilde{q}(T_n - T_m); q \in S(A)\} = \|T_n - T_m\|_\infty$$

for all positive integers m and n and for all $p \in S(A)$, $\{T_n\}_n$ is a fundamental sequence in $B_A(E,F)$ and so it converges to an element $T \in B_A(E,F)$. From

$$|\|T_n\|_\infty - \|T_m\|_\infty| \leq \|T_n - T_m\|_\infty$$

for all positive integers n and m we conclude that $\{\|T_n\|_\infty\}_n$ is a fundamental sequence of positive numbers and so it is bounded. Let $M > 0$ such that $\|T_n\|_\infty \leq M$ for all positive integer n. Then

$$\widetilde{p}(T) = \lim_n \widetilde{p}(T_n) \leq \lim_n \|T_n\|_\infty \leq M$$

for all $p \in S(A)$. This shows that $T \in b(B_A(E,F))$.

To show that $\{T_n\}_n$ converges to T with respect to the norm $\|\cdot\|_\infty$, let $\varepsilon > 0$. Since $\{T_n\}_n$ is a fundamental sequence in $b(B_A(E,F))$, there is a positive integer n_0 such that

$$\|T_n - T_m\|_\infty \leq \varepsilon$$

for all positive integers n and m with $n \geq n_0$ and $m \geq n_0$. Then

$$\begin{aligned} \widetilde{p}(T - T_n) &= \lim_m \widetilde{p}(T_m - T_n) \\ &\leq \lim_m \|T_n - T_m\|_\infty \leq \varepsilon \end{aligned}$$

for all positive integer n with $n \geq n_0$. This means that $\{T_n\}_n$ converges to T with respect to the norm $\|\cdot\|_\infty$. Hence $b(B_A(E,F))$ is a Banach space with respect to the norm $\|\cdot\|_\infty$.

Let $T \in b(B_A(E,F))$. From

$$\overline{p}_F(T\xi) \leq \widetilde{p}(T)\overline{p}_E(\xi) \leq \|T\|_\infty \|\xi\|_\infty$$

for all $p \in S(A)$ and for all $\xi \in b(E)$, we conclude that the restriction $T|_{b(E)}$ of T on $b(E)$ is an element in $B_{b(A)}(b(E), b(F))$ and moreover, $\left\|T|_{b(E)}\right\| \leq \|T\|_\infty$, where $\|\cdot\|$ is the norm on $B_{b(A)}(b(E), b(F))$.

On the other hand, using the fact that T is continuous and $b(E)$ is dense in E, from

$$\begin{aligned} \overline{p}_F(T\xi)^2 &= p(\langle T\xi, T\xi\rangle) = p(\langle T|_{b(E)}\xi, T|_{b(E)}\xi\rangle) \\ &\quad \text{(cf. [\textbf{36}], 2.8)} \\ &\leq p(\left\|T|_{b(E)}\right\|^2 \langle \xi, \xi\rangle) = \left\|T|_{b(E)}\right\|^2 \overline{p}_E(\xi)^2 \end{aligned}$$

for all $p \in S(A)$ and for all $\xi \in b(E)$ we deduce that $\widetilde{p}(T) \leq \left\|T|_{b(E)}\right\|$ for all $p \in S(A)$ and so $\|T\|_\infty \leq \left\|T|_{b(E)}\right\|$. Therefore $\|T\|_\infty = \left\|T|_{b(E)}\right\|$.

Consider the map Ψ from $b(B_A(E,F))$ to $B_{b(A)}(b(E), b(F))$ defined by $\Psi(T) = T|_{b(E)}$. Clearly Ψ is well-defined and moreover, it is a linear isometry from $b(B_A(E,F))$ to $B_{b(A)}(b(E), b(F))$. To prove that $b(B_A(E,F))$ is isometrically isomorphic with $B_{b(A)}(b(E), b(F))$, it remains to prove that Ψ is surjective. Let $S \in B_{b(A)}(b(E), b(F))$. Since

$$\overline{p}_F(S\xi)^2 = p\left(\langle S\xi, S\xi\rangle\right) \leq \|S\|^2 \overline{p}_E(\xi)^2$$

for all $\xi \in b(E)$ and for all $p \in S(A)$, and since $b(E)$ is dense in E, S extends by continuity to a linear map $\widetilde{S}$ from E to F. From Lemma 2.1.3 (1), taking into account that $b(A)$ is dense in A, we conclude that $\widetilde{S}$ is A -linear. Moreover, $\widetilde{p}(S) \leq \|S\|$ for all $p \in S(A)$. Therefore $\widetilde{S} \in b(B_A(E,F))$ and $\Psi(\widetilde{S}) = S$.

2. Let $\{T_n\}_n$ be a sequence in $b(L_A(E,F))$ which converges with respect to the topology induced by the norm $\|\cdot\|_\infty$ to an element T in $b(B_A(E,F))$. Then it is convergent with respect to the topology induced by the family of seminorms $\{\widetilde{p}\}_{p \in S(A)}$. From these facts and taking into account that $L_A(E,F)$ is a closed subspace of $B_A(E,F)$, we conclude that $T \in b(L_A(E,F))$. Therefore $b(L_A(E,F))$ is a closed subspace of $b(B_A(E,F))$.

To show that $b(L_A(E,F))$ is isometrically isomorphic with $L_{b(A)}(b(E),b(F))$, it is enough to show that $\Psi\left(b(L_A(E,F))\right) = L_{b(A)}(b(E),b(F))$, where Ψ is the isomorphism from $b(B_A(E,F))$ onto $B_{b(A)}(b(E),b(F))$ constructed above.

Let $T \in b(L_A(E,F))$. Then $T^* \in b(L_A(F,E))$ and $\left(T|_{b(E)}\right)^* = T^*|_{b(E)}$. Consequently, $\Psi\left(b(L_A(E,F))\right) \subseteq L_{b(A)}(b(E),b(F))$.

Let $S \in L_{b(A)}(b(E),b(F))$. Then there is $\widetilde{S}$ in $b(B_A(E,F))$ and S_0 in $b(B_A(F,E))$ such that $\widetilde{S}|_{b(E)} = S$ and $S_0|_{b(F)} = S^*$. Let $\xi \in E$ and $\{\xi_n\}_n$ in $b(E)$ such that $\xi = \lim\limits_n \xi_n$ and let $\eta \in F$ and $\{\eta_n\}_n$ in $b(F)$ such that $\eta = \lim\limits_n \eta_n$. Then

$$\begin{aligned}\left\langle \eta, \widetilde{S}\xi \right\rangle &= \lim_m \left(\lim_n \langle \eta_m, S\xi_n\rangle\right) = \lim_m \left(\lim_n \langle S^*\eta_m, \xi_n\rangle\right) \\ &= \lim_m \left(\lim_n \langle S\eta_m, \xi_n\rangle\right) = \langle S_0\eta, \xi\rangle .\end{aligned}$$

From this relation, we conclude that $\widetilde{S} \in b(L_A(E,F))$ and so $L_{b(A)}(b(E),b(F)) \subseteq \Psi\left(b(L_A(E,F))\right)$.

3. Let $\{T_n\}_n$ be a sequence in $b(K_A(E,F))$ which converges with respect to the topology induced by the norm $\|\cdot\|_\infty$ to an element T in $b(L_A(E,F))$. Then it is convergent with respect to the topology induced by the family of seminorms $\{\widetilde{p}\}_{p\in S(A)}$, and since $K_A(E,F)$ is a closed subspace of $L_A(E,F)$, T is an element in $K_A(E,F)$. Therefore T is an element in $K_A(E,F) \cap b(L_A(E,F))$ and the assertion 3. is proved. ■

Remark 2.4.3 *From the proof of Theorem 2.4.2, the map Ψ from $b(B_A(E,F))$ to $B_{b(A)}(b(E),b(F))$ defined by $\Psi(T) = T|_{b(E)}$ is an isomorphism of Banach spaces. It is clear that $\Psi^{-1}(K_{b(A)}(b(E),b(F))) \subseteq b(K_A(E,F))$. Therefore $K_{b(A)}$ $(b(E),b(F))$ is isometrically isomorphic with a closed subspace of $b(K_A(E,F))$. In general, $b(K_A(E,F))$ is not isomorphic with $K_{b(A)}(b(E),b(F))$ (see Example 2.4.11).*

Remark 2.4.4 *Since $K_A(E,F)$ is the closure of the vector subspace $\Theta_A(E,F)$ in $L_A(E,F)$ and $K_{b(A)}(b(E),b(F))$ is the closure of the vector subspace $\Theta_{b(A)}(b(E),$ $b(F))$ in $L_{b(A)}(b(E),b(F))$ and since the Hilbert $b(A)$- modules $b(E)$ and $b(F)$*

are dense in E respectiv F, it is not difficult to check that $K_{b(A)}(b(E), b(F))$ is dense in $K_A(E, F)$.

Remark 2.4.5 *If E, F and G are Hilbert A -modules, then*

1. $b(K_A(E, F))b(L_A(G, E)) \subseteq b(K_A(G, F))$;

2. $b(L_A(F, G))b(K_A(E, F)) \subseteq b(K_A(E, G))$.

Corollary 2.4.6 *Let E be a Hilbert A-module. Then:*

1. *$b(B_A(E))$ equipped with the norm $\|\cdot\|_\infty$ is a Banach algebra which is isometrically isomorphic with $B_{b(A)}(b(E))$.*

2. *$b(L_A(E))$ equipped with the norm $\|\cdot\|_\infty$ is a C^*-algebra which is isomorphic with $L_{b(A)}(b(E))$.*

Proof. 1. By Theorem 2.4.2 (1), $b(B_A(E))$ equipped with the norm $\|\cdot\|_\infty$ is a Banach space, and since the norm $\|\cdot\|_\infty$ is submultiplicative, $b(B_A(E))$ is a Banach algebra. From the proof of Theorem 2.4.2 (1), the map Ψ from $b(B_A(E))$ to $B_{b(A)}(b(E))$ defined by $\Psi(T) = T|_{b(E)}$ is an isomorphism of Banach spaces, and since $(ST)|_{b(E)} = S|_{b(E)}T|_{b(E)}$ for all $S, T \in b(B_A(E))$, Ψ is an isomorphism of Banach algebras.

2. Since $\{\widetilde{p}\}_{p\in S(A)}$ is a family of C^*-seminorms on $L_A(E)$, $\|\cdot\|_\infty$ is a C^*-seminorm on $b(L_A(E))$ and by Theorem 2.4.2 (2), $b(L_A(E))$ equipped with the norm $\|\cdot\|_\infty$ is a C^*-algebra. It is not difficult to check that $\left(T|_{b(E)}\right)^* = T^*|_{b(E)}$ and so the map Ψ from $b(L_A(E))$ to $L_{b(A)}(b(E))$ defined by $\Psi(T) = T|_{b(E)}$ is an isomorphism of C^*-algebras. ■

Remark 2.4.7 *If E is a Hilbert A-module, then $b(L_A(E))$ coincides with the set of bounded elements in the locally C^*-algebra $L_A(E)$.*

Corollary 2.4.8 *If A is a strongly spectrally bounded locally C^*-algebra and E is a Hilbert A -module, then $L_A(E)$ is a strongly spectrally bounded locally C^*-algebra.*

Proof. Indeed, if A is strongly spectrally bounded, then $b(E) = E$ as set. Let $T \in L_A(E)$. Then T is a map from $b(E)$ to $b(E)$ and there is a map T^* from $b(E)$ to $b(E)$ such that

$$\langle \eta, T\xi \rangle = \langle T^*\eta, \xi \rangle$$

for all $\xi, \eta \in b(E)$. This implies that $T \in L_{b(A)}(b(E))$. From this fact and Corollary 2.4.6 (2) we conclude that $b(L_A(E)) = L_A(E)$ as set. Therefore, $L_A(E)$ is a strongly spectrally bounded locally C^*-algebra. ■

Remark 2.4.9 *Let E be a Hilbert A -module. From Theorem 2.4.2 (3), Remark 2.4.5 and Corollary 2.4.6(2) we conclude that $b\left(K_A(E)\right)$ is a closed two-side $*$ -ideal of $b\left(L_A(E)\right)$.*

Remark 2.4.10 *By Remarks 2.4.3 and 2.4.9, $K_{b(A)}(b(E))$ is isomorphic to a closed two-side $*$ - ideal of $b\left(K_A(E)\right)$. In general $K_{b(A)}(b(E))$ is not isomorphic with $b\left(K_A(E)\right)$.*

Example 2.4.11 *([**38**], Example 4.9) Let $A = C(\mathbb{Z}^+)$, which is just $\prod\limits_{n=1}^{\infty} \mathbb{C}$, and let $E = \prod\limits_{n=1}^{\infty} \mathbb{C}^n$. We make E into a Hilbert A -module via $\left(\xi_n\right)_n \left(a_n\right)_n = \left(\xi_n a_n\right)_n$ and $\langle \left(\xi_n\right)_n, \left(\eta_n\right)_n \rangle = \left(\langle \xi_n, \eta_n \rangle_n\right)_n$, where $\langle \cdot, \cdot \rangle_n$ denotes the usual $\mathbb{C}$-inner product on $\mathbb{C}^n$.*

For each positive integer n consider the map $p_n : A \to [0, \infty)$ defined by $p_n((a_n)_n) = \sup\{|a_k|; 1 \leq k \leq n\}$. Clearly, p_n is a continuous C^*-seminorm on A and the topology on A is determined by the family of C^*-seminorms $\{p_n\}_n$. It is not difficult to check that A_{p_n} can be identified with the product of the first n factors of A, and E_{p_n} can be identified with the product of the first n factors of E, for each n. Therefore $K_{A_{p_n}}(E_{p_n}) = L_{A_{p_n}}(E_{p_n})$ for each positive integer n. From this fact and Corollaries 2.2.8 and 2.3.5, we conclude that $L_A(E) = K_A(E)$.

Suppose that $b(K_A(E))$ is isomorphic with $K_{b(A)}(b(E))$. Then, since $K_A(E)$ $= L_A(E)$, from Corollary 2.4.6 we deduce that $L_{b(A)}(b(E))$ is isomorphic with $K_{b(A)}(b(E))$. This implies that the Hilbert $b(A)$ -module $b(E)$ is finitely generated.

On the other hand, it is not difficult to check that $b(A) = \{(a_n)_n \in A; \sup\{|a_n|\,;\, n = 1, 2, ...\} < \infty\}$ *and* $b(E) = \{(\xi_n)_n\,; \sup\{|\langle \xi_n, \xi_n\rangle|\,; n = 1, 2, ...\} < \infty\}$ *and so* $b(E)$ *is not finitely generated as* $b(A)$*-modules, a contradiction. Therefore,* $b(K_A(E))$ *is not isomorphic with* $K_{b(A)}(b(E))$

Example 2.4.12 *Let* A *be a locally* C^**-algebra and* $E = \bigoplus_{k=1}^{n} A$ *for some positive integer* n*. We seen that* $b(E) = \bigoplus_{k=1}^{n} b(A)$ *and* E_p *is isomorphic with* $\bigoplus_{k=1}^{n} A_p$ *for all* $p \in S(A)$*. Since* $K_{A_p}(E_p)$ *is isomorphic to* $M_n(\mathbb{C}) \bigotimes A_p$*, where* $M_n(\mathbb{C})$ *is the set of all* $n \times n$ *matrices over* $\mathbb{C}$*, for all* $p \in S(A)$*, from Corollary 2.3.5 and [**2**], we conclude that* $K_A(E)$ *can be identified with* $M_n(\mathbb{C}) \bigotimes A$ *and so* $b(K_A(E))$ *is isomorphic with* $b(M_n(\mathbb{C}) \bigotimes A)$*.*

On the other hand $K_{b(A)}(b(E))$ *can be identified with* $M_n(\mathbb{C}) \bigotimes b(A)$ *and since* $b(M_n(\mathbb{C}) \bigotimes A)$ *is isomorphic with* $M_n(\mathbb{C}) \bigotimes b(A)$*, we conclude that* $b(K_A(E))$ *is isomorphic with* $K_{b(A)}(b(E))$*.*

References for Section 2.4: **[18], [24], [45].**

2.5 Unitary operators on Hilbert modules

In this Section we characterize the unitary operators on Hilbert modules over locally C^*-algebras, and show that a map $\Phi : E \to F$ is an isomorphisms of Hilbert modules if and only if Φ is a unitary operator from E to F. Also we how that the Hilbert A -modules E and F are isomorphic if and only if the Hilbert C^*-modules $b(E)$ and $b(F)$ are isomorphic. Frank [**10**] showed that the two Hilbert C^*-modules structures on a Banach module E over a C^*-algebra A are isomorphic if and only if so are the corresponding C^*-algebras of adjointable operators as well as the corresponding C^*-algebras of compact operators. In this Section we extend the result of Frank in the context of Hilbert modules over locally C^*-algebras.

Let A be a locally C^*-algebra and let E and F be two Hilbert A -modules.

Definition 2.5.1 *An adjointable operator U from E to F is said to be unitary if $U^*U = \mathrm{id}_E$ and $UU^* = \mathrm{id}_F$.*

Remark 2.5.2 *Let $U \in L_A(E,F)$. Then U is unitary if and only if $(\pi_p)_*(U)$ is a unitary operator from E_p to F_p for all $p \in S(A)$.*

Proposition 2.5.3 *Let U be a linear map from E to F. Then the following statements are equivalent:*

1. *U is a unitary operator from E to F;*

2. *$\langle U\xi, U\xi\rangle = \langle \xi,\xi\rangle$ for all $\xi \in E$ and U is surjective;*

3. *$\overline{p}_F(U\xi) = \overline{p}_E(\xi)$ for all $\xi \in E$ and for all $p \in S(A)$ and U is a surjective module homomorphism from E to F.*

Proof. 1. $\Rightarrow$ 2. Suppose that U is unitary. Then clearly U is surjective. Using Remark 2.5.2 and Theorem 3.5 in [**29**], we obtain

$$\begin{aligned}\pi_p\left(\langle U\xi, U\xi\rangle - \langle \xi,\xi\rangle\right) &= \left\langle (\pi_p)_*(U)\sigma_p(\xi), (\pi_p)_*(U)\sigma_p(\xi)\right\rangle - \langle \sigma_p(\xi), \sigma_p(\xi)\rangle \\ &= 0\end{aligned}$$

for all $p \in S(A)$ and for all $\xi \in E$. This implies that $\langle U\xi, U\xi\rangle = \langle \xi,\xi\rangle$ for all $\xi \in E$.

2. $\Rightarrow$ 3. Let $p \in S(A)$ and $\xi \in E$. Then

$$\overline{p}_F(U\xi)^2 = p\left(\langle U\xi, U\xi\rangle\right) = p\left(\langle \xi,\xi\rangle\right) = \overline{p}_E(\xi)^2.$$

By polarization $\langle U\xi_1, U\xi_2\rangle = \langle \xi_1,\xi_2\rangle$ for all ξ_1,$\xi_2 \in E$. Let $\xi \in E$ and $a \in A$. Then

$$\langle U(\xi a) - (U\xi)\,a, U(\xi a) - (U\xi)\,a\rangle = 0$$

and so $U(\xi a) = (U\xi)\,a$. Hence U is a surjective module homomorphism.

3. $\Rightarrow$ 1. Let $\xi \in E$ and $p \in S(A)$. Then, since

$$\begin{aligned}\left\|\pi_p(a^*)\,\pi_p\left(\langle U\xi, U\xi\rangle\right)\pi_p(a)\right\|_p &= p\left(\langle U(\xi a), U(\xi a)\rangle\right) \\ &= \overline{p}_F(U(\xi a))^2 = \overline{p}_E(\xi a)^2 \\ &= p\left(\langle \xi a, \xi a\rangle\right) = \left\|\pi_p(a^*)\,\pi_p\left(\langle \xi,\xi\rangle\right)\pi_p(a)\right\|_p,\end{aligned}$$

for all $a \in A$, by Lemma 3.4 in [**29**], $\pi_p(\langle U\xi, U\xi\rangle) = \pi_p(\langle \xi, \xi\rangle)$. Therefore $\langle U\xi, U\xi\rangle = \langle \xi, \xi\rangle$ for all $\xi \in E$ and by polarization $\langle U\xi_1, U\xi_2\rangle = \langle \xi_1, \xi_2\rangle$ for all $\xi_1, \xi_2 \in E$.

Clearly, U is a bijective module momorphism from E to F. Let U^{-1} be the inverse of U. Then U^{-1} is a module homomorphism from F to E and

$$\langle U\xi, \eta\rangle = \left\langle U\xi, U\left(U^{-1}\eta\right)\right\rangle = \left\langle \xi, U^{-1}\eta\right\rangle$$

for all $\xi \in E$ and for all $\eta \in F$. Therefore U is unitary. ■

Corollary 2.5.4 *Two Hilbert A -modules E and F are isomorphic if and only if there is a unitary operator from E to F.*

Corollary 2.5.5 *If the Hilbert A -modules E and F are isomorphic, then the Hilbert A_p -modules E_p and F_p are isomorphic for all $p \in S(A)$.*

Proposition 2.5.6 *Let E and F be Hilbert A -modules and let $U \in L_A(E, F)$. Then U is a unitary operator from E to F if and only if $U \in b(L_A(E, F))$ and $U|_{b(E)}$ is a unitary operator from $b(E)$ to $b(F)$. Moreover, there is a bijective correspondence between the set $\mathcal{U}_A(E, F)$ of all unitary operators from E to F and the set $\mathcal{U}_{b(A)}(b(E), b(F))$ of all unitary operators from $b(E)$ to $b(F)$.*

Proof. If U is a unitary operator from E to F, then $\widetilde{p}(U) = 1$ for all $p \in S(A)$. Therefore $U \in b(L_A(E, F))$ and since $\mathrm{id}_{b(E)} = (U^*U)\,|_{b(E)} = U^*|_{b(F)}U|_{b(E)}$ and $\mathrm{id}_{b(F)} = (UU^*)\,|_{b(F)} = U|_{b(E)}U^*|_{b(F)}$, $U|_{b(E)}$ is a unitary operator from $b(E)$ to $b(F)$. It is easy to check that the restriction of the isomorphism Ψ from $b(L_A(E, F))$ onto $L_{b(A)}(b(E), b(F))$ defined by $\Psi(T) = T|_{b(E)}$ to $\mathcal{U}_A(E, F)$ is a bijective correspondence between the sets $\mathcal{U}_A(E, F)$ and $\mathcal{U}_{b(A)}(b(E), b(F))$ (see Theorem 2.4.2). ■

Corollary 2.5.7 *Two Hilbert A-modules E and F are isomorphic if and only if the Hilbert $b(A)$-modules $b(E)$ and $b(F)$ are isomorphic.*

Let E be a complex vector space which is also right A-module, compatible with the structure of complex algebra and equipped with two an A -valued inner-products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ which induce either a structure of Hilbert A-module on E. We denote by E_1 the Hilbert A-module $(E,\langle\cdot,\cdot\rangle_1)$ and by E_2 the Hilbert A-module $(E,\langle\cdot,\cdot\rangle_2)$.

The following proposition is a generalization of a result of Frank [**10**] in the context of Hilbert modules over locally C^*-algebras.

Proposition 2.5.8 *Let E be as above. Then the following statements are equivalent:*

1. *E_1 and E_2 are isomorphic as Hilbert A -modules.*

2. *The locally C^*-algebras $K_A(E_1)$ and $K_A(E_2)$ are isomorphic.*

3. *The locally C^*-algebras $L_A(E_1)$ and $L_A(E_2)$ are isomorphic.*

4. *The C^*-algebras $L_{b(A)}(b(E_1))$ and $L_{b(A)}(b(E_2))$ are isomorphic.*

5. *The C^*-algebras $K_{b(A)}(b(E_1))$ and $K_{b(A)}(b(E_2))$ are isomorphic.*

6. *The Hilbert $b(A)$ -modules $b(E_1)$ and $b(E_2)$ are isometrically isomorphic as Banach $b(A)$ -modules.*

7. *$b(E_1)$ and $b(E_2)$ are isomorphic as Hilbert $b(A)$ -modules.*

Proof. 1. $\Rightarrow$ 2. Since E_1 and E_2 are unitarily equivalent, there is a unitary operator U in $L_A(E_1,E_2)$. It is not hard to check that the map Φ from $K_A(E_1)$ to $K_A(E_2)$ defined by $\Phi(T)=UTU^*$ is an isomorphism of locally C^*-algebras.

2. $\Rightarrow$ 3. Let Φ be an isomorphism of locally C^*-algebras from $K_A(E_1)$ onto $K_A(E_2)$. By [**13**, Lemmas 2.4, 2.7 and Corollary //], there is a unique isomorphism of locally C^*-algebras $\overline{\Phi}:M(K_A(E_1))\rightarrow M(K_A(E_2))$ such that $\overline{\Phi}|_{K_A(E_1)}=\Phi$.

On the other hand, the locally C^*-algebras $M(K_A(E_1))$ and $L_A(E_1)$ are isomorphic as well as $M(K_A(E_2))$ and $L_A(E_2)$ (Proposition 2.3.9). Therefore the locally C^*-algebras $L_A(E_1)$ and $L_A(E_2)$ are isomorphic.

3. $\Rightarrow$ 4. If the locally C^*-algebras $L_A(E_1)$ and $L_A(E_2)$ are isomorphic, then the C^*-algebras $b(L_A(E_1))$ and $b(L_A(E_2))$ are isomorphic [**38**, Corollary 2.6]. But $b(L_A(E_i))$ is isomorphic with $L_{b(A)}(b(E_i))$, $i \in \{1,2\}$ (Corollary 2.4.6). Therefore the C^*-algebras $L_{b(A)}(b(E_1))$ and $L_{b(A)}(b(E_2))$ are isomorphic.

The implications 4. $\Rightarrow$ 5. $\Rightarrow$ 6. were proved in [**9**], the equivalence 6. $\Leftrightarrow$ 7. was proved in [**29**] and the implication 7. $\Rightarrow$ 1. was showed in Corollary 2.5.7. ■

References for Section 2.5: **[10], [23], [29], [40].**

Chapter 3

Complemented submodules in Hilbert modules

3.1 Projections on Hilbert modules

In this Section we characterize the projections and the partial isometries on Hilbert modules over locally C^*-algebras.

Let E be a Hilbert module over a locally C^*-algebra A. An adjointable module momorphism P from E to E is said to be a projection in $L_A(E)$ if it is self-adjoint (that is, $P^* = P$) and $PP = P$.

Remark 3.1.1 *Let $P \in L_A(E)$. Then P is a projection in $L_A(E)$ if and only if $(\pi_p)_*(P)$ is a projection in $L_{A_p}(E_p)$ for all $p \in S(A)$.*

Proposition 3.1.2 *Let $P \in L_A(E)$. Then P is a projection in $L_A(E)$ if and only if $P \in b(L_A(E))$ and $P|_{b(E)}$ is a projection in $L_{b(A)}(b(E))$. Moreover, there is a bijective correspondence between the set $\mathcal{P}_A(E)$ of all projections in $L_A(E)$ and the set $\mathcal{P}_{b(A)}(b(E))$ of all projections in $L_{b(A)}(b(E))$.*

Proof. If P is a projection in $L_A(E)$, then $\widetilde{p}(P) = 1$ for all $p \in S(A)$ and so $P \in b(L_A(E))$. It is not hard to check that $P|_{b(E)}$ is a projection in $L_{b(A)}(b(E))$ and the map $P \to P|_{b(E)}$ from $\mathcal{P}_A(E)$ to $\mathcal{P}_{b(A)}(b(E))$ is bijective. ■

An element V in $L_A(E,F)$ is said to be a partial isometry if V^*V is a projection in $L_A(E)$.

Remark 3.1.3 *Let $V \in L_A(E,F)$. Then V is a partial isometry in $L_A(E,F)$ if and only if $(\pi_p)_*(V)$ is a partial isometry in $L_{A_p}(E_p,F_p)$ for all $p \in S(A)$.*

Proposition 3.1.4 *Let $V \in L_A(E,F)$. Then the following statements are equivalent:*

1. *V is a partial isometry in $L_A(E,F)$;*

2. *VV^* is a projection in $L_A(F)$;*

3. *$VV^*V = V$;*

4. *$V^*VV^* = V^*$.*

Proof. It is a simple verification. ■

Proposition 3.1.5 *Let $V \in L_A(E,F)$. Then V is a partial isometry in $L_A(E,F)$ if and only if $V \in b(L_A(E,F))$ and $V|_{b(E)}$ is a partial isometry in $L_{b(A)}(b(E),b(F))$. Moreover, there is a bijective correspondence between the set $\mathcal{I}_A(E,F)$ of a all partial isometries in $L_A(E,F)$ and the set $\mathcal{I}_{b(A)}(b(E),b(F))$ of a all partial isometries in $L_{b(A)}(b(E),b(F))$.*

Proof. If V is a partial isometry from E to F, then $\widetilde{p}(V) \leq 1$ for all $p \in S(A)$. Therefore $V \in b(L_A(E,F))$, and since $(V|_{b(E)})^*V|_{b(E)} = V^*|_{b(F)}V|_{b(E)} = (V^*V)|_{b(E)}$, $V|_{b(E)}$ is a partial isometry in $L_{b(A)}(b(E),b(F))$. It is not hard to check that the map $V \to V|_{b(E)}$ from $\mathcal{I}_A(E,F)$ to $\mathcal{I}_{b(A)}(b(E),b(F))$ is bijective. ■

References for Section 3.1: **[16], [29], [40].**

3.2 Orthogonally complemented submodules

It is well known that the closed submodules of Hilbert C^*-modules do not have in general orthogonal complements. Therefore, the closed submodules of Hilbert modules over locally C^*-algebras are not in general complemented. Mishchenko showed that certain submodules of Hilbert C^*-modules are complemented (see, for example, [**29**], Theorem 3.2). In this Section we show that these results still hold for submodules of a Hilbert module over a locally C^*-algebra.

Let A be a locally C^*-algebra and let E and F be two Hilbert A-modules.

We say that a closed submodule E_0 of E is complemented if $E = E_0 \oplus E_0^{\perp}$, where $E_0^{\perp} = \{\xi \in E; \langle \xi, \eta \rangle = 0 \text{ for every } \eta \in E_0\}$.

Proposition 3.2.1 *Let E be a Hilbert A -module and let E_0 be a closed submodule of E. Then E_0 is complemented if and only if there is a projection P_0 in $L_A(E)$ such that E_0 is the range of P_0.*

Proof. If E_0 is a complemented submodule of E, then any $\xi \in E$ can be uniquely written as sum of two elements ξ_1 and ξ_2, with ξ_1 in E_0 and ξ_2 in $E_0^{\perp}$. It is easy to check that the map P_0 from E to E defined by $P_0(\xi) = \xi_1$ is a projection in $L_A(E)$ whose the range is E_0. Conversely, if P_0 is a projection in $L_A(E)$, then the range ranP_0 of P_0 is a complemented submodule of E and $\operatorname{ran}(P_0)^{\perp} = \operatorname{ran}(\mathrm{id}_E - P_0)$. ■

Remark 3.2.2 *Let E_0 be a closed submodule of E. Then E_0 is complemented if and only if $b(E_0)$ is complemented.*

Lemma 3.2.3 *Let $T \in L_A(E, F)$. Then T^*F, T^*TE, $|T|\, E$, where $|T| = (T^*T)^{\frac{1}{2}}$, and $|T|^{\frac{1}{2}}\, E$ have the same closure.*

Proof. We have

$$\begin{aligned}
\overline{T^*F} &= \lim_{\substack{\leftarrow \\ p}} \overline{\sigma_p^E(T^*F)} \text{ (using Proposition 9 and Corollary at p. 52 in [4])} \\
&= \lim_{\substack{\leftarrow \\ p}} \overline{(\pi_p)_*(T^*)\, \sigma_p^F(F)} = \lim_{\substack{\leftarrow \\ p}} \overline{(\pi_p)_*(T^*)\, F_p} \\
&= \lim_{\substack{\leftarrow \\ p}} \overline{(\pi_p)_*(T)^*\, (\pi_p)_*(T)\, E_p} \text{ (using Proposition 3.7 in [29])} \\
&= \lim_{\substack{\leftarrow \\ p}} \overline{(\pi_p)_*(T^*T)\, \sigma_p^E(E)}
\end{aligned}$$

$$= \lim_{\substack{\leftarrow \\ p}} \overline{\sigma_p^E(T^*TE)}$$
$$= \overline{T^*TE} \text{ (using Proposition 9 and Corollary at p. 52 in [4])},$$

where $\overline{X}$ denotes the closure of a space X with respect to the topology induced by the inner product.

In the same manner, using the fact that $\overline{(\pi_p)_* (T^*T)\, E_p}$, $\overline{(\pi_p)_* (|T|)\, E_p}$ and $\overline{(\pi_p)_* (|T|)^{\frac{1}{2}}\, E_p}$ have the same closure for all $p \in S(A)$, we deduce that T^*TE, $|T|\, E$ and $|T|^{\frac{1}{2}}\, E$ have the same closure. ■

Theorem 3.2.4 *Let $T \in L_A(E,F)$. If T has closed range then:*

1. $\ker T$ *is a complemented submodule of E;*

2. *ranT, the range of T, is a complemented submodule of F.*

Proof. 1. Since $\overline{p}_E(|T|\xi) = \overline{p}_F(T\xi)$ for all $\xi \in E$ and for all $p \in S(A)$, and since T has closed range, $|T|$ has closed range. From

$$\overline{|T|^{\frac{1}{2}}\, E} = \overline{|T|\, E} = |T|\, E \subseteq |T|^{\frac{1}{2}}\, E$$

we conclude that $|T|^{\frac{1}{2}}$ has closed range, and moreover, $|T|\, E = |T|^{\frac{1}{2}}\, E$. Clearly, $|T|^{\frac{1}{2}}\, E \subseteq \left(\ker |T|^{\frac{1}{2}}\right)^{\perp}$.

Let $\xi \in E$. Since $|T|\, E = |T|^{\frac{1}{2}}\, E$, there is $\eta \in E$ such that $|T|\, \eta = |T|^{\frac{1}{2}}\, \xi$. Then $\xi - |T|^{\frac{1}{2}}\, \eta \in \ker(|T|^{\frac{1}{2}})$ and $\xi = \left(\xi - |T|^{\frac{1}{2}}\, \eta\right) + |T|^{\frac{1}{2}}\, \eta$. This shows that $E = \ker |T|^{\frac{1}{2}} \oplus |T|^{\frac{1}{2}}\, E$. From this relation, and taking into account that $\ker(|T|^{\frac{1}{2}}) = \ker(T)$ and $|T|^{\frac{1}{2}}\, E = |T|\, E$, we conclude that $E = \ker T \oplus |T|\, E$.

2. First we will show that T^*T has closed range. For this we show that $|T|\, E = T^*TE$. Clearly, $T^*TE \subseteq |T|\, E$. Let $\xi \in |T|\, E$. Then $\xi = |T|\eta$ and since $\eta = \eta_1 + \eta_2$ with $\eta_1 \in \ker T$ and $\eta_2 \in |T|\, E$, $\xi = |T|\eta_2 \in T^*TE$. Therefore, $|T|\, E = T^*TE = (\ker T)^{\perp}$. By Lemma 4.2.1, $\overline{T^*F} = \overline{T^*TE} = T^*TE \subseteq T^*F$. This implies that T^* has closed range, and according to the assertion 1. of this theorem, $\ker T^*$ is complemented and $(\ker T^*)^{\perp} = |T^*|F$. But, from Lemma 3.2.3 and taking into account that T has closed range we conclude that $|T^*|F = TE$. Hence the range of T is complemented. ■

Remark 3.2.5 *Let T be an element in $L_A(E,F)$ with closed range. By the proof of Theorem 3.2.4 and Lemma 3.2.3 , we deduce that T^* has closed range, and moreover, $E = \ker T \oplus T^*F$ and $F = \ker T^* \oplus TE$.*

Corollary 3.2.6 *A closed submodule E_0 of a Hilbert A -module E is complemented if and only if it is the range of an adjointable operator on E.*

Proposition 3.2.7 *Let E and F be two Hilbert modules over A and let $V \in L_A(E,F)$. Then V is a partial isometry if and only if V has closed range and $V|_{(\ker V)^\perp}$ is a unitary operator from $(\ker V)^\perp$ to VE.*

Proof. Suppose that V is a partial isometry. Then VV^* is a projection in $L_A(F)$ and so VV^*F is a closed submodule of F. From this fact and Lemma 3.2.3 we conclude that V has closed range and then $E = \ker V \oplus V^*F$ and $F = \ker V^* \oplus VE$ (Remark 3.2.5).

Since $(\ker V)^\perp = V^*F = V^*VE$ and since V^*V is a projection in $L_A(E)$, we have:

(**a**) $\langle V(\xi_0), V(\xi_0)\rangle = \langle \xi_0, V^*V\xi_0\rangle = \langle \xi_0, \xi_0\rangle$ for all $\xi_0 \in (\ker V)^\perp$; and

(**b**) $V\left((\ker V)^\perp\right) = VV^*F = VE$.

From these relations and Proposition 2.5.3 we conclude that $V|_{(\ker V)^\perp}$ is a unitary operator from $(\ker V)^\perp$ to VE.

Conversely, suppose that V has closed range and $V|_{(\ker V)^\perp}$ is a unitary operator from $(\ker V)^\perp$ to VE. Then, $V^*|_{VE}$ is a unitary operator from VE to $(\ker V)^\perp$. From this fact and taking into account that $(\ker V)^\perp = V^*F$ (Remark 3.2.5), we have

$$\begin{aligned}
\langle (VV^*V - V)(\xi), (VV^*V - V)(\xi)\rangle &= \langle V(\xi), V(\xi)\rangle - \langle V^*V(\xi), V^*V(\xi)\rangle \\
&= \langle V(\xi), V(\xi)\rangle - \langle V(\xi), V(\xi)\rangle \\
&= 0
\end{aligned}$$

for all $\xi \in E$. This implies that $VV^*V - V$ and so V is a partial isometry. ■

References for Section 3.2: **[16], [29], [40].**

3.3 Polar decomposition of a adjointable operator

We know that the adjointable operators between Hilbert C^*-modules do not generally have a polar decomposition. But, if both T and T^* have the closures of the ranges complemented, then T has a polar decomposition. In this Section we show that this result is valid for adjointable operators between Hilbert modules over locally C^*-algebras.

Let A be a locally C^*-algebra and let E and F be two Hilbert A-modules.

Definition 3.3.1 *An adjointable operator T from E to F has a polar decomposition if there is a partial isometry V from E to F such that $T = V\left|T\right|$, and* $\ker V = \ker T$, *$ranV = \overline{TE}$,* $\ker V^* = \ker T^*$ *and $ranV^* = \overline{\left|T\right| E}$.*

Proposition 3.3.2 *An adjointable operator T from E to F has a polar decomposition if and only if $\left(\pi_p\right)_* (T)$ has a polar decomposition for each $p \in S(A)$. Moreover, if $T = V\left|T\right|$, then $\left(\pi_p\right)_* (T) = \left(\pi_p\right)_* (V) \left|\left(\pi_p\right)_* (T)\right|$ for all $p \in S(A)$.*

Proof. First we suppose that T has a polar decomposition $T = V\left|T\right|$. Let $p \in S(A)$. Since V is a partial isometry, $\left(\pi_p\right)_* (V)$ is a partial isometry. Moreover,

$$\begin{aligned} \operatorname{ran}\left(\left(\pi_p\right)_* (V)\right) &= \sigma_p^F(VE) = \overline{\sigma_p^F(VE)} = \overline{\sigma_p^F(\overline{TE})} \\ &= \overline{\sigma_p^F(TE)} = \overline{\left(\pi_p\right)_* (T)E_p} \end{aligned}$$

and

$$\begin{aligned} \operatorname{ran}\left(\left(\pi_p\right)_* (V^*)\right) &= \sigma_p^E(V^*F) = \overline{\sigma_p^E(V^*F)} = \overline{\sigma_p^E(\overline{T^*F})} \\ &= \overline{\sigma_p^E(T^*F)} = \overline{\left(\pi_p\right)_* (T^*)F_p}. \end{aligned}$$

By functional calculus, $\left(\pi_p\right)_* (\left|T\right|) = \left|\left(\pi_p\right)_* (T)\right|$. Therefore $\left(\pi_p\right)_* (T)$ has a polar decomposition, and $\left(\pi_p\right)_* (T) = \left(\pi_p\right)_* (V) \left|\left(\pi_p\right)_* (T)\right|$.

Conversely, suppose that $(\pi_p)_*(T)$ has a polar decomposition, $(\pi_p)_*(T) = V_p\left|(\pi_p)_*(T)\right|$ for each $p \in S(A)$.

Let $p, q \in S(A)$ with $p \geq q$, and let $\zeta \in E_q$. Then, since the canonical map $\sigma_{pq}^E : E_p \to E_q$ is surjective and since $E_p = \ker V_p \oplus \overline{\left|(\pi_p)_*(T)\right| E_p}$, there is $\xi_0 \in \ker V_p$ and there is a net $\{\xi_i\}_{i \in I}$ such that

$$\zeta = \sigma_{pq}^E(\xi_0) + \lim_i \sigma_{pq}^E\left(\left|(\pi_p)_*(T)\right|(\xi_i)\right).$$

Thus we have:

$$\begin{aligned}
(\pi_{pq})_*(V_p)(\zeta) &= \sigma_{pq}^F\left(V_p\left(\xi_0 + \lim_i \left|(\pi_p)_*(T)\right|(\xi_i)\right)\right) \\
&= \sigma_{pq}^F\left(\lim_i \left|V_p(\pi_p)_*(T)\right|(\xi_i)\right) = \lim_i \sigma_{pq}^F\left((\pi_p)_*(T)(\xi_i)\right) \\
&= \lim_i (\pi_q)_*(T)\left(\sigma_{pq}^E(\xi_i)\right)
\end{aligned}$$

and

$$\begin{aligned}
V_q(\zeta) &= V_q\left(\sigma_{pq}^E(\xi_0) + \lim_i \sigma_{pq}^E\left(\left|(\pi_p)_*(T)\right|(\xi_i)\right)\right) \\
&= V_q\left(\sigma_{pq}^E(\xi_0)\right) + \lim_i V_q\left(\left|(\pi_q)_*(T)\right|\left(\sigma_{pq}^E(\xi_i)\right)\right) \\
&= V_q\left(\sigma_{pq}^E(\xi_0)\right) + \lim_i (\pi_q)_*(T)\left(\sigma_{pq}^E(\xi_i)\right).
\end{aligned}$$

From these relations and taking into account that $\sigma_{pq}(\ker V_p) = \sigma_{pq}(\ker (\pi_p)_*(T)) \subseteq \ker (\pi_q)_*(T) = \ker V_q$ we conclude that $(\pi_{pq})_*(V_p) = V_q$. Hence there is $V \in L_A(E, F)$ such that $(\pi_p)_*(V) = V_p$ for all $p \in S(A)$. Moreover, V is a partial isometry, $T = V|T|$,

$$\begin{aligned}
\mathrm{ran} V &= \lim_{\substack{\leftarrow \\ p}} \overline{\sigma_p^F(VE)} = \lim_{\substack{\leftarrow \\ p}} \overline{V_p E_p} = \lim_{\substack{\leftarrow \\ p}} \overline{(\pi_p)_*(T) E_p} \\
&= \lim_{\substack{\leftarrow \\ p}} \overline{\sigma_p^F(TE)} = \overline{TE}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{ran} V^* &= \lim_{\substack{\leftarrow \\ p}} \overline{\sigma_p^E(V^* F)} = \lim_{\substack{\leftarrow \\ p}} \overline{V_p^* F_p} = \lim_{\substack{\leftarrow \\ p}} \overline{(\pi_p)_*(T^*) F_p} \\
&= \lim_{\substack{\leftarrow \\ p}} \overline{\sigma_p^E(T^* F)} = \overline{T^* F}.
\end{aligned}$$

Hence T has a polar decomposition. ■

Proposition 3.3.3 *An adjointable operator T from E to F has a polar decomposition if and only if $\overline{TE}$ is complemented in F and $\overline{T^*F}$ is complemented in E.*

Proof. Suppose that T has a polar decomposition, $T = V|T|$. Then $E = \ker V \oplus V^*F$ and $F = \ker V^* \oplus VE$, whence, since $V^*F = \overline{T^*F}$ and $VE = \overline{TE}$, we conclude that $\overline{TE}$ is complemented in F and $\overline{T^*F}$ is complemented in E.

Conversely, if $\overline{TE}$ and $\overline{T^*F}$ are complemented, since $(|T|E)^{\perp} = \ker T$ and $(TE)^{\perp} = \ker T^*$, we have

$$E = \ker T \oplus \overline{|T|E} \text{ and } F = \ker T^* \oplus \overline{TE}.$$

Define U from $|T|E$ to TE by $U|T|\xi = T\xi$. Since $\overline{p}_F(T\xi) = \overline{p}_E(|T|\xi)$ for all $\xi \in E$ and for all $p \in S(A)$, U extends by linearity and continuity to a surjective A -linear map, denoted also by U, from $\overline{|T|E}$ to $\overline{TE}$. Moreover, by Proposition 2.5.3, U is unitary. Consider the map V from E to F defined by $V(\xi_1 \oplus \xi_2) = U\xi_2$. By Proposition 3.2.7, V is a partial isometry in $L_A(E, F)$ with $\mathrm{ran}V = \overline{TE}$ and $\mathrm{ran}V^* = \overline{|T|E} = \overline{T^*F}$. It is not difficult to verify that $T = V|T|$. ■

Corollary 3.3.4 *Let $T \in L_A(E, F)$ such that T has closed range. Then T has a polar decomposition.*

Proof. By Theorem 3.2.4 and Remark 3.2.5, TE and T^*F are complemented. Then by Proposition 3.3.3, T has a polar decomposition. ■

Corollary 3.3.5 *Two Hilbert A -modules E and F are isomorphic as Hilbert A -modules if and only if there is an adjointable operator T from E to F such that both T and T^* have dense range.*

Proof. If E and F are isomorphic as Hilbert A -modules, then there is a unitary operator U from E to F.

Conversely, if there is an adjointable operator from E to F such that T and T^* have dense range, then by Proposition 3.3.3, $T = V|T|$, where V is a partial isometry from E to F such that $\mathrm{ran}V = \overline{TE} = F$ and $\mathrm{ran}V^* = \overline{T^*F} = E$. Therefore V is a unitary operator from E to F. ■

References for Section 3.2: **[16], [29].**

Chapter 4

Tensor products of Hilbert modules

4.1 Exterior tensor product

In this Section we define the notion of exterior tensor product of Hilbert modules over locally C^*-algebras and we show that the exterior tensor product of the Hilbert A -module E and the Hilbert B -module F can be identified, up to an isomorphism of Hilbert modules, with the Hilbert $A \otimes B$ -module $\lim\limits_{\substack{\leftarrow \\ (p,q)}} E_p \otimes F_q$.

Let A and B be two locally C^*-algebras, let E be a Hilbert A-module and let F be a Hilbert B-module. The algebraic tensor product $E \otimes_{\text{alg}} F$ of E and F becomes a right $A \otimes_{\text{alg}} B$ -module in the obvious way $(\xi \otimes \eta)(a \otimes b) = \xi a \otimes \eta b$, $\xi \in E, \eta \in F, a \in A$ and $b \in B$.

It is not difficult to check that the map $\langle \cdot, \cdot \rangle$ from $(E \otimes_{\text{alg}} F) \times (E \otimes_{\text{alg}} F)$ to $A \otimes_{\text{alg}} B$ defined by

$$\left\langle \sum_{i=1}^{n} \xi_i \otimes \eta_i, \sum_{j=1}^{m} \xi_j' \otimes \eta_j' \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \left\langle \xi_i, \xi_j' \right\rangle \otimes \left\langle \eta_i, \eta_j' \right\rangle$$

is $\mathbb{C}$- and $A \otimes_{\text{alg}} B$ -linear in its second variable and

$$\left\langle \zeta, \zeta' \right\rangle^* = \left\langle \zeta', \zeta \right\rangle$$

for all $\zeta, \zeta^{'} \in E \otimes_{\text{alg}} F$.

Let $\zeta = \sum\limits_{i=1}^{n} \xi_i \otimes \eta_i \in E \otimes_{\text{alg}} F$. If E_0 is the Hilbert submodule of E generated by $\{\xi_i; i = 1, 2, ..., n\}$ and F_0 is the Hilbert submodule of F generated by $\{\eta_i; i = 1, 2, ..., n\}$, then $E_0 \otimes_{\text{alg}} F_0$ is a submodule of $E \otimes_{\text{alg}} F$ and by Theorem 5.2.7, E_0 can be identified with a Hilbert submodule of H_A and F_0 can be identified with a Hilbert submodule of H_B. Thus we can suppose that for each $i \in \{1, 2, ..., n\}$, $\xi_i = (a_{im})_m$ and $\eta_i = (b_{im})_m$. Then

$$\begin{aligned}
\langle \zeta, \zeta \rangle &= \sum_{i,j=1}^{n} \left\langle (a_{im})_m, (a_{jm})_m \right\rangle \otimes \left\langle (b_{im})_m, (b_{jm})_m \right\rangle \\
&= \sum_{i,j=1}^{n} \left(\sum_m a_{im}^* a_{jm} \right) \otimes \left(\sum_m b_{im}^* b_{jm} \right) \\
&= \sum_m \left(\sum_{i,j=1}^{n} a_{im}^* a_{jm} \otimes b_{im}^* b_{jm} \right) \\
&= \sum_m \left(\sum_{i=1}^{n} a_{im} \otimes b_{im} \right)^* \left(\sum_{i=1}^{n} a_{im} \otimes b_{im} \right) \geq 0.
\end{aligned}$$

Moreover, if $\langle \zeta, \zeta \rangle = 0$, then $\sum\limits_{i=1}^{n} a_{im} \otimes b_{im} = 0$ for all positive integer m and so $\zeta = 0$. Thus, we showed that $E \otimes_{\text{alg}} F$ is a pre-Hilbert module over the pre-locally C^*-algebra $A \otimes_{\text{alg}} B$. Then by Remark 1.2.10, the completion $E \otimes F$ of $E \otimes_{\text{alg}} F$ with respect to the topology induced by the inner-product is a Hilbert $A \otimes B$ -module.

Definition 4.1.1 *The Hilbert $A \otimes B$ -module $E \otimes F$ is said to be the exterior tensor product of E and F.*

For $p \in S(A)$ and $q \in S(B)$ we denote by $E_p \otimes F_q$ the exterior tensor product of the Hilbert C^*-modules E_p and F_q.

Let $p_1, p_2 \in S(A)$ with $p_1 \geq p_2$ and $q_1, q_2 \in S(B)$ with $q_1 \geq q_2$. Then the linear map $\sigma^E_{p_1 p_2} \otimes \sigma^F_{q_1 q_2}$ from $E_{p_1} \otimes_{\text{alg}} F_{q_1}$to $E_{p_2} \otimes_{\text{alg}} F_{q_2}$ defined by $(\sigma^E_{p_1 p_2} \otimes \sigma^F_{q_1 q_2})(\xi \otimes \eta) = \sigma^E_{p_1 p_2}(\xi) \otimes \sigma^F_{q_1 q_2}(\eta)$ may be extended by continuity to a linear map $\sigma^E_{p_1 p_2} \otimes \sigma^F_{q_1 q_2}$ from $E_{p_1} \otimes F_{q_1}$ into $E_{p_2} \otimes F_{q_2}$, since

$$\begin{aligned}\left\langle(\sigma^E_{p_1p_2}\otimes\sigma^F_{q_1q_2})(\xi\otimes\eta),(\sigma^E_{p_1p_2}\otimes\sigma^F_{q_1q_2})(\xi\otimes\eta)\right\rangle &=\\ &=\left\langle\sigma^E_{p_1p_2}(\xi),\sigma^E_{p_1p_2}(\xi)\right\rangle\otimes\left\langle\sigma^F_{q_1q_2}(\eta),\sigma^F_{q_1q_2}(\eta)\right\rangle\\ &=\pi_{p_1p_2}\left(\langle\xi,\xi\rangle\right)\otimes\pi_{q_1q_2}\left(\langle\eta,\eta\rangle\right)\\ &=\left(\pi_{p_1p_2}\otimes\pi_{q_1q_2}\right)\left(\langle\xi\otimes\eta,\xi\otimes\eta\rangle\right)\end{aligned}$$

for all $\xi\in E_{p_1}$ and for all $\eta\in F_{q_1}$. It is not difficult to check that $\{E_p\otimes F_q;\sigma^E_{p_1p_2}\otimes\sigma^F_{q_1q_2};\ A_p\otimes B_q;\ \pi_{p_1p_2}\otimes\pi_{q_1q_2},\ p_1,p_2\in S(A),\ p_1\geq p_2,\ q_1,q_2\in S(B),\ q_1\geq q_2\}$ is an inverse system of Hilbert C^*-modules. We will show that the Hilbert $A\otimes B$ -modules $E\otimes F$ and $\lim\limits_{\substack{\leftarrow\\(p,q)}}(E_p\otimes F_q)$ are isomorphic.

Proposition 4.1.2 *Let A, B, E and F be as above. Then the Hilbert $A\otimes B$ -modules $E\otimes F$ and $\lim\limits_{\substack{\leftarrow\\(p,q)}}(E_p\otimes F_q)$ are isomorphic.*

Proof. First we will show that for each $p\in S(A)$ and $q\in S(B)$ the Hilbert $A_p\otimes B_q$ -modules $(E\otimes F)_{(p,q)}$ and $E_p\otimes F_q$ are isomorphic. Let $p\in S(A)$ and $q\in S(B)$. Since

$$\begin{aligned}t_{(p,q)}\left(\langle\xi\otimes\eta,\xi\otimes\eta\rangle\right) &= \left\|\pi_p\left(\langle\xi,\xi\rangle\right)\otimes\pi_q\left(\langle\eta,\eta\rangle\right)\right\|_{A_p\otimes B_q}\\ &= \left\|\left\langle\sigma^E_p(\xi),\sigma^E_p(\xi)\right\rangle\otimes\left\langle\sigma^F_q(\eta),\sigma^F_q(\eta)\right\rangle\right\|_{A_p\otimes B_q}\\ &= \left\|\left\langle\sigma^E_p(\xi)\otimes\sigma^F_q(\eta),\sigma^E_p(\xi)\otimes\sigma^F_q(\eta)\right\rangle\right\|_{A_p\otimes B_q}\end{aligned}$$

for all $\xi\in E$ and $\eta\in F$, we can define a linear map $U_{(p,q)}:(E\otimes_{\rm alg}F)/N^{E\otimes F}_{(p,q)}\rightarrow E_p\otimes_{\rm alg}F_q$ by

$$U_{(p,q)}\left(\xi\otimes\eta+N^{E\otimes F}_{(p,q)}\right)=\sigma^E_p(\xi)\otimes\sigma^F_q(\eta).$$

Evidently $U_{(p,q)}$ is a surjective $A_p\otimes_{\rm alg}B_q$ -linear map and

$$\left\|U_{(p,q)}\left(\sum_{i=1}^n\xi_i\otimes\eta_i+N^{E\otimes F}_{(p,q)}\right)\right\|_{E_p\otimes F_q}=\left\|\sum_{i=1}^n\xi_i\otimes\eta_i+N^{E\otimes F}_{(p,q)}\right\|_{(E\otimes F)_{(p,q)}}$$

for all $\sum\limits_{i=1}^n\xi_i\otimes\eta_i\in E\otimes_{\rm alg}F$. From these facts, taking into account that $A_p\otimes_{\rm alg}B_q$ is dense in $A_p\otimes B_q$; $(E\otimes_{\rm alg}F)/N^{E\otimes F}_{(p,q)}$ is dense in $(E\otimes F)_{(p,q)}$ and $E_p\otimes_{\rm alg}F_q$ is dense in $E_p \otimes F_q$, we conclude that $U_{(p,q)}$ may be extended by continuity to an isometric, surjective $A_p \otimes B_q$-linear map $U_{(p,q)}$ from $(E \otimes F)_{(p,q)}$ onto $E_p \otimes F_q$. According to Theorem 3.5 in [**29**], $U_{(p,q)}$ is an isomorphism of Hilbert C^*-modules from $(E \otimes F)_{(p,q)}$ onto $E_p \otimes F_q$.

Let $p_1, p_2 \in S(A)$ with $p_1 \geq p_2$ and $q_1, q_2 \in S(B)$ with $q_1 \geq q_2$. Then

$$\begin{aligned}\left(\left(\sigma^E_{p_1p_2} \otimes \sigma^F_{q_1q_2}\right) \circ U_{(p_1,q_1)}\right) \left(\xi \otimes \eta + N^{E\otimes F}_{(p_1,q_1)}\right) &= \left(\sigma^E_{p_1p_2} \otimes \sigma^F_{q_1q_2}\right) \left(\sigma^E_{p_1}(\xi) \otimes \sigma^F_{q_1}(\eta)\right) \\ &= \sigma^E_{p_2}(\xi) \otimes \sigma^F_{q_2}(\eta) = U_{(p_2,q_2)} \left(\xi \otimes \eta + N^{E\otimes F}_{(p_2,q_2)}\right) \\ &= \left(U_{(p_2,q_2)} \circ \sigma^{E\otimes F}_{(p_1,q_1)(p_2,q_2)}\right) \left(\xi \otimes \eta + N^{E\otimes F}_{(p_1,q_1)}\right)\end{aligned}$$

for all $\xi \in E$ and for all $\eta \in F$, and so $\left(U_{(p,q)}\right)_{(p,q)\in S(A)\times S(B)}$ is an inverse system of isomorphisms of Hilbert C^*-modules. Let $U = \lim\limits_{\substack{\leftarrow \\ (p,q)}} U_{(p,q)}$. Then U is an isomorphism of Hilbert modules from $\lim\limits_{\substack{\leftarrow \\ (p,q)}} (E \otimes F)_{(p,q)}$ to $\lim\limits_{\substack{\leftarrow \\ (p,q)}} (E_p \otimes F_q)$. Therefore the Hilbert $A \otimes B$ -modules $E \otimes F$ and $\lim\limits_{\substack{\leftarrow \\ (p,q)}} (E_p \otimes F_q)$ are isomorphic.

■

Remark 4.1.3 *If H is a Hilbert space and E is a Hilbert A -module, then the exterior tensor product $H \otimes E$ of H and E is a Hilbert $\mathbb{C} \otimes A$ -module. But the locally C^*-algebras $\mathbb{C} \otimes A$ and A are isomorphic and then $H \otimes E$ can be regarded as a Hilbert A -module.*

Corollary 4.1.4 *Let E be a Hilbert A -module and let H be a separable infinite dimensional Hilbert space. Then the Hilbert A-modules H_E and $H \otimes E$ are isomorphic.*

Proof. Let $\{\varepsilon_n; i = 1, 2, ...\}$ be an orthonormal basis of H and let $p \in S(A)$. Then the linear map U_p from H_{E_p} from $H \otimes E_p$ defined by $U_p\left((\xi_n)_n\right) = \sum\limits_n \varepsilon_n \otimes \xi_n$ is an isomorphism of Hilbert A_p -modules. It is not difficult to check that $\left(U_p\right)_p$ is an inverse system of isomorphisms of Hilbert C^*-modules. From this fact, Proposition 4.1.2 and Corollary 2.5.4, we conclude that the Hilbert A -modules H_E and $H \otimes E$ are isomorphic. ■

Remark 4.1.5 *Let E be a Hilbert A -module and let F be a Hilbert B -module. Since the C^*-algebras $b(A \otimes B)$ and $b(A) \otimes b(B)$ are not isomorphic in general [**38**], we conclude that the Hilbert C^*-modules $b(E \otimes F)$ and $b(E) \otimes b(F)$ are not isomorphic in general.*

References for Section 4.1: **[20], [29].**

4.2 Interior tensor product

In this Section we define the interior tensor product of two Hilbert modules and we show that the inner tensor product of two Hilbert modules can be identified, up to an isomorphism of Hilbert modules, with an inverse limit of Hilbert C^*-modules which are interior tensor products of Hilbert C^*-modules. Also we study the relation between the bounded part of the interior tensor product of two Hilbert modules E and F and the interior tensor product of the Hilbert C^* -modules $b(E)$ and $b(F)$.

Let A and B be two locally C^*-algebras, let E be a Hilbert A-module, let F be a Hilbert B-module and let Φ be a continuous $*$ -morphism from A to $L_B(F)$. The Hilbert B -module F becomes a left A-module with the action of A on F defined by $(a, \eta) \to \Phi(a)\eta$, $a \in A$, $\eta \in F$. The algebraic tensor product $E \otimes_A F$ of E and F over A, which is the quotient of the algebraic tensor product $E \otimes_{\text{alg}} F$ by the vector subspace N_Φ generated by elements of the form $\xi a \otimes \eta - \xi \otimes \Phi(a)\eta$, $a \in A$, $\xi \in E$, $\eta \in F$, is a right B-module in the obvious way $(\xi \otimes \eta + N_\Phi, b) \to \xi \otimes \eta b + N_\Phi$, $b \in B$, $\xi \in E$, $\eta \in F$. It is not difficult to check that the map $\langle \cdot, \cdot \rangle_\Phi^0$ from $(E \otimes_{\text{alg}} F) \times (E \otimes_{\text{alg}} F)$ to B defined by

$$\left\langle \sum_{i=1}^{n} \xi_i \otimes \eta_i, \sum_{j=1}^{m} \xi_j^{'} \otimes \eta_j^{'} \right\rangle_\Phi^0 = \sum_{i=1}^{n} \sum_{j=1}^{m} \left\langle \eta_i, \Phi\left(\left\langle \xi_i, \xi_j^{'} \right\rangle\right) \eta_j^{'} \right\rangle$$

is $\mathbb{C}$ and B -linear in its second variable. Moreover,

$$\left(\langle \zeta, \zeta' \rangle_\Phi^0\right)^* = \langle \zeta', \zeta \rangle_\Phi^0$$

for all $\zeta, \zeta' \in E \otimes_{\mathrm{alg}} F$.

For a positive integer n we denote by $\Phi^{(n)}$ the continuous $*$ -morphism from $M_n(A)$ to $M_n(L_B(F)))$ defined by $\Phi^{(n)}\left([a_{ij}]_{i,j=1}^n\right) = [\Phi(a_{ij})]_{i,j=1}^n$. Since $M_n(L_B(F))$ can be identified with $L_B(F^n)$ (Corollary 2.2.9), $\Phi^{(n)}\left([a_{ij}]_{i,j=1}^n\right)$ acts on F^n by

$$\Phi^{(n)}\left([a_{ij}]_{i,j=1}^n\right)((\eta_i)_{i=1}^n) = \left(\sum_{j=1}^n \Phi\left(a_{ij}\right)\eta_j\right)_{i=1}^n.$$

Let $\xi_1, ..., \xi_n \in E$ and $\eta_1, ..., \eta_n \in F$. We denote by η the element in F^n whose the components are $\eta_1, ..., \eta_n$ and by X the matrix in $M_n(A)$ with (i,j)-entry $\left\langle \xi_i, \xi_j \right\rangle$. Then

$$\left\langle \sum_{i=1}^n \xi_i \otimes \eta_i, \sum_{i=1}^n \xi_i \otimes \eta_i \right\rangle = \sum_{i,j=1}^n \left\langle \eta_i, \Phi\left(\left\langle \xi_i, \xi_j \right\rangle\right)\eta_j \right\rangle = \left\langle \eta, \Phi^{(n)}(X)\eta \right\rangle \geq 0$$

since X is a positive element in $M_n(A)$.

Lemma 4.2.1 *Let A, B, E, F and Φ be as above. Then*

$$N_\Phi = \{\zeta \in E \otimes_{alg} F; \langle \zeta, \zeta \rangle_\Phi^0 = 0\}.$$

Proof. Let $\xi \in E, \eta \in F$ and $a \in A$. Then

$$\begin{gathered}\langle \xi a \otimes \eta - \xi \otimes \Phi\left(a\right)\eta, \xi a \otimes \eta - \xi \otimes \Phi\left(a\right)\eta \rangle_\Phi^0 = \\ \langle \eta, \Phi\left(\langle \xi a, \xi a \rangle\right)\eta \rangle - \langle \eta, \Phi\left(\langle \xi a, \xi \rangle\right)\Phi\left(a\right)\eta \rangle - \langle \Phi\left(a\right)\eta, \Phi\left(\langle \xi, \xi a \rangle\right)\eta \rangle + \\ \langle \Phi\left(a\right)\eta, \Phi\left(\langle \xi, \xi \rangle\right)\Phi\left(a\right)\eta \rangle = 0.\end{gathered}$$

So $N_\Phi \subseteq \{\zeta \in E \otimes_{\mathrm{alg}} F; \langle \zeta, \zeta \rangle_\Phi^0 = 0\}$. Let $\zeta = \sum\limits_{i=1}^n \xi_i \otimes \eta_i \in E \otimes_{\mathrm{alg}} F$ such that $\langle \zeta, \zeta \rangle_\Phi^0 = 0$. Then, since

$$\langle \zeta, \zeta \rangle_\Phi^0 = \left\langle \eta, \Phi^{(n)}(X)\eta \right\rangle,$$

where η is the element in F^n with the components $\eta_1, ..., \eta_n$ and X is the matrix in $M_n(A)$ with (i,j)-entry $\left\langle \xi_i, \xi_j \right\rangle$, $\Phi^{(n)}(X^{\frac{1}{2}})\eta = 0$ and so $\Phi^{(n)}(X^{\frac{1}{4}})\eta = 0$.

If ξ is the element in E^n with the components $\xi_1, ..., \xi_n$, then in the Hilbert $M_n(A)$ -module $L_A(E^n)$, $\langle \xi, \xi \rangle = X$ and according to Corollary 1.3.11, there is an element $\widetilde{\xi}$ in E^n with the components $\widetilde{\xi_1}, ..., \widetilde{\xi_n}$ such that $\xi = \widetilde{\xi} X^{\frac{1}{4}}$. Suppose that $X^{\frac{1}{4}} = [c_{ij}]_{i,j=1}^n$. Then, for each $j \in \{1, ..., n\}$,

$$\xi_j = \sum_{i=1}^n \widetilde{\xi}_i c_{ij}$$

and for each $i \in \{1, ..., n\}$

$$\sum_{j=1}^n \Phi(c_{ij}) \eta_j.$$

Therefore $\zeta = \sum\limits_{i,j=1}^n \left(\widetilde{\xi}_i c_{ij} \otimes \eta_j - \widetilde{\xi}_i \otimes \Phi(c_{ij}) \eta_j \right)$ and the lemma is proved. ■

In the particular case when $F = B$, the above lemma was proved in [**38**], pp.181.

According to Lemma 4.2.1, we can define a B -valued inner-product $\langle \cdot, \cdot \rangle_\Phi$ on $E \otimes_A F$ by

$$\left\langle \zeta + N_\Phi, \zeta' + N_\Phi \right\rangle_\Phi = \left\langle \zeta, \zeta' \right\rangle_\Phi^0 .$$

Moreover, $E \otimes_A F$ equipped with this B -valued inner-product is a pre-Hilbert B -module. An element $\xi \otimes \eta + N_\Phi$ in $E \otimes_A F$ is denoted by $\xi \otimes_\Phi \eta$.

Definition 4.2.2 *The completion of the pre-Hilbert space $E \otimes_A F$ with respect to the topology induced by the inner-product defined above, denoted by $E \otimes_\Phi F$, is said to be the interior tensor product of E and F using Φ.*

For each $q \in S(B)$, the map $\Phi_q : A \to L_{B_q}(F_q)$ defined by $\Phi_q = (\pi_q)_* \circ \Phi$ is a continuous $*$ -morphism.

Let $q_1, q_2 \in S(B)$ with $q_1 \geq q_2$. Since

$$\begin{aligned}
\left\langle \xi \otimes \sigma_{q_1 q_2}^F(\eta), \xi \otimes \sigma_{q_1 q_2}^F(\eta) \right\rangle_{\Phi_{q_2}}^0 &= \left\langle \sigma_{q_1 q_2}^F(\eta), \Phi_{q_2}(\langle \xi, \xi \rangle) \sigma_{q_1 q_2}^F(\eta) \right\rangle \\
&= \left\langle \sigma_{q_1 q_2}^F(\eta), (\pi_{q_2})_* (\Phi(\langle \xi, \xi \rangle)) \sigma_{q_1 q_2}^F(\eta) \right\rangle \\
&= \left\langle \sigma_{q_1 q_2}^F(\eta), \sigma_{q_1 q_2}^F((\pi_{q_1})_* (\Phi(\langle \xi, \xi \rangle)) \eta) \right\rangle \\
&= \pi_{q_1 q_2}(\langle \eta, \Phi_{q_1}(\langle \xi, \xi \rangle) \eta \rangle) \\
&= \pi_{q_1 q_2}\left(\langle \xi \otimes \eta, \xi \otimes \eta \rangle_{\Phi_{q_1}}^0 \right)
\end{aligned}$$

for all $\xi \in E$ and $\eta \in F_{q_1}$, there is a unique linear map $\chi_{q_1 q_2} : E \otimes_{\mathrm{alg}} F_{q_1}/N_{\Phi_{q_1}} \to E \otimes_{\mathrm{alg}} F_{q_2}/N_{\Phi_{q_2}}$ such that by

$$\chi_{q_1 q_2}\left(\xi \otimes \eta + N_{\Phi_{q_1}}\right) = \xi \otimes \sigma^F_{q_1 q_2}(\eta) + N_{\Phi_{q_2}}$$

for all $\xi \in E$ and $\eta \in F_{q_1}$. Moreover, $\chi_{q_1 q_2}$ is continuous and it extends uniquely to a linear map, denoted also by $\chi_{q_1 q_2}$, from $E \otimes_{\Phi_{q_1}} F_{q_1}$ to $E \otimes_{\Phi_{q_2}} F_{q_2}$ such that

$$\chi_{q_1 q_2}\left(\xi \otimes_{\Phi_{q_1}} \eta\right) = \xi \otimes_{\Phi_{q_2}} \sigma^F_{q_1 q_2}(\eta)$$

for all $\xi \in E$ and $\eta \in F_{q_1}$.

Proposition 4.2.3 *Let A, B, E, F and Φ be as above. Then*

$$\left\{E \otimes_{\Phi_q} F_q;\ B_q; \chi_{q_1 q_2}; \pi_{q_1 q_2},\ q_1 \geq q_2,\ q_1, q_2 \in S(B)\right\}$$

is an inverse system of Hilbert C^*-modules, and the Hilbert B-modules $E \otimes_{\Phi} F$ and $\lim\limits_{\substack{\leftarrow \\ q}} \left(E \otimes_{\Phi_q} F_q\right)$ are isomorphic.

Proof. To show that $\left\{E \otimes_{\Phi_q} F_q;\ B_q;\ \psi_{q_1 q_2},\ q_1 \geq q_2,\ q_1, q_2 \in S(B)\right\}$ is an inverse system of Hilbert C^*-modules, let $q_1, q_2, q_3 \in S(A)$ such that $q_1 \geq q_2 \geq q_3$, $\xi, \xi' \in E$, $\eta, \eta' \in F_{q_1}$ and $b \in B_{q_1}$. Then:

(**a**) $\chi_{q_1 q_2}\left(\left(\xi \otimes_{\Phi_{q_1}} \eta\right) b\right) = \chi_{q_1 q_2}\left(\xi \otimes_{\Phi_{q_1}} \eta b\right) = \xi \otimes_{\Phi_{q_2}} \sigma^F_{q_1 q_2}(\eta b)$
$= \xi \otimes_{\Phi_{q_2}} \sigma^F_{q_1 q_2}(\eta) \pi_{q_1 q_2}(b) = \chi_{q_1 q_2}\left(\xi \otimes_{\Phi_{q_1}} \eta\right) \pi_{q_1 q_2}(b)$;

(**b**) $\left\langle \chi_{q_1 q_2}\left(\xi \otimes_{\Phi_{q_1}} \eta\right), \chi_{q_1 q_2}\left(\xi' \otimes_{\Phi_{q_1}} \eta'\right)\right\rangle_{\Phi_{q_2}} = \left\langle \sigma^F_{q_1 q_2}(\eta), \Phi_{q_2}\left(\left\langle \xi, \xi'\right\rangle\right) \sigma^F_{q_1 q_2}(\eta')\right\rangle$
$= \left\langle \sigma^F_{q_1 q_2}(\eta), \sigma^F_{q_1 q_2}\left(\Phi_{q_1}\left(\left\langle \xi, \xi'\right\rangle\right) \eta'\right)\right\rangle = \pi_{q_1 q_2}\left(\left\langle \eta, \Phi_{q_1}\left(\left\langle \xi, \xi'\right\rangle\right) \eta'\right\rangle\right)$
$= \pi_{q_1 q_2}\left(\left\langle \xi \otimes_{\Phi_{q_1}} \eta, \xi' \otimes_{\Phi_{q_1}} \eta'\right\rangle_{\Phi_{q_1}}\right)$;

(**c**) $\left(\chi_{q_2 q_3} \circ \chi_{q_1 q_2}\right)\left(\xi \otimes_{\Phi_{q_1}} \eta\right) = \chi_{q_2 q_3}\left(\xi \otimes_{\Phi_{q_2}} \sigma^F_{q_1 q_2}(\eta)\right) = \xi \otimes_{\Phi_{q_3}} \sigma^F_{q_1 q_3}(\eta)$
$= \chi_{q_1 q_3}\left(\xi \otimes_{\Phi_{q_1}} \eta\right)$;

(**d**) $\chi_{q_1 q_1}\left(\xi \otimes_{\Phi_{q_1}} \eta\right) = \xi \otimes_{\Phi_{q_1}} \sigma^F_{q_1 q_1}(\eta) = \xi \otimes_{\Phi_{q_1}} \eta$.

From these facts we conclude that $\{E \otimes_{\Phi_q} F_q;\ B_q;\ \psi_{q_1 q_2},\ q_1 \geq q_2,\ q_1, q_2 \in S(B)\}$ is an inverse system of Hilbert C^*-modules.

Let $q \in S(B)$. Since

$$\begin{aligned}
\left\langle \xi \otimes \sigma_q^F(\eta), \xi \otimes \sigma_q^F(\eta) \right\rangle_{\Phi_q}^0 &= \left\langle \sigma_q^F(\eta), \Phi_q\left(\langle \xi, \xi \rangle\right) \sigma_q^F(\eta) \right\rangle \\
&= \left\langle \sigma_q^F(\eta), \sigma_q^F\left(\Phi\left(\langle \xi, \xi \rangle\right) \eta\right) \right\rangle \\
&= \pi_q\left(\langle \eta, \Phi\left(\langle \xi, \xi \rangle\right) \eta \rangle\right) = \pi_q\left(\langle \xi \otimes \eta, \xi \otimes \eta \rangle_\Phi^0\right)
\end{aligned}$$

for all $\xi \in E$ and for all $\eta \in F$, there is a unique linear map U_q^0 from $E \otimes_A F$ to $E \otimes_A F_q$ such that $U_q^0\left(\xi \otimes_\Phi \eta\right) = \xi \otimes_{\Phi_q} \sigma_q^F(\eta)$ for all $\xi \in E$ and for all $\eta \in F$. Moreover, since $U_q^0\left(E \otimes_A F\right) = E \otimes_A F_q$ and $\left\| \xi \otimes_{\Phi_q} \sigma_q^F(\eta) \right\|_{E \otimes_A F_q} \leq \overline{q}_{E \otimes_A F}\left(\xi \otimes_\Phi \eta\right)$ for all $\xi \in E$ and for all $\eta \in F$, U_q^0 extends to a surjective linear map U_q^0 from $E \otimes_\Phi F$ to $E \otimes_{\Phi_q} F_q$. Then there is a surjective linear map U_q from $\left(E \otimes_\Phi F\right)_q$ to $E \otimes_{\Phi_q} F_q$ such that $U_q \circ \sigma_q^{E \otimes_\Phi F} = U_q^0$. It is not difficult to check that U_q is B_q -linear and since

$$\begin{aligned}
\left\langle U_q\left(\sigma_q^{E \otimes_\Phi F}(\xi \otimes_\Phi \eta)\right), U_q\left(\sigma_q^{E \otimes_\Phi F}(\xi \otimes_\Phi \eta)\right) \right\rangle_{\Phi_q} &= \pi_q\left(\langle \xi \otimes_\Phi \eta, \xi \otimes_\Phi \eta \rangle_\Phi\right) \\
&= \left\langle \sigma_q^{E \otimes_\Phi F}(\xi \otimes_\Phi \eta), \sigma_q^{E \otimes_\Phi F}(\xi \otimes_\Phi \eta) \right\rangle
\end{aligned}$$

for all $\xi \in E$ and $\eta \in F$. From these and Theorem 3.5 in [**29**], we conclude that U_q is an isomorphism of Hilbert C^* -modules from $\left(E \otimes_\Phi F\right)_q$ onto $E \otimes_{\Phi_q} F_q$.

Let $q_1, q_2 \in S(B)$ with $q_1 \geq q_2$, $\xi \in E$ and $\eta \in F$. Then

$$\begin{aligned}
\left(\chi_{q_1 q_2} \circ U_{q_1}\right)\left(\sigma_{q_1}^{E \otimes_\Phi F}(\xi \otimes_\Phi \eta)\right) &= \chi_{q_1 q_2}\left(\xi \otimes_{\Phi_{q_1}} \sigma_{q_1}^F(\eta)\right) = \xi \otimes_{\Phi_{q_2}} \sigma_{q_2}^F(\eta) \\
&= U_{q_2}\left(\sigma_{q_2}^{E \otimes_\Phi F}(\xi \otimes_\Phi \eta)\right) \\
&= \left(U_{q_2} \circ \sigma_{q_1 q_2}^{E \otimes_\Phi F}\right)\left(\sigma_{q_1}^{E \otimes_\Phi F}(\xi \otimes_\Phi \eta)\right).
\end{aligned}$$

Therefore $\left(U_q\right)_{q \in S(B)}$ is an inverse system of isomorphism of Hilbert C^* -modules. Let $U = \lim\limits_{\leftarrow q} U_q$. Then U is an isomorphism of Hilbert B -modules from $E \otimes_\Phi F$ onto $\lim\limits_{\leftarrow q}\left(E \otimes_{\Phi_q} F_q\right)$. ■

Proposition 4.2.4 *Let A and B be locally C^*-algebras, let E be a Hilbert A-module, let F be a Hilbert B-module and let $\Phi : A \to L_B(F)$ be a continuous*

$*$-morphism such that $\Phi(A)F$ is dense in F. Then the Hilbert B-modules $H_A \otimes_\Phi F$ and $H \otimes F$, where H is a separable infinite dimensional Hilbert space (as well as $A \otimes_\Phi F$ and F) are unitarily equivalent.

Proof. Let H be a separable infinite dimensional Hilbert space and let $\{\varepsilon_n; n = 1, 2, ...\}$ be an orthonormal bases of H. By Corollary 4.1.4, the Hilbert A-modules H_A and $H \otimes A$ are isomorphic.

The proof is partition in two steps.

Step1. We suppose that B is a C^*-algebra.

The continuity of Φ implies that there is a continuous $*$ -morphism Ψ_p from A_p to $L_B(F)$ such that $\Psi_p \circ \pi_p = \Phi$. Since π_p is surjective, $\Psi_p(A_p)F$ is dense in F. Then, the Hilbert C^*-modules $H_{A_p} \otimes_{\Psi_p} F$ and $H \otimes F$ are isomorphic as well as the Hilbert C^* -modules $A_p \otimes_{\Psi_p} F$ and F (see, for instance, [**29**] pp. 41-42). Moreover, an isomorphism from $H_{A_p} \otimes_{\Psi_p} F$ onto $H \otimes F$ is given by $(\varepsilon_n \otimes a) \otimes_{\Psi_p} \eta \to \varepsilon_n \otimes \Psi_p(a)\eta$ and an isomorphism from $A_p \otimes_{\Psi_p} F$ onto F is given by $a \otimes_{\Psi_p} \eta \to \Psi_p(a)\eta$.

On the other hand, we know that the Hilbert C^*-modules $H_A \otimes_\Phi F$ and $H_{A_p} \otimes_{\Psi_p} F$ are isomorphic as well as the Hilbert C^*-modules $A \otimes_\Phi F$ and $A_p \otimes_{\Psi_p} F$ (see the proof of the Proposition 4.2.3). Moreover, the isomorphism U between $H_A \otimes_\Phi F$ and $H \otimes F$ is defined by $U\left((\varepsilon_n \otimes a) \otimes_\Phi \eta\right) = \varepsilon_n \otimes \Phi(a)\eta$ and the isomorphism V between $A \otimes_\Phi F$ and F is defined by $V(a \otimes_\Phi \eta) = \Phi(a)\eta$. Therefore the proposition is proved in this case.

Step 2. Now we suppose that B is an arbitrary locally C^*-algebra.

For each $q \in S(B)$, $\Phi_q(A)F_q$ is dense in F_q, where Φ_q is a continuous $*$-morphism from A into $L_{B_q}(F_q)$ defined by $\Phi_q = (\pi_q)_* \circ \Phi$, since $\Phi_q(A)F_q = (\pi_q)_* (\Phi(A))\ F_q = \sigma_q^F (\Phi(A) F)$ and $\Phi(A)F$ is dense in F. Then, according to the first step of the proof, the Hilbert C^*-modules $H_A \otimes_{\Phi_q} F_q$ and $H \otimes F_q$ are isomorphic as well as the Hilbert C^*-modules $A \otimes_{\Phi_q} F_q$ and F_q, and moreover, the linear map U_q from $H_A \otimes_{\Phi_q} F_q$ to $H \otimes F_q$ defined by

$$U_q\left((\varepsilon_n \otimes a) \otimes_{\Phi_q} \eta\right) = \varepsilon_n \otimes \Phi_q(a)\eta$$

is an isomorphism of Hilbert C^* -modules as well as the linear map V_q from $A \otimes_{\Phi_q} F_q$ to F_q defined by

$$V_q(a \otimes_{\Phi_q} \eta) = \Phi_q(a)\eta.$$

Let $q_1, q_2 \in S(B)$ with $q_1 \geq q_2$. Then

$$\begin{aligned}
\left(\sigma_{q_1q_2}^{H\otimes F} \circ U_{q_1}\right)\left((\varepsilon_n \otimes a) \otimes_{\Phi_{q_1}} \sigma_{q_1}^F(\eta)\right) &= \sigma_{q_1q_2}^{H\otimes F}\left(\varepsilon_n \otimes \Phi_{q_1}(a)\sigma_{q_1}^F(\eta)\right) \\
&= \varepsilon_n \otimes \sigma_{q_1q_2}^F\left(\Phi_{q_1}(a)\sigma_{q_1}^F(\eta)\right) \\
&= \varepsilon_n \otimes \Phi_{q_2}(a)\sigma_{q_2}^F(\eta) \\
&= U_{q_2}\left((\varepsilon_n \otimes a) \otimes_{\Phi_{q_2}} \sigma_{q_2}^F(\eta)\right) \\
&= \left(U_{q_2} \circ \sigma_{q_1q_2}^{H_A\otimes_\Phi F}\right)\left((\varepsilon_n \otimes a) \otimes_{\Phi_{q1}} \sigma_{q_1}^F(\eta)\right)
\end{aligned}$$

and

$$\begin{aligned}
\left(\sigma_{q_1q_2}^F \circ V_{q_1}\right)\left(a \otimes_{\Phi_{q_1}} \sigma_{q_1}^F(\eta)\right) &= \sigma_{q_1q_2}^F\left(\Phi_{q_1}(a)\sigma_{q_1}^F(\eta)\right) = \Phi_{q_2}(a)\sigma_{q_2}^F(\eta) \\
&= V_{q_2}\left(a \otimes_{\Phi_{q_2}} \sigma_{q_2}^F(\eta)\right) \\
&= \left(V_{q_2} \circ \sigma_{q_1q_2}^{A\otimes_\Phi F}\right)\left(a \otimes_{\Phi_{q1}} \sigma_{q_1}^F(\eta)\right)
\end{aligned}$$

for all $a \in A$, for all $\eta \in F$ and for all positive integer n. Therefore $(U_q)_q$ is an inverse system of isomorphisms of Hilbert C^* -modules as well as $(V_q)_q$. Then the Hilbert B -modules $\lim\limits_{\substack{\leftarrow \\ q}}\left(H_A \otimes_{\Phi_q} F_q\right)$ and $\lim\limits_{\substack{\leftarrow \\ q}}\left(H \otimes F_q\right)$ are isomorphic as well as the Hilbert B -modules $\lim\limits_{\substack{\leftarrow \\ q}}\left(A \otimes_{\Phi_q} F_q\right)$ and $\lim\limits_{\substack{\leftarrow \\ q}} F_q$. From these facts and Propositions 4.2.3 and 1.3.10, we conclude that the Hilbert B -modules $H_A \otimes_\Phi F$ and H_E are isomorphic as well as the Hilbert B -modules $A \otimes_\Phi F$ and F. ■

Remark 4.2.5 *Putting $F = B$ in Proposition 4.2.4 and using Corollary 4.1.4, we deduce that the Hilbert B-modules $H_A \otimes_\Phi B$ and H_B are isomorphic as well as the Hilbert B -modules $A \otimes_\Phi B$ and B.*

Remark 4.2.6 *If Φ is a continuous $*$ -morphism from A to $L_B(F)$, then $\Phi\left(b(A)\right) \subseteq b(L_B(F))$, and since the C^*-algebras $b(L_B(F))$ and $L_{b(B)}(b(F))$ are isomorphic,* *we can regard the restriction $\Phi|_{b(A)}$ of Φ on $b(A)$ as a $*$ -morphism from $b(A)$ to $L_{b(B)}(b(F))$. In general, the Hilbert $b(B)$ -modules $b\left(E \otimes_\Phi F\right)$ and $b(E) \otimes_{\Phi|_{b(A)}} b(F)$, where E is a Hilbert A -module, are not isomorphic.*

Example 4.2.7 *Let $A = C_{cc}([0,1])$, $B = C\left(\mathbb{Z}^+\right)$ and let Φ be a unital continuous $*$ -morphism from A to B. Then $\Phi\left(A\right)B = B$ and by Remark 4.2.5, the Hilbert B -modules $H_A \otimes_\Phi B$ and H_B are isomorphic. From this and Corollary 2.5.7, we conclude that the Hilbert $b(B)$ -modules $b\left(H_A \otimes_\Phi B\right)$ and $b(H_B)$ are isomorphic.*

Suppose that the Hilbert $b(B)$ -modules $b\left(H_A \otimes_\Phi B\right)$ and $b(H_A) \otimes_{\Phi|_{b(A)}} b(B)$ are isomorphic. But, by Example 1.3.6, the Hilbert $b(A)$-modules $b(H_A)$ and $H_{b(A)}$ coincides and by Remark 4.2.5, the Hilbert $b(B)$ -modules $H_{b(A)} \otimes_{\Phi|_{b(A)}} b(B)$ and $H_{b(B)}$ are isomorphic. Therefore the Hilbert $b(B)$ -modules $b\left(H_A \otimes_\Phi B\right)$ and $H_{b(B)}$ are isomorphic.

From these facts, we conclude that the Hilbert $b(B)$ -modules $H_{b(B)}$ and $b(H_B)$ are isomorphic, a contradiction (Example 1.3.5). Therefore the Hilbert $b(B)$ -modules $b\left(H_A \otimes_\Phi B\right)$ and $b(H_A) \otimes_{\Phi|_{b(A)}} b(B)$ are not isomorphic.

Example 4.2.8 *Let A and B be two locally C^*-algebras with A unital, let F be a Hilbert B -module and let Φ be a unital continuous $*$ -morphism from A to $L_B(F)$. Then, clearly $\Phi(A)F$ is dense in F and $\Phi|_{b(A)}(b(A))b(F)$ is dense in $b(F)$. By Remark 4.2.5, the Hilbert B -modules $A \otimes_\Phi F$ and F are isomorphic as well as the Hilbert $b(B)$ -modules $b(A) \otimes_{\Phi|_{b(A)}} b(F)$ and $b(F)$. Since the Hilbert B -modules $A \otimes_\Phi F$ and F are isomorphic, by Corollary 2.5.7, the Hilbert $b(B)$ -modules $b\left(A \otimes_\Phi F\right)$ and $b(F)$ are isomorphic. Therefore, the Hilbert $b(B)$ -modules $b\left(A \otimes_\Phi F\right)$ and $b(A) \otimes_{\Phi|_{b(A)}} b(F)$ are isomorphic.*

References for Section 4.2: **[20], [29],[38].**

4.3 Operators on tensor products of Hilbert modules

In this Section, by analogy with the case of Hilbert C^*-modules, we study the relation between the locally C^*-algebras $L_A(E) \otimes L_B(F)$ and $L_A(E) \otimes L_B(F)$

respectively $K_A(E) \otimes K_B(F)$ and $K_{A\otimes B}(E \otimes F)$. Also we study the relation between the locally C^*-algebras $L_A(E)$ and $L_B(E \otimes_\Phi F)$, respectively $K_A(E)$ to $K_B(E \otimes_\Phi F)$.

Proposition 4.3.1 *Let A and B be locally C^*-algebras, let E be a Hilbert A-module and let F be a Hilbert B-module. Then the locally C^*-algebras $L_{A\otimes B}(E\otimes F)$ and $\lim\limits_{\substack{\leftarrow \\ (p,q)}} L_{A_p\otimes B_q}(E_p\otimes F_q)$ as well as $K_{A\otimes B}(E\otimes F)$ and $\lim\limits_{\substack{\leftarrow \\ (p,q)}} K_{A_p\otimes B_q}(E_p\otimes F_q)$ are isomorphic.*

Proof. By Corollaries 2.2.8 and 2.3.5, the locally C^*-algebras $L_{A\otimes B}(E \otimes F)$ and $\lim\limits_{\substack{\leftarrow \\ (p,q)}} L_{A_p\otimes B_q}((E \otimes F)_{(p,q)})$ are isomorphic as well as the locally C^*-algebras $K_{A\otimes B}(E \otimes F)$ and $\lim\limits_{\substack{\leftarrow \\ (p,q)}} K_{A_p\otimes B_q}((E \otimes F)_{(p,q)})$. From these facts, Propositions 4.1.2, 2.2.7 and 2.3.4, we conclude that the locally C^* -algebras $L_{A\otimes B}(E \otimes F)$ and $\lim\limits_{\substack{\leftarrow \\ (p,q)}} L_{A_p\otimes B_q}(E_p \otimes F_q)$ as well as $K_{A\otimes B}(E \otimes F)$ and $\lim\limits_{\substack{\leftarrow \\ (p,q)}} K_{A_p\otimes B_q}(E_p \otimes F_q)$ are isomorphic. ■

Proposition 4.3.2 *Let A and B be locally C^*-algebras, let E be a Hilbert A-module and let F be a Hilbert B-module. Then there is a continuous $*$ -morphism j from $L_A(E) \otimes L_B(F)$ into $L_{A\otimes B}(E \otimes F)$ such that*

$$j(T \otimes S)(\xi \otimes \eta) = T\xi \otimes S\eta,\ T \in L_A(E),\ S \in L_B(F),\ \xi \in E,\ \eta \in F.$$

Moreover, j is injective and $j(K_A(E)\otimes K_B(F)) = K_{A\otimes B}(E \otimes F)$.

Proof. Let $p \in S(A)$ and $q \in S(B)$. Then, since A_p and B_q are C^*-algebras, E_p is a Hilbert A_p-module and F_q is a Hilbert B_q-module, there is an injective morphism of C^*-algebras $j_{(p,q)}$ from $L_{A_p}(E_p)\otimes L_{B_q}(F_q)$ to $L_{A_p\otimes B_q}(E_p\otimes F_q)$ such that

$$j_{(p,q)}(T_p \otimes S_q)(\xi_p \otimes \eta_q) = T_p\xi_p \otimes S_q\eta_q$$

for all $T_p \in L_{A_p}(E_p),\ S_q \in L_{B_q}(F_q),\ \xi_p \in E_p,\ \eta_q \in F_q$ and

$$j_{(p,q)}\left(K_{A_p}(E_p) \otimes K_{B_q}(F_q)\right) = K_{A_p\otimes B_q}(E_p \otimes F_q)$$

(see, for instance, [**29**] pp. 35-37).

Since

$$\left(\left(j_{(p_2,q_2)}\circ((\pi_{p_1p_2})_*\otimes(\pi_{q_1q_2})_*)\right)(T_{p_1}\otimes S_{q_1})\right)(\sigma^E_{p_2}(\xi)\otimes\sigma^F_{q_2}(\eta))$$

$$\begin{aligned} &= (\pi_{p_1p_2})_*(T_{p_1})\left(\sigma^E_{p_2}(\xi)\right)\otimes(\pi_{q_1q_2})_*(S_{q_1})\left(\sigma^F_{q_2}(\eta)\right)\\ &= \sigma^E_{p_1p_2}\left(T_{p_1}\left(\sigma^E_{p1}(\xi)\right)\right)\otimes\sigma^F_{q_1q_2}\left(S_{q_1}\left(\sigma^F_{q1}(\eta)\right)\right)\end{aligned}$$

and

$$\left(\left(\left(\pi_{(p_1,q_1)(p_2,q_2)}\right)_*\circ j_{(p_1,q_1)}\right)(T_{p_1}\otimes S_{q_1})\right)(\sigma^E_{p_2}(\xi)\otimes\sigma^F_{q_2}(\eta))$$

$$\begin{aligned} &= \sigma^{E\otimes F}_{(p_1,q_1)(p_2,q_2)}\left(j_{(p_1,q_1)}\left((T_{p_1}\otimes S_{q_1})\right)(\sigma^E_{p_1}(\xi)\otimes\sigma^F_{q_1}(\eta))\right)\\ &= \sigma^{E\otimes F}_{(p_1,q_1)(p_2,q_2)}\left(T_{p_1}\sigma^E_{p_1}(\xi)\otimes S_{q_1}\sigma^F_{q_1}(\eta)\right)\\ &= \sigma^E_{p_1p_2}\left(T_{p_1}\left(\sigma^E_{p1}(\xi)\right)\right)\otimes\sigma^F_{q_1q_2}\left(S_{q_1}\left(\sigma^F_{q1}(\eta)\right)\right)\end{aligned}$$

for all $T_{p_1}\in L_{A_{p_1}}(E_{p_1})$, for all $S_{q_1}\in L_{B_{q_1}}(F_{q_1})$, for all $\xi\in E$, for all $\eta\in F$ and for all $p_1,p_2\in S(A)$ with $p_1\geq p_2$ and $q_1,q_2\in S(B)$ with $q_1\geq q_2$, $\left(j_{(p,q)}\right)_{(p,q)\in S(A)\times S(B)}$ is an inverse system of injective morphisms of C^* -algebras. Moreover, $\left(j_{(p,q)}|_{K_{A_P}(E_p)\otimes K_{B_q}(F_q)}\right)_{(p,q)\in S(A)\times S(B)}$ is an inverse system of isomorphisms of C^*-algebras. Let $j=\lim\limits_{\substack{\leftarrow\\(p,q)}} j_{(p,q)}$. Then j is an injective continuous $*$ -morphism from $\lim\limits_{\substack{\leftarrow\\(p,q)}} L_{A_p}(E_p)\otimes L_{B_q}(F_q)$ to $\lim\limits_{\substack{\leftarrow\\(p,q)}} L_{A_p\otimes B_p}(E_p\otimes F_q)$, and since the locally C^* -algebras $\lim\limits_{\substack{\leftarrow\\(p,q)}} L_{A_p}(E_p)\otimes L_{B_q}(F_q)$ and $L_A(E)\otimes L_B(F)$ can be identified up to an isomorphism (Corollary 4.11, [**8**]) as well as the locally C^*-algebras $\lim\limits_{\substack{\leftarrow\\(p,q)}} L_{A_p\otimes B_p}(E_p\otimes F_q)$ and $L_{A\otimes B}(E\otimes F)$ (Proposition 4.3.1), j can be regarded as an injective continuous $*$ -morphism from $L_A(E)\otimes L_B(F)$ to $L_{A\otimes B}(E\otimes F)$ such that

$$j(T\otimes S)(\xi\otimes\eta)=T\xi\otimes S\eta,\ T\in L_A(E),\ S\in L_B(F),\ \xi\in E,\ \eta\in F.$$

Moreover, since $j|_{\lim\limits_{\substack{\leftarrow\\(p,q)}} K_{A_p\otimes B_q}(E_p\otimes F_q)}=\lim\limits_{\substack{\leftarrow\\(p,q)}} j_{(p,q)}|_{K_{A_P}(E_p)\otimes K_{B_q}(F_q)}$, and since the locally C^* -algebras $\lim\limits_{\substack{\leftarrow\\(p,q)}} K_{A_p}(E_p)\otimes L_{B_q}(F_q)$ and $K_A(E)\otimes L_B(F)$ can be identified up to an isomorphism (Corollary 4.11, [**8**]) as well as the locally C^*-algebras $\lim\limits_{\substack{\leftarrow \\ (p,q)}} K_{A_p \otimes B_p}(E_p \otimes F_q)$ and $K_{A\otimes B}(E \otimes F)$ (Proposition 4.3.1), we have $j(K_A(E) \otimes K_B(F)) = K_{A\otimes B}(E \otimes F)$. ■

Proposition 4.3.3 *Let E be a Hilbert module over the locally C^*-algebra A, let F be a Hilbert module over the locally C^* -algebra B, and let Φ be a morphism of locally C^* -algebras from A to $L_B(F)$. Then the locally C^*-algebras $L_B\left(E \otimes_\Phi F\right)$ and $\lim\limits_{\substack{\leftarrow \\ q}} L_{B_q}\left(E \otimes_{\Phi_q} F_q\right)$ as well as $K_B\left(E \otimes_\Phi F\right)$ and $\lim\limits_{\substack{\leftarrow \\ q}} K_{B_q}\left(E \otimes_{\Phi_q} F_q\right)$ are isomorphic.*

Proof. By Corollaries 2.2.8 and 2.3.5, the locally C^*-algebras $L_A(E \otimes_\Phi F)$ and $\lim\limits_{\substack{\leftarrow \\ q}} L_{B_q}(\left(E \otimes_\phi F\right)_q)$ are isomorphic as well as the locally C^*-algebras $K_A(E \otimes_\Phi F)$ and $\lim\limits_{\substack{\leftarrow \\ q}} K_{B_q}((E \otimes_\Phi F)_p)$. From these facts, Propositions 4.2.2, 2.2.7 and 2.3.4, we conclude that the locally C^* -algebras $L_A(E \otimes_\Phi F)$ and $\lim\limits_{\substack{\leftarrow \\ q}} L_{B_q}(E \otimes_{\Phi_q} F_q)$ as well as $K_B(E \otimes_\Phi F)$ and $\lim\limits_{\substack{\leftarrow \\ q}} K_{B_q}(E \otimes_{\Phi_q} F_q)$ are isomorphic. ■

Proposition 4.3.4 *Let A and B be locally C^*-algebras, let E be a Hilbert A-module, let F be a Hilbert B-module and let $\Phi : A \to L_B(F)$ be a continuous $*$-morphism.*

1. *Then there is a continuous $*$-morphism $\Phi_* : L_A(E) \to L_B\left(E \otimes_\Phi F\right)$ such that*

$$\Phi_*(T)(\xi \otimes_\Phi \eta) = T(\xi) \otimes_\Phi \eta,\ \xi \in E,\ \eta \in F,\ T \in L_A(E).$$

Moreover, if Φ is injective, then Φ_* is injective.

2. *If $\Phi(A) \subseteq K_B(F)$, then $\Phi_*\left(K_A(E)\right) \subseteq K_B\left(E \otimes_\Phi F\right)$. Moreover, if $\Phi(A)$ is dense in $K_A(F)$, then $\Phi_*\left(K_A(E)\right)$is dense in $K_B\left(E \otimes_\Phi F\right)$.*

Proof. We partition the proof in two steps.

Step 1. We suppose that B is a C^* -algebra.

1. The continuity of Φ implies that there is a continuous $*$ -morphism Ψ_p from A_p to $L_B(F)$ such that $\Psi_p \circ \pi_p = \Phi$. Then, from the theory of Hilbert C^* -modules (see, for instance, [**29**]), there is a morphism of C^*-algebras $(\Psi_p)_*$from $L_{A_p}(E_p)$ to $L_B(E_p \otimes_{\Psi_p} F)$ such that $(\Psi_p)_*(T)\left(\sigma_p^E(\xi) \otimes_{\Psi_p} \eta\right) = T\left(\sigma_p^E(\xi)\right) \otimes_{\Psi_p} \eta$ for all $\xi \in E$, $\eta \in F$ and $T \in L_{A_p}(E_p)$. We will show that the Hilbert C^* -modules $E \otimes_\Phi F$ and $E_p \otimes_{\Psi_p} F$ are isomorphic. For this, we define a linear map U from $E \otimes_\Phi F$ to $E_p \otimes_{\Psi_p} F$ by $U(\xi \otimes_\Phi \eta) = \sigma_p^E(\xi) \otimes_{\Psi_p} \eta$. Clearly, U is B -linear and $U(E \otimes_\Phi F) = E_p \otimes_{\Psi_p} F$. Since

$$\begin{aligned}
\left\langle \sigma_p^E(\xi) \otimes_{\Psi_p} \eta, \sigma_p^E(\xi) \otimes_{\Psi_p} \eta \right\rangle_{\Psi_p} &= \left\langle \eta, \Psi_p\left(\left\langle \sigma_p^E(\xi), \sigma_p^E(\xi)\right\rangle \eta\right)\right\rangle \\
&= \left\langle \eta, (\Psi_p \circ \pi_p)(\langle \xi, \xi\rangle \eta)\right\rangle \\
&= \langle \eta, \Phi(\langle \xi, \xi\rangle)\eta\rangle = \langle \xi \otimes_\Phi \eta, \xi \otimes_\Phi \eta\rangle_\Phi
\end{aligned}$$

for all $\xi \in E$ and for all $\eta \in F$, U extends to an isometric, surjective linear B -map U from $E \otimes_\Phi F$ onto $E_p \otimes_{\Psi_p} F$, and by Theorem 3.5 in [**29**], the Hilbert B -modules $E \otimes_\Phi F$ and $E_p \otimes_{\Psi_p} F$ are isomorphic.

We consider the map Φ_* from $L_A(E)$ to $L_B(E \otimes_\Phi F)$ defined by $\Phi_*(T) = U^*(\Psi_p)_*\left((\pi_p)_*(T)\right)U$. Clearly, Φ_* is a morphism of locally C^*-algebras. Moreover,

$$\begin{aligned}
\Phi_*(T)(\xi \otimes_\Phi \eta) &= U^*(\Psi_p)_*\left((\pi_p)_*(T)\right)\left(\sigma_p^E(\xi) \otimes_{\Psi_p} \eta\right) \\
&= U^*\left((\pi_p)_*(T)\sigma_p^E(\xi) \otimes_{\Psi_p} \eta\right) \\
&= U^*\left(\sigma_p^E(T\xi) \otimes_{\Psi_p} \eta\right) = T\xi \otimes_\Phi \eta
\end{aligned}$$

for all $\xi \in E$, for all $\eta \in F$ and for all $T \in L_A(E)$.

Suppose that Φ is injective. Let $T \in L_A(E)$ such that $\Phi_*(T) = 0$. Then $T\xi \otimes_\Phi \eta = 0$ for all $\xi \in E$ and for all $\eta \in F$ and so

$$0 = \left\langle T\xi \otimes_\Phi \eta, T\xi \otimes_\Phi \eta' \right\rangle_\Phi = \left\langle \eta, \Phi(\langle T\xi, T\xi\rangle)\eta' \right\rangle$$

for all $\xi \in E$ and for all $\eta, \eta' \in F$. This implies that $\Phi(\langle T\xi, T\xi\rangle) = 0$ for all for all $\xi \in E$, and since Φ is injective, T is the null operator.

2. If $\Phi(A) \subseteq K_B(F)$, then $\Psi_p(A_p) \subseteq K_B(F)$ and according to Proposition 4.7 of [**29**], $(\Psi_p)_*\left(K_{A_p}(E_p)\right) \subseteq K_B(E_p \otimes_{\Psi_p} F)$. Since $(\pi_p)_*(K_A(E)) = K_{A_p}(E_p)$, we have

$$\begin{aligned}\Phi_*(K_A(E)) &= U^*(\Psi_p)_*\left((\pi_p)_*(K_A(E))\right)U \\ &\subseteq U^* K_B(E_p \otimes_{\Psi_p} F)U = K_B(E \otimes_\Phi F).\end{aligned}$$

If $\Phi(A)$ is dense in $K_A(F)$, then $\Psi_p(A_p) = K_B(F)$ and according to Proposition 4.7 of [**29**], $(\Psi_p)_*\left(K_{A_p}(E_p)\right) = K_B(E_p \otimes_{\Psi_p} F)$. Then

$$\begin{aligned}\Phi_*(K_A(E)) &= U^*(\Psi_p)_*\left((\pi_p)_*(K_A(E))\right)U \\ &= U^* K_B(E_p \otimes_{\Psi_p} F)U = K_B(E \otimes_\Phi F).\end{aligned}$$

Step 2. Now we suppose that B is an arbitrary locally C^* -algebra.

1. For each $q \in S(B)$, the map Φ_q from A to $L_{B_q}(F_q)$ defined by $\Phi_q = (\pi_q)_* \circ \Phi$ is a continuous $*$ -morphism, and by the first step of the proof, there is a continuous $*$ -morphism $(\Phi_q)_*$ from $L_A(E)$ to $L_{B_q}(E \otimes_{\Phi_q} F_q)$ such that

$$(\Phi_q)_*(T)\left(\xi \otimes_{\Phi_q} \sigma_q^F(\eta)\right) = T(\xi) \otimes_{\Phi_q} \sigma_q^F(\eta)$$

for all $\xi \in E$ for all $\eta \in F$ and for all $T \in L_A(E)$.

Let Ψ_q be the map from $L_A(E)$ to $L_{B_q}((E \otimes_\Phi F)_q)$ defined by $\Psi_q(T) = U_q^*(\Phi_q)_*(T)U_q$, where U_q is the isomorphism of Hilbert B_q -modules from $(E \otimes_\Phi F)_q$ onto $E \otimes_{\Phi_q} F_q$ defined in the proof of Proposition 4.2.3, and let $q_1, q_2 \in S(B)$ with $q_1 \geq q_2$. Since

$$\begin{aligned}\left(\left((\pi_{q_1 q_2})_* \circ \Psi_{q_1}\right)(T)\right)\left(\sigma_{q_1}^{E \otimes_\Phi F}(\xi \otimes_\Phi \eta)\right) &= \sigma_{q_1 q_2}^{E \otimes_\Phi F}\left(\Psi_{q_1}(T)\left(\sigma_{q_1}^{E \otimes_\Phi F}(\xi \otimes_\Phi \eta)\right)\right) \\ &= \sigma_{q_1 q_2}^{E \otimes_\Phi F}\left(\sigma_{q_1}^{E \otimes_\Phi F}(T\xi \otimes_\Phi \eta)\right) \\ &= \sigma_{q_2}^{E \otimes_\Phi F}(T\xi \otimes_\Phi \eta) \\ &= \Psi_2(T)\left(\sigma_{q_1}^{E \otimes_\Phi F}(T\xi \otimes_\Phi \eta)\right)\end{aligned}$$

for all $T \in L_A(E)$ and for all $\xi \in E$ and $\eta \in F$, $(\pi_{q_1 q_2})_* \circ \Psi_{q_1} = \Psi_{q_2}$. Therefore there is a continuous $*$ -morphism Φ_* from $L_A(E)$ to $L_{B_q}(E \otimes_{\Phi_q} F_q)$ such that

$(\pi_q)_* \circ \Phi_* = \Psi_q$ for all $q \in S(B)$. Moreover, $\Phi_*(T)(\xi \otimes_\Phi \eta) = T\xi \otimes_\Phi \eta$, for all $\xi \in E$ for all $\eta \in F$ and for all $T \in L_A(E)$, since

$$\begin{aligned}\sigma_q^{E\otimes_\Phi F}\left(\Phi_*(T)(\xi \otimes_\Phi \eta)\right) &= (\pi_q)_*\left(\Phi_*(T)\sigma_q^{E\otimes_\Phi F}\left(\xi \otimes_\Phi \eta\right)\right) \\ &= \Psi_q\left(T\right)\left(\sigma_q^{E\otimes_\Phi F}\left(\xi \otimes_\Phi \eta\right)\right) \\ &= \sigma_q^{E\otimes_\Phi F}\left(T\xi \otimes_\Phi \eta\right).\end{aligned}$$

It is easy to check that if Φ is injective, then Φ_* is injective.

2. If $\Phi(A) \subseteq K_B(F)$, then $\Phi_q(A) \subseteq K_{B_q}(F_q)$ for each $q \in S(B)$, and according to the first part of the proof, $(\Phi_q)_*\left(K_A(E)\right) \subseteq K_{B_q}(E \otimes_{\Phi_q} F_q)$ and so $\Psi_q\left(K_A(E) \subseteq K_{B_q}((E \otimes_\Phi F)_q)\right.$. From these, since $(\pi_q)_* \circ \Phi_* = \Psi_q$ for each $q \in S(B)$, we conclude that $\Phi_*\left(K_A(E)\right) \subseteq K_B\left(E \otimes_\Phi F\right)$.

If $\Phi(A)$ is dense in $K_A(F)$, then for each $q \in S(B)$, $\Phi_q(A)$ is dense in $K_{B_q}(F_q)$ and according to the first half of this proof, $(\Phi_q)_*\left(K_A(E)\right)$ is dense in $K_{B_q}(E \otimes_{\Phi_q} F_q)$ and so $\Psi_q\left(K_A(E)\right.$ is dense in $K_{B_q}((E \otimes_\Phi F)_q)$. Thus we have

$$\begin{aligned}\overline{\Phi_*\left(K_A(E)\right)} &= \lim_{\substack{\leftarrow \\ q}} \overline{\Psi_q\left(K_A(E)\right)} \\ &= \lim_{\substack{\leftarrow \\ q}} K_{B_q}((E \otimes_\Phi F)_q) = K_B\left(E \otimes_\Phi F\right).\end{aligned}$$

■

In the case when B is a C^*-algebra and $F = B$, the above proposition was proved in [**38**], pp.184-185.

Corollary 4.3.5 *Let A and B be locally C^*-algebras, let E be a Hilbert A-module, let F be a Hilbert B-module and let $\Phi : A \to L_B(F)$ be a continuous $*$-morphism such that $\Phi(A) = K_B(F)$. If for each $q \in S(B)$ there is $p_q \in S(A)$ such that $\widetilde{q}(\Phi(a)) = p_q(a)$ for all $a \in A$ and if $\{p_q; q \in S(B)\}$ is a cofinal subset of $S(A)$, then $\Phi_*(K_A(E)) = K_B(E \otimes_\Phi F)$.*

Proof. According to Proposition 4.3.4, $\Phi_*(K_A(E))$ is dense in $K_B(E \otimes_\Phi F)$. To show that $\Phi_*(K_A(E))$ is closed, let $q \in S(A)$ and $p_q \in S(A)$ such that $\widetilde{q}(\Phi(a)) = p_q(a)$ for all $a \in A$. Then there is a continuous $*$ -morphism Φ_{p_q} :

$A_{p_q} \to L_{B_q}(F_q)$ such that $\Phi_{p_q} \circ \pi_{p_q} = (\pi_q)_* \circ \Phi$. Moreover, $\Phi_{p_q}\left(A_{p_q}\right) = K_{B_q}(F_q)$ and then according to Proposition 4.7 of [**29**], $\left\|\left(\Phi_{p_q}\right)_*(T)\right\| = \|T\|$ for all T in $K(E_{p_q})$. If U_{p_q} is the isomorphism of Hilbert B_q -module from E It is easy to verify that $\left(\Phi_{p_q}\right)_* \circ \left(\pi_{p_q}\right)_* = (\pi_q)_* \circ (\Phi)_*$. Then for each $T \in K_A(E)$ we have

$$\begin{aligned}\widetilde{q}\left((\Phi)_*(T)\right) &= \left\|(\pi_q)_*\left((\Phi)_*(T)\right)\right\| \ = \left\|\left(\Phi_{p_q}\right)_*\left(\left(\pi_{p_q}\right)_*(T)\right)\right\| \\ &= \left\|\left(\pi_{p_q}\right)_*(T)\right\| = \widetilde{p_q}(T).\end{aligned}$$

From this, since $\{p_q; q \in S(B)\}$ is a cofinal subset of $S(A)$, it follows that $\Phi_*($ $K_A(E))$ is closed. ■

References for Section 4.3: **[20], [29], [38].**

Chapter 5

Stabilisation

5.1 Full Hilbert modules

In this Section we characterize the full Hilbert modules over locally C^*-algebras and we show that, in general, a full Hilbert module E does not induce a structure of full Hilbert C^*- module on $b(E)$.

Let A be a locally C^*-algebra.

Definition 5.1.1 *A Hilbert A -module E is full if the closed two-sided $*$ -ideal $\langle E, E\rangle$ coincides with A.*

Remark 5.1.2 *Since A has an approximate unit, the Hilbert A -module A is full.*

Proposition 5.1.3 *Let A be a locally C^*-algebra and let E be a Hilbert A-module. Then E is full if and only if E_p is full for all $p \in S(A)$.*

Proof. First we suppose that E is full. Let $p \in S(A)$. Since $\sigma_p^E(E) = E_p$, and since $\left\langle \sigma_p^E(\xi), \sigma_p^E(\eta)\right\rangle = \pi_p\left(\langle \xi, \eta\rangle\right)$ for all $\xi, \eta \in E$, the closed vector subspace $\langle E_p, E_p\rangle$ of A_p coincides with $\pi_p\left(\langle E, E\rangle\right)$. But $\langle E, E\rangle = A$ and $\pi_p(A) = A_p$. Therefore E_p is full.

Conversely, if E_p is full for all $p \in S(A)$, then we have

$$\begin{aligned}\overline{\langle E, E\rangle} &= \lim_{\overleftarrow{p}} \overline{\pi_p\left(\langle E, E\rangle\right)} \\ &\quad \text{(cf.[\textbf{33}], Lemma III.3.2)} \\ &= \lim_{\overleftarrow{p}} \overline{\langle E_p, E_p\rangle} = \lim_{\overleftarrow{p}} A_p = A\end{aligned}$$

Therefore E is full. ■

Corollary 5.1.4 *For any locally C^*-algebras A, the Hilbert A-module H_A is full.*

Let E and F be Hilbert A -modules. It is not difficult to check that $K_A(E,F)$ is a pre-Hilbert $K_A(E)$ -module with the action of $K_A(E)$ on $K_A(E,F)$ defined by $(T,S) \to TS$, $T \in K_A(E,F)$ and $S \in K_A(E)$ and the inner-product defined by $\langle T,S\rangle = T^*S$, $T,S \in K_A(E,F)$. Since the topology on $K_A(E,F)$ induced by the inner-product is the same as the topology on $K_A(E,F)$ induced by the family of seminorms $\{\widetilde{p}\}_{p\in S(A)}$, $K_A(E,F)$ is a Hilbert $K_A(E)$ -module. A simple calculus shows that $\{K_{A_p}(E_p,F_p); (\pi_{pq})_*; K_{A_p}(E_p), p \geq q, p, q \in S(A)\}$ is an inverse system of Hilbert C^*-modules.

Let $p \in S(A)$. Define a linear map U_p from $(K_A(E,F))_p$ to $K_{A_p}(E_p,F_p)$ by

$$U_p(T + N_p^{K_A(E,F)}) = (\pi_p)_*(T)$$

$T \in K_A(E,F)$. It is not difficult to check that U_p is a unitary element in $L_{B_p}((K_A(E,\ F))_p, K_{A_p}(E_p,F_p))$, where $B_p = K_{A_p}(E_p)$. Also it is not difficult to check that

$$(\pi_{p_1p_2})_* \circ U_{p_1} = U_{p_2} \circ \sigma_{p_1p_2}^{K_A(E,F)}$$

for all $p_1, p_2 \in S(A)$ with $p_1 \geq p_2$. Therefore $(U_p)_p$ is an inverse system of isomorphisms of Hilbert C^*-modules. Let $U = \lim_{\overleftarrow{p}} U_p$. Then U is an isomorphism of Hilbert modules from $\lim_{\overleftarrow{p}} (K_A(E,F))_p$ onto $\lim_{\overleftarrow{p}} K_{A_p}(E_p,F_p))$, and so the Hilbert $K_A(E)$ -modules $K_A(E,F)$ and $\lim_{\overleftarrow{p}} K_{A_p}(E_p,F_p)$ are isomorphic.

Remark 5.1.5 *By Corollary 3.2.8, the locally C^*-algebras $L_{K_A(E)}(K_A(E,F))$ and $\lim\limits_{\leftarrow p} L_{B_p}(K_{A_p}(E_p, F_p))$ as well as $K_{K_A(E)}(K_A(E,F))$ and $\lim\limits_{\leftarrow p} K_{B_p}(K_{A_p}(E_p, F_p))$ (Corollary 3.3.5) are isomorphic.*

Remark 5.1.6 *Let E and F be two Hilbert A -modules such that F is full. Since F is full, $E\langle F, F\rangle$ is dense in E (Corollary 1.2.13) and so $K_A(E)$ is generated by the elements in the form $\theta_{\xi\langle\eta,\mu\rangle,\zeta}$, $\xi, \zeta \in E$, $\eta, \mu \in F$. From this fact and taking into account that $\theta_{\xi\langle\eta,\mu\rangle,\zeta} = \theta_{\xi,\mu}\theta_{\eta,\zeta}$ for all $\xi, \zeta \in E$ and for all $\eta, \mu \in F$, we conclude that $\langle K_A(E,F), K_A(E,F)\rangle$ coincides with $K_A(E)$. Therefore, if F is full, then the Hilbert $K_A(E)$ -module $K_A(E,F)$ is full.*

Remark 5.1.7 *If E is a Hilbert A -module such that $b(E)$ is a full Hilbert $b(A)$ -module, then, taking into account that $b(A)$ is dense in A and $b(E)$ is dense in E, we deduce that E is full. In general, the converse implication is not valid.*

Example 5.1.8 *Let $A = C(\mathbb{Z}^+)$ and $E = \prod\limits_n \mathbb{C}^n$. Then E is a Hilbert A-module (see, Example 3.4.10). It is not difficult to check that the Hilbert modules E and $b(E)$ are full.*

Example 5.1.9 *Let A and E as above and let $F = K_A(E,A)$. Then F is a full Hilbert $K_A(E)$ -module. We will show that $b(F)$ is not full.*

Suppose that $b(F)$ is full. Then the C^*-subalgebra of $b(K_A(E))$ generated by $b(K_A(E,A)^*)b(K_A(E,A))$ coincides with $b(K_A(E))$. On the other hand, since A is unital, $K_A(E,A) = L_A(E,A)$ and $L_{b(A)}(b(E), b(A)) = K_{b(A)}(b(E), b(A))$, and then by Theorem 3.4.2, the C^*-algebras $b(K_A(E,A))$ and $K_{b(A)}(b(E), b(A))$ are isomorphic. Therefore the C^* -subalgebra of $b(K_A(E))$ generated by $b(K_A(E,A)^*)$ $b(K_A(E,A))$ is isomorphic with $K_{b(A)}(b(E))$. From these facts, we conclude that the C^*-algebras $b(K_A(E))$ and $K_{b(A)}(b(E))$ are isomorphic, a contradiction (see Example 3.4.10). Therefore $b(F)$ is not full.

Remark 5.1.10 *Let E be a full Hilbert A -module and let F be a full Hilbert B -module. Then $E \otimes F$ is full, since the closed ideal of $A \otimes B$ generated by $\langle E \otimes F, E \otimes F\rangle$ coincides with the ideal of $A \otimes B$ generated by $\langle E, E\rangle \otimes_{alg} \langle F, F\rangle$.*

Remark 5.1.11 *Let E be a full Hilbert A -module, let F be a full Hilbert B -module and let $\Phi : A \to L_B(F)$ be a morphism of locally C^* -algebras such that $\Phi(A)F$ is dense in F. Then $E \otimes_\Phi F$ is full, since the closed ideal of B generated by $\langle E \otimes_\Phi F, E \otimes_\Phi F \rangle$ coincides with the ideal of B generated by $\langle F, \Phi(\langle E, E \rangle) F \rangle$.*

References for Section 5.1: **[21], [29].**

5.2 Countably generated Hilbert modules

In this Section we show that the Kasparov stabilisation theorem for countably Hilbert C^*-modules is valid in the context of Hilbert modules over locally C^*-algebras and we prove a criterion for a Hilbert module over a Fréchet locally C^*-algebra be countably generated. Also, we show that if E is a full countably Hilbert module over a separable Fréchet locally C^*-algebra, then the Hilbert A-modules H_A and H_E are unitarily equivalent.

Let A be a locally C^*-algebra.

Definition 5.2.1 *A Hilbert A -module E is countably generated if there is a countable subset G of E such that the Hilbert -submodule of E generated by G is exactly E.*

Lemma 5.2.2 *Let E be a countably generated Hilbert A -module. Then there is a countable subset G_0 of $b(E)$ such that G_0 is a generating set for E.*

Proof. Let $\{\xi_n; n = 1, 2, ...\}$ be a generating set for E. By Theorem 1.3.2 (2), for each positive integer n there is a sequence $\{\xi_{nm}\}_m$ of elements in $b(E)$ which converges to ξ_n. Let $G_0 = \{\xi_{nm}; n, m = 1, 2, ...\}$. Clearly, G_0 is a countable subset of $b(E)$ and $\{\xi_n; n = 1, 2, ...\}$ is a subset of the Hilbert A -module generated by G_0. Therefore G_0 is a generating set for E. ■

Remark 5.2.3 *If G is a generating set for the Hilbert A -module E, then $\sigma_p(G)$ is a generating set for E_p for each $p \in S(A)$. Therefore if E is a countably generated Hilbert A -module, then the Hilbert A_p-module E_p is countably generated for each $p \in S(A)$.*

Example 5.2.4 *If A has a countable approximate unit $\{u_n\}_n$, then the Hilbert A-module A is countably generated.*

*Indeed, let $h = \sum\limits_n 2^{-n}u_n$. Then $\pi_p(h)$ is a strictly positive element in A_p for each $p \in S(A)$, since $\{\pi_p(u_n)\}_n$ is a countable approximate unit for A_p (see, [**37**], Proposition 3.10.5), and by Lemma III 3.2, [**33**],*

$$\overline{hA} = \lim_{\substack{\leftarrow \\ p}} \overline{\pi_p(hA)} = \lim_{\substack{\leftarrow \\ p}} \overline{\pi_p(h)A_p} = \lim_{\substack{\leftarrow \\ p}} A_p = A.$$

Therefore $\{h\}$ is a generating set for A.

Example 5.2.5 *If A has a countable approximate unit $\{u_n\}_n$, then the Hilbert A-module H_A is countably generated.*

Example 5.2.6 *If E is a countably generated Hilbert A -module, then the Hilbert A-module H_E is countably generated.*

If A is a C^*-algebra, Kasparov [**26**] shoved that H_A is big enough to absorb any countably generated Hilbert A -module. This result is known as Kasparov's stabilization theorem, and it is also true in the context of Hilbert modules over locally C^*-algebras.

Theorem 5.2.7 *Let E be a countably generated Hilbert A -module. Then the Hilbert A -modules $E \oplus H_A$ and H_A are isomorphic.*

Proof. First we suppose that A is unital. Then $\{e_n; n = 1, 2, ...\}$, where e_n is an element in H_A whose all the components are zero except at the n^{th} component which is 1, is a generating set for H_A.

Let $\{\xi_n; n = 1, 2, ...\}$ be a generating set for E with each element repeated infinitely often. By Lemma 5.2.2, we can suppose that for each positive integer n, ξ_n is an element in $b(E)$ and $\|\xi_n\|_\infty \leq 1$. Clearly, $\{\xi_n \oplus e_n; n = 1, 2, ...\}$ is a generating set for $E \oplus H_A$.

Let $p \in S(A)$. Then $\{\sigma_p^E(\xi_n) \oplus \sigma_p^{H_A}(e_n); n = 1, 2, ...\}$ is a generating set for $E_p \oplus H_{A_p}$ and according to the proof of Kasparov's stabilization theorem for countably generated Hilbert C^*-modules (see, for example, [**35**]), the linear operator T_p from H_{A_p} to $E_p \oplus H_{A_p}$ defined by

$$T_p = \sum_n 2^{-n} \theta_{\sigma_p^E(\xi_n) \oplus 2^{-n} \sigma_p^{H_A}(e_n), \sigma_p^{H_A}(e_n)}$$

and its adjoint have dense range. It is not difficult to check that $(T_p)_p$ is a coherent sequence in $L_{A_p}(H_{A_p}, E_p \oplus H_{A_p})$. Let $T \in L_A(H_A, E \oplus H_A)$ such that $(\pi_p)_*(T) = T_p$ for each $p \in S(A)$. Then:

$$\begin{aligned} \overline{TH_A} &= \lim_{\stackrel{\leftarrow}{p}} \overline{(\sigma_p^E \oplus \sigma_p^{H_A})(TH_A)} = \lim_{\stackrel{\leftarrow}{p}} (\pi_p)_*(T)\left(\sigma_p^{H_A}(H_A)\right) \\ &= \lim_{\stackrel{\leftarrow}{p}} \overline{T_p(H_{A_p})} = \lim_{\stackrel{\leftarrow}{p}} E_p \oplus H_{A_p} = E \oplus H_A \end{aligned}$$

and

$$\begin{aligned} \overline{T^*(E \oplus H_A)} &= \lim_{\stackrel{\leftarrow}{p}} \overline{\sigma_p^{H_A}(T^*(E \oplus H_A))} \\ &= \lim_{\stackrel{\leftarrow}{p}} (\pi_p)_*(T^*)\left(\left(\sigma_p^E \oplus \sigma_p^{H_A}\right)(E \oplus H_A)\right) \\ &= \lim_{\stackrel{\leftarrow}{p}} \overline{T_p^*(E_p \oplus H_{A_p})} = \lim_{\stackrel{\leftarrow}{p}} H_{A_p} = H_A. \end{aligned}$$

Therefore T and T^* have dense range and by Corollary 3.3.5, the Hilbert A -modules H_A and $E \oplus H_A$ are isomorphic.

If A is not unital, let A^+ be the unitization of A. Since E can be regarded as Hilbert A^+ -module, according to the first part of the proof, there is a unitary operator U^+ from H_{A^+} onto $E \oplus H_{A^+}$. Clearly , the restriction U of U^+ on $\overline{H_{A^+}A}$ is a unitary operator from $\overline{H_{A^+}A}$ onto $\overline{(E \oplus H_{A^+})A}$. It is not difficult to check that the linear maps: $(a_n)_n a \rightarrow (a_n a)_n$ from $\overline{H_{A^+}A}$ to H_A and $(\xi \oplus (a_n)_n)\, a \rightarrow (\xi \oplus (a_n a)_n)$ from $\overline{(E \oplus H_{A^+})A}$ to $E \oplus H_A$ are unitary operators. From these facts we conclude that the Hilbert A -modules H_A and $E \oplus H_A$ are isomorphic. ■

For countably generated Hilbert modules over Fréchet locally C^*-algebras, this theorem was proved in [**38**], Theorem 5.12.

Remark 5.2.8 *If E is a Hilbert A-module such that the Hilbert $b(A)$-module $b(E)$ is countably generated, then E is countably generated.*

Corollary 5.2.9 *If A is a locally C^*-algebra such that $b(H_A) = H_{b(A)}$, and E is a countably generated Hilbert A-module, then the Hilbert $b(A)$-module $b(E)$ is countably generated.*

Proof. By Theorem 5.2.7, the Hilbert A-modules $E \oplus H_A$ and H_A are isomorphic. From this, using Corollary 2.5.7 and Remark 1.3.7, we conclude that the Hilbert $b(A)$-module $b(E) \oplus H_{b(A)}$ and $H_{b(A)}$ are isomorphic, and since $H_{b(A)}$ is countably generated, $b(E)$ is countably generated. ∎

It is known that a Hilbert C^*-module is countably generated if and only if the C^*-algebra $K_A(E)$ has a countable approximate unit (see, for example, [**29**], Proposition 6.7). This result can be extended in the context of Hilbert modules over Fréchet locally C^*-algebras.

Proposition 5.2.10 *Let A be a locally C^*-algebra and let E be a Hilbert A -module. Then:*

1. *If E is countably generated, then $K_A(E)$ has a countable approximate unit.*

2. *If A is a Fréchet locally C^*-algebra and if $K_A(E)$ has a countable approximate unit, then E is countably generated.*

Proof. 1. Suppose that A is unital. Then, for each $p \in S(A)$, H_{A_p} is countably generated and by [**29**], Proposition 6.7, $K_{A_p}(H_{A_p})$ has a countable approximate unit. Moreover, $\{U_n^p\}_n$, where $U_n^p = \left(\sum\limits_m 2^{-n}\theta_{\sigma_p^{H_A}(e_m),\sigma_p^{H_A}(e_m)}\right)^{\frac{1}{n}}$ is an approximate unit for $K_{A_p}(H_{A_p})$. It is not difficult to check that, for each positive integer n, $(U_n^p)_p$ is a coherent sequence in $K_{A_p}(H_{A_p})$. For each positive integer n, let $U_n = (U_n^p)_p$. Then $\{U_n\}_n$ is an approximate unit in $K_A(H_A)$. Since E is countably generated, by Theorem 5.2.7, there is an element P in $L_A(H_A, E)$ such that $PP^* = \mathrm{id}_E$. Then $\{PU_nP^*\}_n$ is an approximate unit for $K_A(E)$.

2. Let $\{V_n\}_n$ be a countable approximate unit of $K_A(E)$ and let $T = \sum\limits_n 2^{-n}$ V_n. For each $p \in S(A)$, $\{(\pi_p)_*(V_n)\}_n$ is a countable approximate unit for $K_{A_p}(E_p)$ and $(\pi_p)_*(T)$ is an element in $K_{A_p}(E_p)$ with dense range (see, for instance, [**12**]). Then

$$\begin{aligned}\overline{TE} &= \lim_{\overleftarrow{p}} \overline{\sigma_p^E(TE)} = \lim_{\overleftarrow{p}} \overline{(\pi_p)_*(T)(E_p)} \\ &= \lim_{\overleftarrow{p}} E_p = E.\end{aligned}$$

Therefore T has dense range. Let $\{p_n\}_n$ be a cofinal subset of $S(A)$. Since $T \in K_A(E)$, for each positive integer n, there are $\xi_1^n, ..., \xi_{m_n}^n$ $\eta_1^n, ..., \eta_{m_n}^n$ in E such that

$$\widetilde{p_n}\left(T - \sum_{k=1}^{m_n} \theta_{\xi_k^n, \eta_k^n}\right) < \frac{1}{2n}.$$

We show that $\{\xi_k^n; 1 \leq k \leq m_n, n = 1, 2, ...\}$ is a system of generators for E.

Let $\xi \in E$, $\varepsilon > 0$ and let n_0 be a positive integer. Since T has dense range, there is $\eta \in E$ such that $\overline{p_{n_0}}(\xi - T\eta) < \frac{\varepsilon}{2}$. Let $n = \max\{n_0, [\frac{\overline{p_{n_0}}(\eta)}{\varepsilon}] + 1\}$, where $[t]$ means the integer part of the positive number t. Then $p_{n_0} \leq p_n$ and

$$\begin{aligned}\overline{p_{n_0}}\left(\xi - \sum_{k=1}^{m_n} \xi_k^n \left\langle \eta_k^n, \eta \right\rangle\right) &\leq \overline{p_{n_0}}\left(\xi - T\eta\right) + \overline{p_{n_0}}\left(T\eta - \sum_{k=1}^{m_n} \theta_{\xi_k^n, \eta_k^n}(\eta)\right) \\ &< \frac{\varepsilon}{2} + \overline{p_{n_0}}\left(\eta\right) \widetilde{p_{n_0}}\left(T - \sum_{k=1}^{m_n} \theta_{\xi_k^n, \eta_k^n}\right) \\ &< \frac{\varepsilon}{2} + \overline{p_{n_0}}\left(\eta\right) \widetilde{p_n}\left(T - \sum_{k=1}^{m_n} \theta_{\xi_k^n, \eta_k^n}\right) < \varepsilon.\end{aligned}$$

This shows that $\{\xi_k^n; 1 \leq k \leq m_n, n = 1, 2, ...\}$ is a system of generators for E and therefore E is countably generated. ■

Corollary 5.2.11 *Let A be a Fréchet locally C^*-algebra and let E be a Hilbert A -module. Then E is countably generated if and only if $K_A(E)$ has a countable approximate unit.*

The following theorem extends Theorem 1.9, [**35**] in the context of Hilbert modules over locally C^*-algebras.

Theorem 5.2.12 *Let A be a Fréchet locally C^*-algebra with countable approximate unity and let E be a full Hilbert A -module.*

1. *There is a Hilbert A -module F such that the Hilbert A -modules H_E and $A \oplus F$ are isomorphic.*

2. *If E is countably generated, then the Hilbert A -modules H_E and H_A are isomorphic.*

To prove this theorem we will use the same arguments as in the proof of Theorem 1.9, [**35**]. Thus, first we extend Lemma 1.7, [**35**] which plays an important role in the proof of Theorem 1.9.

Lemma 5.2.13 *Let A be a Fréchet locally C^*-algebra with countable approximate unit and let E be a Hilbert A -module. If E is full, then there is a sequence $\{\xi_n\}_n$ in E such that $p\left(\sum\limits_{k=1}^{n}\langle\xi_k,\xi_k\rangle a-a\right)\to 0$ for all $p\in S(A)$ and for all $a\in A$.*

Proof. First we show that A has a countable approximate unit contained in the C^*-subalgebra $\langle b(E),b(E)\rangle$ of $b(A)$.

Let $\{e_n\}_n$ be a countable approximate unit for A and let $\{u_i\}_{i\in I}$ be an approximate unit for $\langle b(E),b(E)\rangle$.

We show that $\{u_i\}_{i\in I}$ is an approximate unit for A. Let $a\in A$, $p\in S(A)$ and $\varepsilon>0$. Since E is full, there are $\xi_1,...,\xi_n,\eta_1,...,\eta_n\in E$ such that

$$p(a-\sum_{k=1}^{n}\langle\xi_k,\eta_k\rangle)<\varepsilon/8,$$

and since $b(E)$ is dense in E, there is $\widetilde{\xi}_1,...,\widetilde{\xi}_n\in E$ such that

$$\widetilde{p}(\xi_k-\widetilde{\xi}_k)<\varepsilon/8(\sum_{k=1}^{n}\widetilde{p}(\xi_k)+1)$$

for all positive integer k with $1\le k\le n$, and there are $\widetilde{\eta}_1,...,\widetilde{\eta}_n\in E$ such that

$$\widetilde{p}(\eta_k-\widetilde{\eta}_k)<\varepsilon/8(\sum_{k=1}^{n}\widetilde{p}(\widetilde{\xi}_k)+1)$$

for all positive integer k with $1\le k\le n$. Then

$$p(a-\sum_{k=1}^{n}\left\langle\widetilde{\xi}_k,\widetilde{\eta}_k\right\rangle)\ \le\ p(a-\sum_{k=1}^{n}\langle\xi_k,\eta_k\rangle)+p(\sum_{k=1}^{n}\left\langle\xi_k-\widetilde{\xi}_k,\eta_k\right\rangle)$$

$$
\begin{aligned}
&+p(\sum_{k=1}^{n}\left\langle\widetilde{\xi}_k,\widetilde{\eta}_k-\eta_k\right\rangle)\\
<&\ \varepsilon/8+\sum_{k=1}^{n}\widetilde{p}(\xi_k-\widetilde{\xi}_k)\widetilde{p}(\eta_k)+\sum_{k=1}^{n}\widetilde{p}(\eta_k-\widetilde{\eta}_k)\widetilde{p}(\widetilde{\xi}_k)\\
<&\ 3\varepsilon/8.
\end{aligned}
$$

On the other hand, since $\{u_i\}_{i\in I}$ is an approximate unit for $\langle b(E),b(E)\rangle$, there is $i_0\in I$ such that

$$
\left\|\sum_{k=1}^{n}\left\langle\widetilde{\xi}_k,\widetilde{\eta}_k\right\rangle u_i-\sum_{k=1}^{n}\left\langle\widetilde{\xi}_k,\widetilde{\eta}_k\right\rangle\right\|_\infty<\varepsilon/4
$$

for all $i\in I$ with $i\geq i_0$. From these relations we obtain

$$
\begin{aligned}
p(au_i-a)\ \leq&\ p((a-\sum_{k=1}^{n}\left\langle\widetilde{\xi}_k,\widetilde{\eta}_k\right\rangle)u_i)+p(\sum_{k=1}^{n}\left\langle\widetilde{\xi}_k,\widetilde{\eta}_k\right\rangle u_i-\sum_{k=1}^{n}\left\langle\widetilde{\xi}_k,\widetilde{\eta}_k\right\rangle)\\
&+p(a-\sum_{k=1}^{n}\left\langle\widetilde{\xi}_k,\widetilde{\eta}_k\right\rangle)\\
<&\ 3\varepsilon/8+\varepsilon/4+3\varepsilon/8=\varepsilon
\end{aligned}
$$

for all $i\in I$ with $i\geq i_0$ and so $\{u_i\}_{i\in I}$ is an approximate unit for A.

We choose a countable subnet $\{v_n\}_n$ of $\{u_i\}_{i\in I}$ such that with $\{p_n\}_n$ a cofinal subset of $S(A)$, we have $p_n(v_ne_k-e_k)+$ $p_n(e_kv_n-e_k)<\frac{1}{n}$ for all $1\leq k\leq n$.

Let $a\in A$, $p_m\in S(A)$ and $\varepsilon>0$. Then there is a positive integer k_0 such that

$$
p_m(ae_{k_0}-a)+p_m(e_{k_0}a-a)<\frac{\varepsilon}{3}
$$

and there is a positive integer n_0, $n_0=\max\{k_0,m,[\frac{3(p_m(a)+1)}{\varepsilon}]\}$, where $[t]$ is the integer part of the real number t, such that

$$
\begin{aligned}
p_m(av_n-a)+p_m(v_na-a)\ \leq&\ p_m(v_n)\left(p_m(ae_{k_0}-a)+p_m(e_{k_0}a-a)\right)+\\
&+p_m(a)\left(p_m(v_ne_{k_0}-e_{k_0})+p_m(e_{k_0}v_n-e_{k_0})\right)+\\
&+\left(p_m(ae_{k_0}-a)+p_m(e_{k_0}a-a)\right)\\
\leq&\ 2\left(p_m(ae_{k_0}-a)+p_m(e_{k_0}a-a)\right)+\\
&+p_m(a)\left(p_n(v_ne_{k_0}-e_{k_0})+p_n(e_{k_0}v_n-e_{k_0})\right)\\
<&\ 2\frac{\varepsilon}{3}+p_m(a)\frac{1}{n}\leq\varepsilon
\end{aligned}
$$

for all positive integer n, $n \geq n_0$. This shows that $\{v_n\}_n$ is an approximate unit for A.

Let $h = \sum_n 2^{-n} v_n$. Since $\{\pi_p(v_n)\}_n$ is an approximate unit for A_p, $\pi_p(h)$ is a strictly positive element in A_p for each $p \in S(A)$ and so $\pi_p(h)A_p$ is dense in A_p for each $p \in S(A)$. Then

$$\overline{hA} = \lim_{\underset{p}{\leftarrow}} \overline{\pi_p(hA)} = \lim_{\underset{p}{\leftarrow}} \overline{\pi_p(h)A_p} = \lim_{\underset{p}{\leftarrow}} A_p = A.$$

Therefore hA is dense in A.

On the other hand, since h is an element in $\langle b(E), b(E)\rangle$ and $b(E)$ is a full Hilbert $\langle b(E), b(E)\rangle$ -module, according to the proof of Lemma 1.7 in [**35**], there is a sequence $\{\xi_n\}_n$ in $b(E)$ such that $\left\| \sum_{k=1}^{n} \langle \xi_k, \xi_k \rangle h - h \right\|_\infty \to 0$ and $\left\| \sum_{k=1}^{n} \langle \xi_k, \xi_k \rangle \right\|_\infty \leq 1$ for all positive integers n. Then

$$p(\sum_{k=1}^{n} \langle \xi_k, \xi_k \rangle ha - ha) \to 0$$

for all $a \in A$ and for all $p \in S(A)$.

Let $a \in A$, $p \in S(A)$ and $\varepsilon > 0$. Since hA is dense in A, there is $b \in A$ such that $p(a - hb) < \varepsilon/3$. Let n_0 be a positive integer such that $p(\sum_{k=1}^{n} \langle \xi_k, \xi_k \rangle hb - hb) < \varepsilon/3$ for all positive integer n with $n \geq n_0$. Then

$$\begin{aligned} p(\sum_{k=1}^{n} \langle \xi_k, \xi_k \rangle a - a) &\leq p(\sum_{k=1}^{n} \langle \xi_k, \xi_k \rangle (a - hb)) + p(\sum_{k=1}^{n} \langle \xi_k, \xi_k \rangle hb - hb) + p(a - hb) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

for all positive integer n with $n \geq n_0$. ■

The proof of Theorem 5.2.12.

1. Let $\{\xi_n\}_n$ be a sequence in E such that $p\left(\sum_{k=1}^{n} \langle \xi_k, \xi_k \rangle a - a \right) \to 0$ for all $p \in S(A)$ and for all $a \in A$. Thus we can define a map T from A to H_E by $T(a) = (\xi_n a)_n$.

Let $(\eta_n)_n \in H_E$ and $p \in S(A)$. Then

$$\begin{aligned} p\left(\sum_{k=n}^{m} \langle \xi_k, \eta_k \rangle\right)^2 &\leq p\left(\sum_{k=n}^{m} \langle \xi_k, \xi_k \rangle\right) p\left(\sum_{k=n}^{m} \langle \eta_k, \eta_k \rangle\right) \\ &\quad \text{(cf. Cauchy -Schwartz Inequality)} \\ &\leq \left\| \sum_{k=1}^{m} \langle \xi_k, \xi_k \rangle \right\|_\infty p\left(\sum_{k=n}^{m} \langle \eta_k, \eta_k \rangle\right) \\ &\leq p\left(\sum_{k=n}^{m} \langle \eta_k, \eta_k \rangle\right) \end{aligned}$$

for all positive integers n and m with $m > n$. This shows that $\sum\limits_n \langle \xi_n, \eta_n \rangle$ converges in A and so we can define a map S from H_E to A by $S\left((\eta_n)_n\right) = \sum\limits_n \langle \xi_n, \eta_n \rangle$. From

$$\langle T(a), (\eta_n)_n \rangle = a^* \sum_n \langle \xi_n, \eta_n \rangle = \langle a, S\left((\eta_n)_n\right) \rangle$$

for all $a \in A$ and for all $(\eta_n)_n \in H_E$, we deduce that $T \in L_A(A, H_E)$ and $T^* = S$. Since $\langle T(a), T(a) \rangle = \langle a, a \rangle$ for all $a \in A$, T has closed range and moreover, the Hilbert A -modules A and $\text{ran}(T)$ are isomorphic. Then by Remark 3.2.5, H_E is isomorphic with $\ker(T^*) \oplus A$. Therefore there is a Hilbert A -module $F = \ker(T^*)$ such that the Hilbert A -modules H_E and $F \oplus A$ are isomorphic.

2. If E is countably generated, since A is also countably generated and since the Hilbert A-modules H_E and $F \oplus A$ are isomorphic, F is countably generated and so H_F is countably generated. Then, by Theorem 5.2.7, the Hilbert A-modules $H_F \oplus H_A$ and H_A are isomorphic.

On the other hand, since the Hilbert A -modules H_E and $F \oplus A$ are isomorphic, the Hilbert A -modules H_E and $H_F \oplus H_A$ are isomorphic. Therefore the Hilbert A -modules H_E and H_A are isomorphic. ■

We don't know if Theorem 5.2.12 is valid in the general case. If A is an arbitrary locally C^*-algebra we have the following result:

Proposition 5.2.14 *Let A be a unital locally C^*-algebra such that $b(H_A) = H_{b(A)}$, and let E be a countably generated Hilbert A -module such that $b(E)$ is full. Then the Hilbert A -modules H_E and H_A are isomorphic.*

Proof. By Corollary 5.2.9, the Hilbert $b(A)$ -module $b(E)$ is countably generated, and then by Theorem 1.9, [**35**], the Hilbert $b(A)$ -modules $H_{b(E)}$ and $H_{b(A)}$ are isomorphic. From this and Lemma 1.3.8, we conclude that the Hilbert A -modules H_E and H_A are isomorphic. ■

References for Section 5.2: **[14], [16], [19], [23], [26], [29], [35], [38].**

5.3 Strong Morita equivalence

In this Section we extend the concept of strong Morita equivalence in the context of locally C^*-algebras and we show that a well-known result of Brown, Green and Rieffel [**5**] which states that two C^*-algebras are stably isomorphic if and only if they are strongly Morita equivalent is valid for Fréchet locally C^*-algebras.

Definition 5.3.1 *Two locally C^*-algebras A and B are strongly Morita equivalent, written $A \sim_M B$, if there is a full Hilbert A-module E such that the locally C^*-algebras $K_A(E)$ and B are isomorphic.*

Proposition 5.3.2 *Strong Morita equivalence is an equivalence relation in the set of all locally C^*-algebras.*

To prove this proposition, the following lemma will be necessary.

Lemma 5.3.3 *Let A be a locally C^*-algebra and let E and F be Hilbert A -modules. If E is full, then the locally C^*-algebras $L_A(F)$ and $L_B(G)$ respectively $K_A(F)$ and $K_B(G)$ are isomorphic, where $G = K_A(E, F)$ and $B = K_A(E)$.*

Proof. Let $p \in S(A)$. By Proposition 5.1.3, E_p is full and then the C^*-algebras $L_{A_p}(F_p)$ and $L_{B_p}(G_p)$ are isomorphic as well as $K_{A_p}(F_p)$ and $K_{B_p}(G_p)$, where $B_p = K_{A_p}(E_p)$ and $G_p = K_{A_p}(E_p, F_p)$ (see, for example, [**29**], Proposition 7.1). Moreover, the isomorphism is given by $\alpha_p : L_{A_p}(F_p) \to L_{B_p}(G_p)$, $\alpha_p(T_p)(S_p) = T_pS_p$, $T_p \in L_{A_p}(F_p)$, $S_p \in G_p$ respectively $\alpha_p|_{K_{A_p}(F_p)}$.

Let $p, q \in S(A)$ with $p \geq q$. Then

$$
\begin{aligned}
\left((\pi_{pq})_* \circ \alpha_p\right)(T_p)\left(\sigma_q^G(S)\right) &= \sigma_{pq}^G\left(\alpha_p(T_p)\left(\sigma_p^G(S)\right)\right) \\
&= \sigma_{pq}^G\left(T_p \circ \sigma_p^G(S)\right) = (\pi_{pq})_*(T_p) \circ \left(\sigma_q^G(S)\right) \\
&= \left(\alpha_q \circ (\pi_{pq})_*\right)(T_p)\left(\sigma_q^G(S)\right)
\end{aligned}
$$

for all $T_p \in L_{A_p}(F_p)$ and for all $S \in G$. This implies that $(\alpha_p)_p$ is an inverse system of isomorphisms of C^*-algebras as well as $\left(\alpha_p|_{K_{A_p}(F_p)}\right)_p$ and so the locally C^* -algebras $\lim\limits_{\leftarrow p} L_{A_p}(F_p)$ and $\lim\limits_{\leftarrow p} L_{B_p}(G_p)$ are isomorphic as well as $\lim\limits_{\leftarrow p} K_{A_p}(F_p)$ and $\lim\limits_{\leftarrow p} K_{B_p}(G_p)$. From these facts, Corollaries 2.2.8 and 2.3.5, we conclude that the locally C^*-algebras $L_A(F)$ and $L_B(G)$ are isomorphic as well as $K_A(F)$ and $K_B(G)$, and the lemma is proved. ■

The proof of Proposition 5.3.2.

Proof. Since A is a full Hilbert A -module and the locally C^*-algebras A and $K_A(A)$ are isomorphic, we have $A \sim_M A$. So the relation is reflexive.

If $A \sim_M B$, then there is a full Hilbert A -module E such that the locally C^*-algebras B and $K_A(E)$ are isomorphic. By Lemma 5.3.3, the locally C^*-algebras A and $K_B(G)$ are isomorphic, where G is the Hilbert $K_A(E)$ -module $K_A(E, A)$, and since G is full (Remark 5.1.6), $B \sim_M A$. Therefore the relation is symmetric.

To show that the relation is transitive, we suppose that A, B, C are locally C^*-algebras such that B is isomorphic to $K_A(E)$ for some full Hilbert A -module E and C is isomorphic to $K_B(F)$ for some full Hilbert B -module F. By Remark 1.2.11, F can be regarded as a Hilbert $K_A(E)$ -module and so the C^*-algebras C and $K_{k(A)}(F)$ are isomorphic.

Let i be the natural embedding of $K_A(E)$ into $L_A(E)$ and let $F \otimes_i E$ be the interior tensor product of F and E using i. By Proposition 4.2.3, the Hilbert A_p-modules $(F \otimes_i E)_p$ and $F \otimes_{(\pi_p)_* \circ i} E_p$ are isomorphic for each $p \in S(A)$. On the other hand, from the proof of Proposition 4.3.4 (1) (**Step1**), the Hilbert A_p-modules $F \otimes_{(\pi_p)_* \circ i} E_p$ and $F_p \otimes_{i_p} E_p$, where i_p is the natural embedding of $K_{A_p}(E_p)$ into $L_{A_p}(E_p)$ are isomorphic for each $p \in S(A)$. Therefore, for each

$p \in S(A)$, the Hilbert A_p -modules$(F \otimes_i E)_p$ and $F_p \otimes_{i_p} E_p$ are isomorphic and since $F_p \otimes_{i_p} E_p$ is full [**29**], according to Proposition 5.1.3, $F \otimes_i E$ is full.

Let $p \in S(A)$. Then the map $(i_p)_*$ from $K_{K_{A_p}(E_p)}(F_p)$ to $K_{A_p}\left(F_p \otimes_{i_p} E_p\right)$ defined by $(i_p)_*(T_p)\left(\eta_p \otimes_{i_p} \xi_p\right) = T_p\left(\eta_p\right) \otimes_{i_p} \xi_p$ is an isomorphism of C^*-algebras (see [**29**], Proposition 4.7). It is not difficult to check that $\left((i_p)_*\right)_p$ is an inverse system of isomorphisms of C^*-algebras and so the locally C^*-algebras $\lim\limits_{\leftarrow p} K_{K_{A_p}(E_p)}(F_p)$ and $\lim\limits_{\leftarrow p} K_{A_p}\left(F_p \otimes_{i_p} E_p\right)$ are isomorphic. But the locally C^*-algebras $\lim\limits_{\leftarrow p} K_{K_{A_p}(E_p)}(F_p)$ and $K_{K_A(E)}(F)$ (Corollary 2.3.5) are isomorphic as well as $\lim\limits_{\leftarrow p} K_{A_p}\left(F_p \otimes_{i_p} E_p\right)$ and $K_A(F \otimes_i E)$ (Proposition 4.3.3). From these facts, we conclude that the locally C^*-algebras C and $K_A(F \otimes_i E)$ are isomorphic, and since $F \otimes_i E$ is full, $A \sim_M C$. Thus we showed that the relation is transitive. ■

Proposition 5.3.4 *Let A_1, A_2, B_1 and B_2 be locally C^*-algebras. If $A_1 \sim_M B_1$ and $A_2 \sim_M B_2$, then $A_1 \otimes A_2 \sim_M B_1 \otimes B_2$.*

Proof. Since $A_1 \sim_M B_1$ and $A_2 \sim_M B_2$, there is a full Hilbert A_1-module E_1 such that the locally C^*-algebras $K_{A_1}(E_1)$ and B_1 are isomorphic and there is a full Hilbert A_2-module E_2 such that the locally C^*-algebras $K_{A_2}(E_2)$ and B_2 are isomorphic. Then by Proposition 3.4 in [**13**], the locally C^*-algebras $B_1 \otimes B_2$ and $K_{A_1}(E_1) \otimes K_{A_2}(E_2)$ are isomorphic. But according to Proposition 4.3.2, the locally C^*-algebras $K_{A_1}(E_1) \otimes K_{A_2}(E_2)$ and $K_{A_1 \otimes A_2}(E_1 \otimes E_2)$ are isomorphic, where $E_1 \otimes E_2$ is the exterior tensor product of Hilbert modules E_1 and E_2. From these facts and taking into account that $E_1 \otimes E_2$ is full (Remark 5.1.10) we conclude that $A_1 \otimes A_2 \sim_M B_1 \otimes B_2$. ■

Corollary 5.3.5 *Let A be a locally C^*-algebra. Then the locally C^*-algebras A and $\mathcal{K} \otimes A$ are strongly Morita equivalent.*

Proof. From $A \sim_M A$, $\mathcal{K} \sim_M \mathbb{C}$ and Proposition 5.3.4, we conclude that $\mathcal{K} \otimes A \sim_M \mathbb{C} \otimes \mathbb{A}$ and since the locally C^*-algebras A and $\mathbb{C} \otimes \mathbb{A}$ are isomorphic the corollary is proved. ■

Definition 5.3.6 *We will say that two locally C^*-algebras are stably isomorphic if the locally C^*-algebras $\mathcal{K} \otimes A$ and $\mathcal{K} \otimes B$ are isomorphic.*

Proposition 5.3.7 *Let A and B be locally C^*-algebras. If A and B are stably isomorphic, then A and B are strongly Morita equivalent.*

Proof. By Corollaries 4.1.4 and 2.3.6 and Proposition 4.3.2, $K_B(H_B)$ is isomorphic with $\mathcal{K}\otimes B$, and since A and B are stably isomorphic, $K_B(H_B)$ is isomorphic with $\mathcal{K} \otimes A$. From these facts and taking into account that H_B is a full Hilbert B-module, we conclude that $\mathcal{K} \otimes A \sim_M B$. But, by Corollary 5.3.5 $\mathcal{K} \otimes A \sim_M A$. Therefore $A \sim_M B$. ■

The following theorem is a generalization of a well-known theorem of Brown, Green and Rieffel ([**5**], Theorem 1.2) for Fréchet locally C^*-algebras.

Theorem 5.3.8 *Let A and B be Fréchet locally C^*-algebras with countable approximate units. Then A is strongly Morita equivalent to B if and only if A and B are stably isomorphic.*

Proof. If A and B are stably isomorphic, then by Proposition 5.3.7, A and B are strongly Morita equivalent.

To show the converse implication, let E be a full Hilbert A -module such that $K_A(E)$ is isomorphic with B. Since A has a countably approximate unit, E is countably generated (Corollary 5.2.11) and then by Theorem 5.2.12, the Hilbert A -modules H_E and H_A are isomorphic. Therefore the locally C^*-algebra $K_A(H_E)$ is isomorphic with the locally C^*-algebra $K_A(H_A)$ which is $K_A(H_A)$ is isomorphic with $\mathcal{K} \otimes A$.

On the other hand, $\mathcal{K} \otimes B$ is isomorphic with $\mathcal{K} \otimes K_A(E)$ which is isomorphic with $K_A(H_E)$ (Proposition 4.3.2). From these facts we conclude that the locally C^*-algebras $\mathcal{K} \otimes A$ and $\mathcal{K} \otimes B$ are isomorphic and so the locally C^*-algebras A and B are stably isomorphic. ■

References for Section 5.3: **[5], [21], [29],[42], [43].**

Chapter 6

Representations of locally C^*-algebras on Hilbert modules

6.1 Representations of locally C^*-algebras

In this Section we introduce the notion of representation of a locally C^*-algebra on a Hilbert module and we characterize the non-degenerate representations of locally C^*-algebras on Hilbert modules. Also we show that given a locally C^*-algebra B, then any separable locally C^*-algebra A admits a non-degenerate representation on the Hilbert module H_B.

Let A and B be two locally C^*-algebras.

Definition 6.1.1 *A representation of A on a Hilbert B -module E is a continuous $*$-morphism Φ from A into $L_B(E)$. We say that the representation Φ is non-degenerate if $\Phi(A)E$ is dense in E.*

Definition 6.1.2 *Two representations Φ_1 and Φ_2 of A on the Hilbert B -modules E_1 respectively E_2 are said to be unitarily equivalent if there is a unitary operator U in $L_B(E_1, E_2)$ such that $U\Phi_1(a) = \Phi_2(a)U$ for all a in A.*

Remark 6.1.3 *If A is a Fréchet locally C^*-algebra, then any $*$ -morphism from A to $L_B(E)$ is a representation of A on E.*

Proposition 6.1.4 *Let A be a locally C^*-algebra, let Φ be a representation of A on a Hilbert B -module E. Then the following statements are equivalent:*

1. *Φ is non-degenerate;*

2. *there is a unique unital continuous $*$-morphism $\overline{\Phi}$ from $M(A)$ into $L_B(E)$ such that:*

   (**a**) *$\overline{\Phi}|_A = \Phi$,*

   (**b**) *$\overline{\Phi}|_C$ is strictly continuous whenever C is a bounded selfadjoint subset of $M(A)$;*

3. *for some approximate unit $\{e_i\}_{i\in I}$ of A, $\{\Phi(e_i)\}_{i\in I}$ converges strictly to $1_{L_B(E)}$, the identity map on E.*

Proof. 1. $\Rightarrow$ 2. Let $c \in M(A)$. We consider the map $\overline{\Phi}(c)$ from $\Phi(A)E$ into E defined by

$$\overline{\Phi}(c)\left(\sum_{j=1}^{n}\Phi(a_j)\xi_j\right) = \sum_{j=1}^{n}\Phi(ca_j)\xi_j.$$

Let $\{u_i\}_{i\in I}$ be an approximate unit for A, $\sum\limits_{j=1}^{n}\Phi(a_j)\xi_j \in \Phi(A)E$ and $q \in S(B)$. Then

$$\begin{aligned}
\overline{q}\left(\overline{\Phi}(c)\left(\sum_{j=1}^{n}\Phi(a_j)\xi_j\right)\right) &= \lim_i \overline{q}\left(\sum_{j=1}^{n}\Phi(cu_ia_j)\xi_j\right)\\
&\leq \lim_i \widetilde{q}\left(\Phi(ce_i)\right)\overline{q}\left(\sum_{j=1}^{n}\phi(a_j)\xi_j\right)\\
&\leq p_q(c)\overline{q}\left(\sum_{j=1}^{n}\phi(a_j)\xi_j\right)
\end{aligned}$$

for some $p_q \in S(A)$. Hence $\overline{\Phi}(c)$ can be extended on E by continuity. It is easy to check that $\overline{\Phi}(c)$ is an element in $L_B(E)$ and $\overline{\Phi}(c)^* = \overline{\Phi}(c^*)$. Also it is not difficult to that the map $\overline{\Phi}$ is a unital continuous $*$ -morphism from $M(A)$ to $L_B(E)$. Evidently, $\overline{\Phi}|_A = \Phi$.

Let C be a bounded selfadjoint subset of $M(A)$. To show that $\overline{\Phi}|_C$ is strictly continuous, let $\{c_i\}_{i\in I}$ be a net in C which converges strictly to an element c in C, $\xi \in E$, $q \in S(B)$ and $\varepsilon > 0$. Since $\overline{\Phi}$ is a morphism of locally C^* -algebras, there is $p_q \in S(A)$ such that $\widetilde{q}\left(\overline{\Phi}(b)\right) \leq p_q(b)$ for all $b \in M(A)$, and since $\Phi(A)E$ is dense in E, there is $\sum\limits_{j=1}^{n} \Phi(a_j)\xi_j \in \Phi(A)E$ such that

$$\overline{q}\left(\xi - \sum_{j=1}^{n} \Phi(a_j)\xi_j\right) < \frac{\varepsilon}{3M_{p_q}},$$

where $M_{p_q} = \sup\{p_q(c); c \in C\}$. Let $M = \sup\{\overline{q}(\xi_j); j = 1, 2, ..., n\}$. Since the net $\{c_i\}_{i\in I}$ converges strictly to c, there is $i_0 \in I$ such that

$$p_q\left(c_i a_j - c a_j\right) + p_q\left(c_i^* a_j - c^* a_j\right) < \frac{\varepsilon}{3nM}$$

for all $j \in \{1, 2, ..., n\}$ and for all $i \in I$ with $i \geq i_0$. Then

$$\begin{aligned}
&\overline{q}\left(\overline{\Phi}(c_i)\xi - \overline{\Phi}(c)\xi\right) + \overline{q}\left(\overline{\Phi}(c_i^*)\xi - \overline{\Phi}(c^*)\xi\right) \\
&\qquad \leq \overline{q}\left(\overline{\Phi}(c_i - c)\left(\xi - \sum_{j=1}^{n}\Phi(a_j)\xi_j\right)\right) \\
&\qquad\quad + \overline{q}\left(\overline{\Phi}(c_i^* - c^*)\left(\xi - \sum_{j=1}^{n}\Phi(a_j)\xi_j\right)\right) \\
&\qquad\quad + \overline{q}\left(\overline{\Phi}(c_i - c)\sum_{j=1}^{n}\Phi(a_j)\xi_j\right) + \overline{q}\left(\overline{\Phi}(c_i^* - c^*)\sum_{j=1}^{n}\Phi(a_j)\xi_j\right) \\
&\qquad \leq \left(p_q(c_i - c) + p_q(c_i^* - c^*)\right)\overline{q}\left(\xi - \sum_{j=1}^{n}\Phi(a_j)\xi_j\right) \\
&\qquad\quad + \sum_{j=1}^{n}\left(p_q(c_i a_j - c a_j) + p_q(c_i^* a_j - c^* a_j)\right)\overline{q}\left(\xi_j\right) \\
&\qquad < 2M_{p_q}\tfrac{\varepsilon}{3M_{p_q}} + nM\tfrac{\varepsilon}{3nM} = \varepsilon
\end{aligned}$$

for all $i \in I$ with $i \geq i_0$. Therefore the net $\{\overline{\Phi}(c_i)\}_{i\in I}$ converges strictly to $\overline{\Phi}(c)$.

To show that $\overline{\Phi}$ is unique, let $\widetilde{\Phi}$ be another unital continuous $*$-morphism from $M(A)$ to $L_B(E)$ which verifies the conditions (a) and (b) and $c \in M(A)$. Then

$$\begin{aligned}\widetilde{\Phi}(c)\left(\sum_{j=1}^{n}\Phi(a_j)\xi_j\right) &= \sum_{j=1}^{n}\widetilde{\Phi}(c)\Phi(a_j)\xi_j \\ &= \sum_{j=1}^{n}\Phi(ca_j)\xi_j = \overline{\Phi}(c)\left(\sum_{j=1}^{n}\phi(a_j)\xi_j\right)\end{aligned}$$

for all $\sum\limits_{j=1}^{n}\Phi(a_j)\xi_j \in \Phi(A)E$. From this, since $\Phi(A)E$ is dense in E, we conclude that $\overline{\Phi}(c) = \widetilde{\Phi}(c)$ and so $\overline{\Phi} = \widetilde{\Phi}$.

2. $\Rightarrow$ 3. Let $\{e_i\}_{i\in I}$ be an approximate unit for A. Then $C = \{e_i\}_{i\in I} \cup \{1_{M(A)}\}$ is a bounded selfadjoint subset of $M(A)$ and since $\{e_i\}_{i\in I}$ converges strictly to $1_{M(A)}$, and $\overline{\Phi}|_C$ is strictly continuous, $\{\Phi(e_i)\}_{i\in I}$ converges strictly to $\overline{\Phi}(1_{M(A)}) = 1_{L_B(E)}$.

3. $\Rightarrow$ 1. Let $\{e_i\}_{i\in I}$ be an approximate unit of A such that $\{\Phi(e_i)\}_{i\in I}$ converges strictly to $1_{L_B(E)}$ and let $\xi \in E$. Then the net $\{\Phi(e_i)\xi\}_{i\in I}$ converges to ξ. This shows that $\Phi(A)E$ is dense in E. ■

Remark 6.1.5 *If Φ is a non-degenerate representation of A on a Hilbert B -module E, then for any approximate unit $\{e_i\}_{i\in I}$ for A, the net $\{\Phi(e_i)\}_{i\in I}$ converges strictly to $1_{L_B(E)}$.*

We know that given a C^*-algebra B, then any separable C^*-algebra admits a faithful non-degenerate representation on the Hilbert B -module H_B (see, for example, [**29**], Lemma 6.4). Two natural questions arise: (1) Given two locally C^*-algebras A and B, is there a representation of A on the Hilbert B -module H_B? (2) If there is a such representation, is it faithful? The following proposition gives an answer to these questions.

Proposition 6.1.6 *Let A and B be two locally C^*-algebras with A separable.*

1. *There is a non-degenerate representation of A on H_B.*

2. *If A is a strong spectrally bounded Fréchet locally C^*-algebra, then there is a faithful, non-degenerate representation of A on H_B.*

Proof. Since A is separable, the C^*-algebras A_p, $p \in S(A)$ are all separable.

1. Let $p \in S(A)$. Since A_p is separable, by Corollary 3.7.5 in [**37**], there is a faithful, non-degenerate representation φ_p of A_p on an infinite separable Hilbert space H. Then $\varphi = \varphi_p \circ \pi_p$ is a non-degenerate representation of A on H. According to Corollary 4.1.4, the Hilbert B -modules H_B and $H \otimes B$ are isomorphic. Thus to prove the proposition, we will construct a representation of A on $H \otimes B$. Consider the linear map $\mathrm{i}_{L(H)}$ from $L(H)$ to $L(H) \otimes L_B(B)$ defined by

$$\mathrm{i}_{L(H)}(T) = T \otimes \mathrm{id}_B.$$

It is not difficult to check that $\mathrm{i}_{L(H)}$ is a morphism of locally C^*-algebras. Let $\Phi = j \circ \mathrm{i}_{L(H)} \circ \varphi$, where j is the the morphism of locally C^*-algebras constructed in Proposition 4.3.2. Then Φ is a representation of A on $H \otimes B$. Moreover, since

$$\Phi(A)(H \otimes_{\mathrm{alg}} B) = j((\varphi(A) \otimes \mathrm{id}_B))(H \otimes_{\mathrm{alg}} B) = \varphi(A) H \otimes_{\mathrm{alg}} B,$$

Φ is non-degenerate.

2. We will construct a faithful, non -degenerate representation of A on a separable Hilbert space H (see, for example, [**7**]). Let $\{p_n; n = 1, 2, ...\}$ be a cofinal subset of $S(A)$. Then, for each positive integer n, there is a faithful non-degenerate representation φ_n of A_{p_n} on an infinite separable Hilbert space H_n (see, for example, [**36**]). Let $H = \bigoplus_n H_n$. Then H is an infinite separable Hilbert space. For each $a \in A$, define a linear map $\varphi(a)$ from H to H by $\varphi(a)\left(\bigoplus_n \xi_n\right) = \bigoplus_n \varphi_n(\pi_n(a))\xi_n$.

From

$$\begin{aligned}\left\langle \varphi(a)\left(\bigoplus_n \xi_n\right), \varphi(a)\left(\bigoplus_n \xi_n\right)\right\rangle &= \sum_n \langle \varphi_n(\pi_n(a))\xi_n, \varphi_n(\pi_n(a))\xi_n\rangle \\ &\leq \sum_n \|\pi_n(a)\|_{p_n} \langle \xi_n, \xi_n\rangle\end{aligned}$$

$$\leq \sup\{p_n(a); n = 1, 2, ...\} \left\langle \bigoplus_n \xi_n, \bigoplus_n \xi_n \right\rangle$$

for all $\bigoplus\limits_n \xi_n \in H$, we conclude that $\varphi(a) \in L(H)$. In this way we have defined a map φ from A to $L(H)$. It is not hard to check that φ is a representation of A on H. Let $a \in A$ such that $\varphi(a) = 0$. Then $\varphi_n(\pi_n(a)) = 0$ for all positive integer n. From this fact and taking into account that φ_n is injective for each positive integer n, we conclude that $\pi_n(a) = 0$ for all positive integer n and so $a = 0$. Therefore φ is a faithful representation of A on H . Moreover, since

$$\varphi(A)H = \bigoplus_n \varphi_n(A_{p_n})H_n$$

and since $\varphi_n(A_{p_n})H_n$ is dense in H_n for all positive integer n, φ is non-degenerate.

Let $\Phi = j \circ \mathrm{i}_{L(H)} \circ \varphi$. Clearly, Φ is a non-degenerate representation of A on $H \otimes B$. Let $a \in A$ such that $\Phi(a) = 0$. Then

$$\begin{aligned} 0 &= \langle \Phi(a)(h \otimes b), \Phi(a)(h \otimes b) \rangle = \langle \varphi(a)h \otimes b, \varphi(a)h \otimes b \rangle \\ &= \langle \varphi(a)h, \varphi(a)h \rangle \, b^* b \end{aligned}$$

for all $h \in H$ and for all $b \in B$. This implies that $\langle \varphi(a)h, \varphi(a)h \rangle = 0$ and so $\varphi(a)h = 0$ for all $h \in H$. From this fact and taking into account that φ is injective, we conclude that $a = 0$. Therefore Φ is a faithful, non-degenerate representation of A on $H \otimes B$. ■

References for Section 6.1: [**13**], [**17**], [**19**],[**26**], [**29**].

6.2 Completely positive linear maps

In this section we present some properties of the positive linear maps between locally C^*-algebras.

Let A and B be two locally C^*-algebras. We say that a linear map ρ from A into B is completely positive if for all positive integers n, the linear maps $\rho^{(n)} : M_n(A) \to M_n(B)$ defined by $\rho^{(n)}([a_{ij}]_{i,j=1}^n) = [\rho(a_{ij})]_{i,j=1}^n$ are positive.

Proposition 6.2.1 *Let $\rho : A \to B$ be a continuous linear map between locally C^*-algebras. Then the following statements are equivalent:*

1. *ρ is completely positive and for some approximate unit $\{e_i\}_{i\in I}$ for A, $\{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(B)$;*

2. *$\rho(b(A)) \subseteq b(B)$ and $\rho|_{b(A)} : b(A) \to b(B)$ is completely positive.*

Proof. 1. $\Rightarrow$ 2. Let $\{e_i\}_{i\in I}$ be an approximate unit of A such that $\{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(B)$. Then there is $M > 0$ such that $\|\rho(e_i)\|_\infty \leq M$ for all $i \in I$.

To show that $\rho(b(A)) \subseteq b(B)$ it is sufficient to prove that $\rho(P(b(A))) \subseteq P(b(B))$, since an arbitrary element of $b(A)$ may be written as a linear combination of positive elements in $b(A)$.

Let $i_0 \in I$ and $a \in P(b(A))$. Since $e_{i_0} a e_{i_0} \leq \|a\|_\infty e_{i_0}$,

$$q(\rho(e_{i_0} a e_{i_0})) \leq \|a\|_\infty q(\rho(e_{i_0})) \leq M \|a\|_\infty$$

for every $q \in S(B)$. Therefore $\{\rho(e_i a e_i)\}_{i\in I}$ is a bounded net in $b(B)$ and since ρ is continuous and $p(e_i a e_i - a) \to 0$ for every $p \in S(A)$, $\rho(a) \in b(B)$. Clearly, $\rho|_{b(A)} : b(A) \to b(B)$ is completely positive.

2. $\Rightarrow$ 1. Let $\{e_i\}_{i\in I}$ be an approximate unit of $b(A)$. Then $\{e_i\}_{i\in I}$ is an approximate unit of A and $\{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(B)$ since $\|\rho(e_i)\|_\infty \leq \left\|\rho|_{b(A)}\right\|$ for all $i \in I$.

Let n be a positive integer. Since $M_n(b(A)) = b(M_n(A))$, ([**2**], Lemma 2.1), we have

$$\rho^{(n)}(b(M_n(A))) = \rho^{(n)}(M_n(b(A))) \subseteq M_n(b(B)) = b(M_n(B)).$$

From this relation and taking into account that $P(b(M_n(A)))$ is dense in $P(M_n(A))$ and $\rho^{(n)}$ is continuous, we conclude that $\rho^{(n)}$ is positive. Hence ρ is completely positive. ■

Remark 6.2.2 *Let $\rho : A \to B$ be a continuous completely positive linear map between locally C^*-algebras. If A is unital and $\rho(1_A) \in b(B)$ or if $b(B) = B$ as sets, then ρ is completely positive if and only if by restriction, it defines a completely positive linear map between C^*-algebras $b(A)$ and $b(B)$.*

Remark 6.2.3 *In the particular case when $\rho : A \to B$ is a unital continuous linear map between locally C^*-algebras we obtain the Corollary 2.3 in [**2**].*

Corollary 6.2.4 *Let $\rho : A \to B$ be a continuous completely positive linear map between locally C^*-algebras such that for some approximate unit $\{e_i\}_{i\in I}$ of A, $\{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(B)$. Then there is $M > 0$ such that*

$$\rho^{(n)}(([a_{jk}]_{j,k=1}^n)^*)\rho^{(n)}([a_{jk}]_{j,k=1}^n) \leq M\rho^{(n)}(([a_{jk}]_{j,k=1}^n)^* [a_{jk}]_{j,k=1}^n)$$

for every $[a_{jk}]_{i,j=1}^n \in M_n(A)$, and consequently, $[\rho(a_j^*)\rho(a_k)]_{j,k=1}^n \leq M[\rho(a_j^* a_k)]_{j,k=1}^n$ for every $a_1, ..., a_n \in A$.

Proof. By Proposition 6.2.1, $\rho|_{b(A)}$ is a completely positive linear map from $b(A)$ to $b(B)$. Then there is $M > 0$ such that

$$\rho^{(n)}(([a_{jk}]_{j,k=1}^n)^*)\rho^{(n)}([a_{jk}]_{j,k=1}^n) \leq M\rho^{(n)}(([a_{jk}]_{j,k=1}^n)^* [a_{jk}]_{j,k=1}^n)$$

for every $[a_{jk}]_{i,j=1}^n \in M_n(b(A))$, and

$$[\rho(a_j^*)\rho(a_k)]_{j,k=1}^n \leq M[\rho(a_j^* a_k)]_{j,k=1}^n$$

for every $a_1, ..., a_n \in b(A)$ (see, for example, [**29**], Lemmas 5.3 and 5.4), and since $M_n(b(A))$ is dense in $M_n(A)$, $b(A)$ is dense in A and ρ is continuous the corollary is proved. ∎.

Proposition 6.2.5 *Let $\rho : A \to B$ be a continuous completely positive linear map between locally C^*-algebras such that for some approximate unit $\{e_i\}_{i\in I}$ of $A, \{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(B)$. Then there is a continuous completely positive linear map ρ^+ from A^+ into B^+ such that $\rho^+|_A = \rho$, where A^+ (respectively B^+) is the unitization of A (respectively B).*

Proof. According to the Proposition 6.2.1, $\rho|_{b(A)} : b(A) \to b(B)$ is a completely positive linear map between C^*-algebras. Let $q \in S(B)$. The continuity of ρ implies that there is $K_q > 0$ and $p_q \in S(A)$ such that $q(\rho(a)) \leq K_q p_q(a)$ for all $a \in A$. Hence there is a continuous linear map $\rho_q : A_{p_q} \to B_q$ such that $\rho_q \circ \pi_{p_q} = \pi_q \circ \rho$. Clearly, $\rho_q^{(n)} \circ \pi_{p_q}^{(n)} = \pi_q^{(n)} \circ \rho^{(n)}$ for all positive integers n and so ρ_q is a completely positive linear map between C^*-algebras. Since $\|\rho_q\| \leq \|\rho|_{b(A)}\|$, the map $\widetilde{\rho}_q : A_{p_q}^+ \to B_q^+$ defined by $\widetilde{\rho}_q(a, \lambda) = \rho_q(a) + \lambda \|\rho|_{b(A)}\|$ is a completely positive linear map between C^*-algebras. Then the map $\rho_q^+ : A^+ \to B_q^+$ defined by $\rho_q^+ = \widetilde{\rho}_q \circ \pi_{p_q}^+$, where $\pi_{p_q}^+$ is the canonical map from A^+ into $A_{p_q}^+$, is a continuous completely positive linear map from A^+ into B_q^+. It is easy to verify that $\pi_{qr}^+ \circ \rho_q^+ = \rho_r^+$ for all $q, r \in S(B)$, $q \geq r$, where π_{qr}^+, $q, r \in S(B)$, $q \geq r$ are the connecting maps of the inverse system $\{B_q^+\}_{q \in S(B)}$. This implies that there is a continuous linear map ρ^+ from A^+ into B^+ such that $\pi_q^+ \circ \rho^+ = \rho_q^+$ for all $q \in S(B)$, where π_q^+ is the canonical map from B^+ into B_q^+. Evidently ρ^+ is completely positive and $\rho^+|_A = \rho$. ■

References for Section 6.2: [**2**], [**15**], [**17**], [**29**].

6.3 The KSGNS construction

In this Section we give a construction of type KSGNS(Kasparov, Stinespring, Gel'fand, Naimark, Segal) for strict continuous completely positive linear maps between locally C^*-algebras. Also we extend the generalized Stinespring theorem on dilatation of completely positive linear maps between C^*-algebras [**26**] in the context of locally C^*-algebras.

Definition 6.3.1 *Let A and B be locally C^*-algebras and let E be a Hilbert B-module. We say that a continuous completely positive linear map $\rho : A \to L_B(E)$ is strict if for some approximate unit $\{e_i\}_{i\in I}$ of A, $\{\rho(e_i)\}_{i\in I}$ is strictly Cauchy in $L_B(E)$.*

Remark 6.3.2 *The condition of strictness is automatically satisfied if A is unital or $B = \mathbb{C}$.*

Proposition 6.3.3 *Let A and B be locally C^*-algebras, let E and F be Hilbert B-modules, let Φ be a non-degenerate representation of A on F and let V be an element in $L_B(E, F)$. Then the map $\rho : A \to L_B(E)$ defined by*

$$\rho(a) = V^*\Phi(a)V, \quad a \in A$$

is a continuous strict completely positive linear map.

Proof. It is easy to check that ρ is a continuous completely positive linear map from A to $L_B(F)$. Let $\{e_i\}_{i\in I}$ be an approximate unit for A. By Proposition 6.1.4, the net $\{\Phi(e_i)\}_{i\in I}$ converges strictly to $1_{L_B(F)}$, and then the net $\{\rho(e_i)\}_{i\in I}$ converges strictly to V^*V. Therefore ρ is strict. ■

If A and B are two C^*-algebras, then any strict completely positive linear map ρ from A to $L_B(E)$, where E is a Hilbert B -module, induces a non-degenerate representation Φ_ρ of A on a Hilbert B -module E_ρ. Moreover, this representation is unique up to unitary equivalence and

- (**a**) $E_\rho = A \otimes_\rho E$, $\Phi_\rho(a)\left(c \otimes_\rho \xi\right) = ac \otimes_\rho \xi$ for all $a, c \in A$ and for all $\xi \in E$;
- (**b**) $\rho\,(a) = V_\rho^*\Phi_\rho(a)V_\rho$ for all $a \in A$, where V_ρ is an adjointable operator from E to E_ρ defined by $V_\rho\xi = \lim\limits_i \left(e_i \otimes_\rho \xi\right)$, $\{e_i\}_{i\in I}$ being an approximate unit for A such that the net $\{\rho\,(e_i)\}_{i\in I}$ is strictly Cauchy in $L_B(E)$, and
- (**c**) $\Phi_\rho(A)V_\rho E$ is dense in E_ρ.

The construction of this representation is known that the KSGNS (Kasparov, Sitinespring, Ge'lfand, Naimark, Segal) construction associated with ρ (see, for example, [**29**]).We extend this construction in the context of locally C^*-algebras.

Construction 6.3.4 *Let A and B be locally C^*-algebras, let E be a Hilbert B-module and let ρ be a continuous completely positive linear map from A to $L_B(E)$. The algebraic tensor product $A \otimes_{alg} E$ is a right B -module in the obvious way*

$(a \otimes \xi)\, b = a \otimes \xi b$ and the map $\langle \cdot, \cdot \rangle_\rho^0$ from $(A \otimes_{alg} E) \times (A \otimes_{alg} E)$ to B defined by

$$\left\langle \sum_{i=1}^{n} a_i \otimes \xi_i, \sum_{j=1}^{m} c_j \otimes \eta_j \right\rangle_\rho^0 = \sum_{i=1}^{n}\sum_{j=1}^{m} \left\langle \xi_i, \rho\left(a_i^* c_j\right) \eta_j \right\rangle$$

is $\mathbb{C}$ - and A -linear in its second variable, and

$$\left(\left\langle \zeta, \zeta' \right\rangle_\rho^0\right)^* = \left\langle \zeta', \zeta \right\rangle_\rho^0$$

for all $\zeta, \zeta' \in A \otimes_{alg} E$. Since ρ is completely positive linear map and the locally C^*-algebras $M_n\left(L_B(E)\right)$ and $L_B(E^n)$ can be identified, we conclude that

$$\langle \zeta, \zeta \rangle_\rho^0 \geq 0$$

for all $\zeta \in A \otimes_{alg} E$. Then $\left(A \otimes_{alg} E\right)/N_\rho$, where $N_\rho = \{\zeta \in A \otimes_{alg} E; \langle \zeta, \zeta \rangle_\rho^0 = 0\}$, is a pre-Hilbert B -module with the inner-product defined by

$$\left\langle \zeta + N_\rho, \zeta' + N_\rho \right\rangle_\rho = \left\langle \zeta, \zeta' \right\rangle_\rho^0.$$

The Hilbert B -module obtained by the completion of $\left(A \otimes_{alg} E\right)/N_\rho$ with respect to the topology induced by the inner -product is denoted by $A \otimes_\rho E$.

Let $q \in S(B)$. Then $\rho_q = \left(\pi_q\right)_* \circ \rho$ is a continuous completely positive linear map from A to $L_B(E_q)$. For each $q_1, q_2 \in S(B)$ with $q_1 \geq q_2$, the linear map $\chi_{q_1 q_2}$ from $A \otimes_{alg} E_{q_1}$ to $A \otimes_{alg} E_{q_2}$, defined by $\chi_{q_1 q_2}(a \otimes \xi) = a \otimes \sigma^E_{q_1 q_2}(\xi)$, extends to a linear map $\chi_{q_1 q_2}$ from $A \otimes_{\rho_{q_1}} E_{q_1}$ to $A \otimes_{\rho_{q_2}} E_{q_2}$ such that $\chi_{q_1 q_2}\left(a \otimes_{\rho_{q_1}} \xi\right) = a \otimes_{\rho_{q_2}} \sigma^E_{q_1 q_2}(\xi)$ and in the same way as in the proof of Proposition 4.2.3, we deduce that $\{A \otimes_{\rho_q} E_q; B_q; \chi_{q_1 q_2}; \pi_{q_1 q_2}, q, q_1, q_2 \in S(B)$ with $q_1 \geq q_2\}$ is an inverse system of Hilbert C^* -modules and the Hilbert B -modules $A \otimes_\rho E$ and $\lim\limits_{\leftarrow q} A \otimes_{\rho_q} E_q$ are isomorphic. Moreover, the Hilbert B_q -modules $\left(A \otimes_\rho E\right)_q$ and $A \otimes_{\rho_q} E_q$, $q \in S(B)$ are isomorphic and so the locally C^*-algebras $L_B\left(A \otimes_\rho E\right)$ and $\lim\limits_{\leftarrow q} L_{B_q}\left(A \otimes_{\rho_q} E_q\right)$ are isomorphic.

Theorem 6.3.5 *Let A and B be two locally C^*-algebras, let E be a Hilbert B-module and let $\rho : A \to L_B(E)$ be a continuous strict completely positive linear map.*

1. *Then there is a Hilbert B-module E_ρ, a representation Φ_ρ of A on E_ρ and an element V_ρ in $L_B(E, E_\rho)$ such that:*

   (**a**) *$\rho(a) = V_\rho^* \Phi_\rho(a) V_\rho$ for every $a \in A$,*

   (**b**) *$\Phi_\rho(A) V_\rho E$ is dense in E_ρ.*

2. *If F is a Hilbert B-module, Φ is a representation of A on F, W is an element in $L_B(E, F)$ such that:*

   (**a**) *$\rho(a) = W^* \Phi(a) W$ for every $a \in A$,*

   (**b**) *$\Phi(A) W E$ is dense in F,*

   then there is a unitary operator U in $L_B(E_\rho, F)$ such that

$$\Phi(a) = U \Phi_\rho(a) U^*$$

for every $a \in A$ and $W = U V_\rho$.

The triple $(E_\rho, \Phi_\rho, V_\rho)$ constructed in the Theorem 6.3.5 will be called the KSGNS (Kasparov, Stinespring, Gel'fand, Naimark, Segal) construction associated with the continuous strict completely positive linear map ρ.

Remark 6.3.6 *The above construction is a generalization of the ordinary KSGNS construction. In particular we obtain the Kasparov's construction in [**26**] as well as the ordinary GNS construction.*

*On the other hand, we also obtain the Stinespring's construction for locally C^*-algebras (Theorem 2.2, [**15**]).*

The proof of Theorem 6.3.5. We partition the proof in two steps.

Step1. We suppose that B is a C^* -algebra.

1. The continuity of ρ implies that there is $p \in S(A)$ and $M > 0$ such that $\|\rho(a)\| \leq M p(a)$ for all $a \in A$ and so there is a linear map ρ_p from A_p to $L_B(E)$ such that $\rho_p \circ \pi_p = \rho$. Clearly ρ_p is a strict completely positive linear map between C^* -algebras A_p and $L_B(E)$. Moreover, ρ_p is strict, since if $\{e_i\}_{i \in I}$ is an

approximate unit for A such that the net $\{\rho(e_i)\}_{i\in I}$ is strictly Cauchy in $L_B(E)$, then $\{\pi_p(e_i)\}_{i\in I}$ is an approximate unit for A_p and the net $\{\rho_p(\pi_p(e_i))\}_{i\in I}$ is strictly Cauchy in $L_B(E)$. Let $(E_{\rho_p}, \Phi_{\rho_p}, V_{\rho_p})$ be the ordinary KSGNS construction associated with ρ_p (see, Theorem 5.6, [**29**]). Define a linear map U_p from $A \otimes_{\text{alg}} E$ to $A_p \otimes_{\text{alg}} E$ by $U_p(a \otimes \xi) = \pi_p(a) \otimes \xi$. Since

$$\begin{aligned} \langle U_p(a\otimes\xi), U_p(a\otimes\xi)\rangle^0_{\rho_p} &= \left\langle \xi, \rho_p(\pi_p(a^*a))\xi\right\rangle \\ &= \langle \xi, \rho(a^*a)\xi\rangle = \langle a\otimes\xi, a\otimes\xi\rangle^0_\rho \end{aligned}$$

for all $a \in A$ and for all $\xi \in E$, $U_p((a\otimes\xi)b) = U_p(a\otimes\xi)b$ for all $b \in B$, for all $a \in A$ and for all $\xi \in E$, and since $U_p(A\otimes_{\text{alg}} E) = A_p \otimes_{\text{alg}} E$, U_p extends to an isometric, surjective, B -linear map U_p from $A \otimes_\rho E$ onto E_{ρ_p} such that

$$U_p\left(a \otimes_\rho \xi\right) = \pi_p(a) \otimes_{\rho_p} \xi$$

for all $a \in A$ and for all $\xi \in E$. Therefore U is a unitary element in $L_B(A \otimes_\rho E, E_{\rho_p})$.

Let $E_\rho = A \otimes_\rho E$, $V_\rho = U_p^* V_{\rho_p}$ and let Φ_ρ be a map from A to $L_B(E_\rho)$ defined by $\Phi_\rho(a) = U_p^*\Phi_{\rho_p}(\pi_p(a))U_p$. It is not difficult to check that Φ_ρ is a representation of A on E_ρ and $\Phi_\rho(a)\left(c\otimes_\rho\xi\right) = ac\otimes_\rho \xi$ for all $a, c \in A$ and for all $\xi \in E$. Moreover,

$$\begin{aligned} V_\rho^*\Phi_\rho(a)V_\rho &= V_{\rho_p}^* U_p U_p^* \Phi_{\rho_p}(\pi_p(a))U_pU_p^*V_{\rho_p} = V_{\rho_p}^*\Phi_{\rho_p}(\pi_p(a))V_{\rho_p} \\ &= \rho_p(\pi_p(a)) = \rho(a) \end{aligned}$$

for all $a \in A$. From

$$\Phi_\rho(A)V_\rho E = U_p^*\Phi_{\rho_p}(A_p)V_{\rho_p}E$$

and taking into account that $\Phi_{\rho_p}(A_p)V_{\rho_p}E$ is dense in E_{ρ_p}, we conclude that $\Phi_\rho(A)V_\rho E$ is dense in E_ρ.

2. Since Φ and ρ are continuous, and since $S(A)$ is directed, we can choose $p \in S(A)$ such that $\|\Phi(a)\| \le p(a)$ for all $p \in S(A)$ and $\|\rho(a)\| \le Mp(a)$ for some $M > 0$ and for all $a \in A$. Then there is a strictly continuous completely positive

linear map ρ_p from A_p to $L_B(E)$ such that $\rho_p \circ \pi_p = \rho$ and there is a continuous $*$-morphism Φ_p from A_p to $L_B(F)$ such that $\Phi_p \circ \pi_p = \Phi$. Moreover, $\rho_p(a) = W^*\Phi_p(a)W$ for all $a \in A_p$ and $\Phi_p(A_p)WE$ is dense in F, since $\Phi_p(A_p)WE = \Phi(A)WE$.

If $(E_{\rho_p}, \Phi_{\rho_p}, V_{\rho_p})$ is the ordinary KSGNS construction associated with the strict completely positive linear map ρ_p, then there is a unitary operator U_0 in $L_B\left(E_{\rho_p}, F\right)$ such that

$$\Phi_p(a) = U_0\Phi_{\rho_p}(a)U_0^*$$

for all $a \in A_p$ and $W = U_0V_{\rho_p}$.

Let $U = U_0U_p$, where U_p is the unitary operator in $L_B(E_\rho, E_{\rho_p})$ constructed in the proof of the assertion 1. Then U is a unitary operator in $L_B(E_\rho, F)$, and moreover,

$$\begin{aligned}\Phi(a) &= \Phi_p\left(\pi_p(a)\right) = U_0\Phi_{\rho_p}\left(\pi_p(a)\right)U_0^* \\ &= U_0U_p\Phi_\rho(a)U_p^*U_0^* = U\Phi_\rho(a)U^*\end{aligned}$$

for all $a \in A$ and $W = U_0V_{\rho_p} = U_0U_pV_\rho = UV_\rho$.

Step 2. Now we suppose that B is an arbitrary locally C^*-algebra.

1. Let $q \in S(B)$ and let $\rho_q = (\pi_q)_* \circ \rho$. Then ρ_q is a strictly continuous completely positive linear map from A to $L_{B_q}(E_q)$. Let $(E_{\rho_q}, \Phi_{\rho_q}, V_{\rho_q})$ be the KSGNS construction associated with ρ_q by the step 1 of the proof. By Construction 6.3.4, $\{E_{\rho_q}; B_q; \chi_{q_1,q_2}; \pi_{q_1q_2}; q, q_1, q_2 \in S(B), q_1 \geq q_2\}$ is an inverse system of Hilbert C^*-modules. Let $E_\rho = \lim\limits_{\substack{\leftarrow \\ q}} E_{\rho_q}$. For $q_1,\ q_2 \in S(B)$ with $q_1 \geq q_2$, $a, c \in A$, $\xi \in E_{q_2}$ and $\eta \in E_{q_1}$ such that $\sigma^E_{q_1q_2}(\eta) = \xi$ we have:

$$\begin{aligned}(\pi_{q_1q_2})_*\left(\Phi_{\rho_{q_1}}(a)\right)\left(c \otimes_{\rho_{q_2}} \xi\right) &= \chi_{q_1q_2}\left(\Phi_{\rho_{q_1}}(a)\left(c \otimes_{\rho_{q_1}} \eta\right)\right) \\ &= \chi_{q_1q_2}\left(ac \otimes_{\rho_{q_1}} \eta\right) = ac \otimes_{\rho_{q_2}} \xi \\ &= \Phi_{\rho_{q_2}}(a)\left(c \otimes_{\rho_{q_2}} \xi\right)\end{aligned}$$

and

$$\begin{aligned}(\pi_{q_1q_2})_*\left(V_{\rho_{q_1}}\right)\xi &= \chi_{q_1q_2}\left(V_{\rho_{q_1}}\eta\right)=\lim_i \chi_{q_1q_2}\left(e_i\otimes_{\rho_{q_1}}\eta\right)\\ &= \lim_i\left(e_i\otimes_{\rho_{q_2}}\xi\right)=V_{\rho_{q_2}}\xi.\end{aligned}$$

These implies that $\left(\Phi_{\rho_q}(a)\right)_q$ is a coherent sequence in $L_{B_q}(E_{\rho_q})$ for all $a\in A$ and $\left(V_{\rho_q}\right)_q$ is a coherent sequence in $L_{B_q}(E_q,E_{\rho_q})$. Let $V_\rho\in L_B(E,E_\rho)$ such that $(\pi_q)_*\left(V_\rho\right)=V_{\rho_q}$ and for each $a\in A$, let $\Phi_\rho(a)\in L_B(E_\rho)$ such that $(\pi_q)_*\left(\Phi_\rho(a)\right)=\Phi_{\rho_q}(a)$. Thus we have defined a map Φ_ρ from A to $L_B(E_\rho)$. It is not difficult to check that Φ_ρ is a representation of A on E_ρ. Moreover, since

$$\begin{aligned}(\pi_q)_*\left(V_\rho^*\Phi_\rho(a)V_\rho\right) &= (\pi_q)_*\left(V_\rho\right)^*(\pi_q)_*\left(\Phi_\rho(a)\right)(\pi_q)_*\left(V_\rho\right)\\ &= V_{\rho_q}^*\Phi_{\rho_q}(a)V_{\rho_q}=\rho_q(a)=(\pi_q)_*(\rho(a))\end{aligned}$$

for all $q\in S(B)$ and for all $a\in A$, $V_\rho^*\Phi_\rho(a)V_\rho=\rho(a)$ for all $a\in A$.

From

$$\begin{aligned}\overline{\Phi_\rho(A)V_\rho E} &= \lim_{\substack{\leftarrow\\ q}}\overline{\chi_q\left(\Phi_\rho(A)V_\rho E\right)}\\ &= \lim_{\substack{\leftarrow\\ q}}\overline{(\pi_q)_*\left(\Phi_\rho(A)\right)(\pi_q)_*\left(V_\rho\right)\sigma_q^E(E)}\\ &= \lim_{\substack{\leftarrow\\ q}}\overline{\Phi_{\rho_p}(A)V_{\rho_q}E_q}=\lim_{\substack{\leftarrow\\ q}}E_{\rho_q}=E_\rho\end{aligned}$$

we conclude that $\Phi_\rho(A)V_\rho E$ is dense in E_ρ.

2. Let $q\in S(B)$, $\Phi_q=(\pi_q)_*\circ\Phi$ and let $W_q=(\pi_q)_*(W)$. Then Φ_q is a representation of A on F_q;

$$\rho_q(a)=(\pi_q)_*(\rho(a))=(\pi_q)_*(W^*\Phi(a)W)=W_q^*\Phi_q(a)W_q$$

for all $a\in A$ and $\Phi_q(A)W_qE_q$ is dense in F_q, since

$$\Phi_q(A)W_qE_q=(\pi_q)_*(\Phi(A)W)\sigma_q^E(E)=\sigma_q^F(\Phi(A)WE)$$

and $\Phi(A)WE$ is dense in F. Thus, by the first step of the proof, there is a unitary operator U_q in $L_{B_q}(E_{\rho_q},F_q)$ such that

$$\Phi_q(a)=U_q\Phi_{\rho_p}(a)U_q^*$$

for all $a \in A$ and $W_q = U_q V_{\rho_q}$. Let $q_1,\ q_2 \in S(B)$ with $q_1 \geq q_2$, $a \in A$ and $\xi \in E$. Then

$$\begin{aligned}
\left(\pi_{q_1 q_2}\right)_* \left(U_{q_1}\right)\left(\Phi_{\rho_{q_2}}(a)V_{\rho_{q_2}}\sigma_{q_2}^E(\xi)\right) &= \left(\pi_{q_1 q_2}\right)_* \left(U_{q_1}\right)\left(\left(\pi_{q_1 q_2}\right)_* \left(\Phi_{\rho_{q_1}}(a)V_{\rho_{q_1}}\right)\sigma_{q_2}^E(\xi)\right) \\
&= \left(\pi_{q_1 q_2}\right)_* \left(U_{q_1}\Phi_{\rho_{q_1}}(a)V_{\rho_{q_1}}\right)\sigma_{q_2}^E(\xi) \\
&= \left(\pi_{q_1 q_2}\right)_* \left(\Phi_{q_1}(a)\, W_{q_1}\right)\sigma_{q_2}^E(\xi) \\
&= \Phi_{q_2}(a)\, W_{q_2}\sigma_{q_2}^E(\xi) = U_{q_2}\left(\Phi_{\rho_{\rho_2}}(a)V_{\rho_{q_2}}\sigma_{q_2}^E(\xi)\right).
\end{aligned}$$

From this relation, since $\Phi_{\rho_{q_2}}(A)V_{\rho_{q_2}}\sigma_{q_2}^E(E)$ is dense in $E_{\rho_{q_2}}$, we conclude that $\left(U_q\right)_q$ is a coherent sequence in $L_{B_q}(E_{\rho_q}, F_q)$. Let $U \in L\left(E_\rho, F\right)$ such that $\left(\pi_q\right)_*(U) = U_q$ for all $q \in S(B)$. From

$$\begin{aligned}
\left(\pi_q\right)_*\left(\Phi\left(a\right)\right) &= \Phi_q\left(a\right) = U_q \Phi_{\rho_p}(a) U_q^* \\
&= \left(\pi_q\right)_*(U)\left(\pi_q\right)_*\left(\Phi_\rho(a)\right)\left(\pi_q\right)_*(U^*) \\
&= \left(\pi_q\right)_*\left(U\Phi_\rho(a)U^*\right)
\end{aligned}$$

for all $q \in S(B)$ and for all $a \in A$ and

$$\begin{aligned}
\left(\pi_q\right)_*(W) &= W_q = U_q V_{\rho_q} = \left(\pi_q\right)_*(U)\left(\pi_q\right)_*\left(V_\rho\right) \\
&= \left(\pi_q\right)_*\left(UV_\rho\right)
\end{aligned}$$

for all $q \in S(B)$, we conclude that

$$\Phi\left(a\right) = U\Phi_\rho(a)U^*$$

for all $a \in A$ and $W = UV_\rho$. ■

If $\rho : A \to L_B(E)$ is a continuous completely positive linear map which is not strict but $\{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(L_B(E))$ for some approximate unit $\{e_i\}_{i\in I}$ of A, then we can find a representation Φ_ρ of A on a Hilbert B-module E_ρ and an element V_ρ in $L_B(E, E_\rho)$ such that $\rho(a) = V_\rho^* \Phi_\rho(a) V_\rho$ for every $a \in A$. In this case the $*$-representation Φ_ρ is not non-degenerate.

Corollary 6.3.7 *Let A and B be two locally C^*-algebras, let E be a Hilbert B-module and let $\rho : A \to L_B(E)$ be a continuous completely positive linear map* *such that for some approximate unit $\{e_i\}_{i\in I}$ of $A, \{\rho(e_i)\}_{i\in I}$ is a bounded net in $b(L_B(E))$. Then there is a Hilbert B-module E_ρ, a representation Φ_ρ of A on E_ρ, and an element V_ρ in $L_B(E, E_\rho)$ such that*

$$\rho(a) = V_\rho^* \Phi_\rho(a) V_\rho$$

for every $a \in A$.

Proof. According to Proposition 6.2.5, there is a continuous completely positive linear map ρ^+ from A^+ into $L_B(E)$ such that $\rho^+|_A = \rho$ which is strict by Remark 6.3.2. Then, according to Theorem 6.3.5, there is a Hilbert B-module E_ρ, a representation Φ_{ρ^+} of A^+ on E_ρ, and an element V_ρ in $L_B(E, E_\rho)$ such that

$$\rho^+(a) = V_\rho^* \Phi_{\rho^+}(a) V_\rho$$

for every $a \in A^+$. Let $\Phi_\rho = \Phi_{\rho^+}|_A$. Then Φ_ρ is a representation of A on E_ρ and $\rho(a) = V_\rho^* \Phi_\rho(a) V_\rho$ for every $a \in A$. ■

Corollary 6.3.8 *Let A and B be two locally C^*-algebras, let E be a Hilbert B-module and let $\rho : A \to L_B(E)$. Then the following statements are equivalent:*

1. *ρ is a continuous strict completely positive linear map;*

2. *There is a unique continuous completely positive linear map $\overline{\rho} : M(A) \to L_B(E)$ such that:*

  (**a**) *$\overline{\rho}|_A = \rho$;*

  (**b**) *$\overline{\rho}|_C$ is strictly continuous whenever C is a bounded selfadjoint subset of $M(A)$.*

Proof. 1. $\Rightarrow$ 2. Let $(E_\rho, \Phi_\rho, V_\rho)$ be the KSGNS construction associated with ρ. Since Φ_ρ is non-degenerate, there is a unique continuous $*$-morphism $\overline{\Phi_\rho}$: $M(A) \to L_B(E_\rho)$ such that $\overline{\Phi_\rho}|_A = \Phi_\rho$ and $\overline{\Phi_\rho}|_C$ is strictly continuous whenever C is a bounded selfadjoint subset of $M(A)$. Evidently the map $\overline{\rho} : M(A) \to$

$L_B(E)$ defined by $\overline{\rho}(a) = V_\rho^* \overline{\Phi_\rho}(a) V_\rho$ is a continuous completely positive linear map which satisfies the conditions (a) and (b).

To show the uniqueness of $\overline{\rho}$, let $\widetilde{\rho} : M(A) \to L_B(E)$ be another continuous completely positive linear map which satisfies the conditions (a) and (b). Let $\{e_i\}_{i \in I}$ be an approximate unit of A and let a be a selfadjoint element in $M(A)$. Then, since $\{e_i a e_i\}_{i \in I}$ is a bounded selfadjoint net in A and since it converges strictly to a, $\overline{\rho}(a) = \widetilde{\rho}(a)$. Therefore $\overline{\rho} = \widetilde{\rho}$.

2. $\Rightarrow$ 1. From (a) it follows that ρ is a continuous completely positive linear map. If $\{e_i\}_{i \in I}$ is an approximate unit for A, then, from (b), since $\{e_i\}_{i \in I} \cup \{1_{M(A)}\}$ is a bounded selfadjoint subset of $M(A)$, and since the net $\{e_i\}_{i \in I}$ converges strictly to $1_{M(A)}$, the $\{\rho(e_i)\}_{i \in I}$ converges strictly to $\overline{\rho}(1_{M(A)})$. This shows that ρ is a continuous strict completely positive linear map. ■

Remark 6.3.9 *If ρ is a strict, continuous completely positive linear map from A to $L_B(E)$, then for any approximate unit $\{e_i\}_{i \in I}$ for A, the net $\{\rho(e_i)\}_{i \in I}$ is strictly Cauchy in $L_B(E)$.*

Definition 6.3.10 *Let A and B be two locally C^*-algebras, let E be a Hilbert B-module. A continuous completely positive linear map ρ from A to $L_B(E)$ is non-degenerate if the net $\{\rho(e_i)\}_{i \in I}$ converges strictly to id_E, for some approximate unit $\{e_i\}_{i \in I}$ for A.*

Remark 6.3.11 *Any non-degenerate, continuous completely positive linear map ρ from A to $L_B(E)$ is strict.*

From Corollary 6.3.8 and Remark 6.3.9 we obtain the following corollary.

Corollary 6.3.12 *Let A and B be two locally C^*-algebras, let E be a Hilbert B-module and let ρ be a continuous completely positive linear map from A to $L_B(E)$. Then the following statements are equivalent:*

1. *ρ is non-degenerate;*

2. *There is a unique unital, continuous completely positive linear map $\overline{\rho}$: $M(A) \to L_B(E)$ such that:*

(a) $\overline{\rho}|_A = \rho$;

(b) $\overline{\rho}|_C$ *is strictly continuous whenever C is a bounded selfadjoint subset of $M(A)$.*

Remark 6.3.13 *If ρ is a non-degenerate, continuous completely positive linear map from A to $L_B(E)$, then for any approximate unit $\{e_i\}_{i\in I}$ for A, the net $\{\rho(e_i)\}_{i\in I}$ converges strictly to id_E.*

The following theorem is an analogue of the generalized Stinespring theorem on dilatations of completely positive maps between C^*-algebras (Theorem 3 in [**26**]) in the context of locally C^*-algebras.

Theorem 6.3.14 *Let A and B be two locally C^*-algebras such that A is separable and B has a countable approximate unit and let ρ be a non-degenerate, continuous completely positive linear map from A to $L_B(H_B)$. Then there is a non-degenerate, continuous $*$-morphism Φ from A to the 2×2 matrix algebra $M_2(L_B(H_B))$ such that $\rho(a) = (\Phi(a))_{11}$, the $(1,1)$ -entry of the matrix $\Phi(a)$, for all $a \in A$. If moreover, A is metrizable and $b(A) = A$ as set, the morphism Φ is faithful.*

Proof. To proof this theorem, we use the same arguments as in the proof of the generalized Stinespring theorem on dilatations of completely positive maps between C^*-algebras (see, for example, [**29**] Theorem 6.5).

Let $(E_\rho, \Phi_\rho, V_\rho)$ be the KSGNS construction associated with ρ. Since ρ is non-degenerate, $V_\rho^* V_\rho = \mathrm{id}_{H_B}$ and so V_ρ is a partial isometry from H_B to E_ρ. Then, by Corollary 3.2.7, the range of V_ρ is a complemented submodule of E_ρ. On the other hand, from the construction of E_ρ (Construction 6.3.4) and taking into account that A is separable and H_B is countably generated we deduce that E_ρ is countably generated, and so the range of V_ρ is a countably generated Hilbert

B -module. Let F =ranV_ρ and let $P \in L_B(E_\rho, F)$ be the projection of E_ρ on F. By Theorem 5.2.7, the Hilbert B -modules $F^\perp \oplus H_B$ and H_B are isomorphic. Let W be a unitary operator from H_B onto $F^\perp \oplus H_B$. Then $PV_\rho \oplus W$ is a unitary operator from $H_B \oplus H_B$ onto $E_\rho \oplus H_B$. Let Φ_1 be the non-degenerate representation of A on H_B constructed in Proposition 6.1.6. Then the map Φ from A to $L_B(H_B \oplus H_B)$ defined by

$$\Phi(a) = \left(V_\rho^* P^* \oplus W^*\right)\left(\Phi_\rho(a) \oplus \Phi_1(a)\right)\left(PV_\rho \oplus W\right)$$

is a non-degenerate representation of A on $H_B \oplus H_B$. Identifying $L_B(H_B \oplus H_B)$ with $M_2(L_B(H_B))$, it is not difficult to see that

$$(\Phi(a))_{11} = V_\rho^* \Phi_\rho(a) V_\rho = \rho(a)$$

for all $a \in A$.

If A is metrizable and $A = b(A)$ as set, then by Proposition 6.1.6, the representation Φ_1 of A on H_B is faithful and since Φ is unitarily equivalent with $\Phi_\rho \oplus \Phi_1$, Φ is faithful. ■

As in the case of C^*-algebras, if ρ is degenerate, then the representation Φ is not non-degenerate.

Proposition 6.3.15 *Let A and B be two locally C^*-algebras such that A is separable and B has a countable approximate unit and let ρ be a continuous completely positive linear map from A to $L(H_B)$ such that $\{\rho(e_i)\}_i$ is a bounded net in $b(L(H_B))$ for some approximate unit $\{e_i\}_i$ of A. Then there is a continuous $*$ -morphism Φ from A to $M_2(L(H_B))$ such that $\rho(a) = (\Phi(a))_{11}$ for all $a \in A$. If moreover, A is metrizable and $A = b(A)$ as set, then Φ is faithful.*

Proof. By Proposition 6.2.5, ρ extends to a continuous completely positive linear map ρ^+ from A^+ to $L_B(E)$ such that $\rho^+(1) = \left\|\rho|_{b(A)}\right\| \mathrm{id}_{H_B}$. Let $\widetilde{\rho} = \frac{1}{\left\|\rho|_{b(A)}\right\|}\rho^+$. Then $\widetilde{\rho}$ is a non-degenerate, continuous completely positive linear map from A_1 to $L_B(H_B)$ and by Theorem 6.3.14 there is a non-degenerate, continuous $*$-morphism $\widetilde{\Phi}$ from A_1 to the 2×2 matrix algebra $M_2(L_B(H_B))$

such that $\widetilde{\rho}(a) = \left(\widetilde{\Phi}(a)\right)_{11}$, the $(1,1)$ -entry of the matrix $\widetilde{\Phi}(a)$, for all $a \in A_1$. If moreover, A is metrizable and $b(A) = A$ as set, the morphism $\widetilde{\Phi}$ is faithful.

Let $\Phi = \widetilde{\Phi}|_A$. Then Φ is a continuous $*$ -morphism Φ from A to $M_2(L(H_B))$ such that $\rho(a) = (\Phi(a))_{11}$ for all $a \in A$. If moreover, A is metrizable and $A = b(A)$ as set, then Φ is faithful. ■

References for Section 6.3 : **[2]**, **[15]**, **[17]**, **[19]**, **[26]**, **[29]**,**[44]**.

Chapter 7

Induced representations of locally C^*-algebras

7.1 Definitions, notation

In this Section we give some facts about representations of locally C^*- algebras on Hilbert spaces which will be necessary for the study of induced representations of locally C^*-algebras.

Proposition 7.1.1 *Let A be a locally C^* -algebra and φ be a representation of A on a Hilbert space H. Then there is $p \in S(A)$ and a representation φ_p of A_p on H such that $\varphi = \varphi_p \circ \pi_p$.*

Proof. Since φ is a continuous $*$- morphism from A to $L(H)$, there is $p \in S(A)$ such that

$$\|\varphi(a)\| \leq p(a)$$

for all $a \in A$. Then there is a map $\varphi_p : A_p \to L(H)$ such that

$$\varphi_p(\pi_p(a)) = \varphi(a)$$

for all $a \in A$. Clearly φ_p is a representation of A_p on H. ■

We say that φ_p is a representation of A_p associated to φ.

Remark 7.1.2 *If φ_p is a representation of A_p on H, then $\varphi_p \circ \pi_p$ is a representation of A on H.*

Remark 7.1.3 *Any locally C^* -algebra admits a representation on a Hilbert space.*

Remark 7.1.4 *Let φ be a representation of A on a Hilbert space H and let φ_p be the representation of A_p associated to φ. Then φ is non-degenerate if and only if φ_p is non-degenerate.*

A representation φ of A on H is faithful if $\varphi(a) = 0$ implies that $a = 0$.

Remark 7.1.5 *If φ is a faithful representation of A on H and φ_p is a representation of A_p associated to φ, then φ_p is faithful.*

Definition 7.1.6 *Let φ be a representation of A on the Hilbert space H. We say that φ is irreducible if the only invariant subspaces of H under $\varphi(A)$ are H itself and $\{0\}$.*

Remark 7.1.7 *Let φ be a representation of A on a Hilbert space H and let φ_p be a representation of A_p associated to φ. Then φ is irreducible if and only if φ_p is irreducible.*

Definition 7.1.8 *Let φ and ψ be two representations of A on the Hilbert spaces H_φ respectively H_ψ. We say that φ and ψ are unitarily equivalent if there is a unitary operator U from H_φ to H_ψ such that $U\varphi(a) = \psi(a)U$ for all $a \in A$.*

Proposition 7.1.9 *Let φ and ψ be two representations of A on the Hilbert spaces H_φ respectively H_ψ,which are unitarily equivalent. Then there is $p \in S(A)$ and there are two representations φ_p and ψ_q of A_p associated to φ respectively ψ such that φ_p is unitarily equivalent to ψ_q.*

Proof. Let U be a unitary operator from H_φ to H_ψ such that $U\varphi(a) = \psi(a)U$ for all $a \in A$, let φ_r be a representation of A_r associated to φ and let ψ_q be a representation of A_q associated to ψ.

Since $S(A)$ is directed, there is $p \in S(A)$ such that $p \geq r$ and $p \geq q$. Let $\varphi_p = \varphi_r \circ \pi_{pr}$ and $\psi_p = \psi_q \circ \pi_{pq}$. Clearly, φ_p and ψ_p are representations of A_p on H_φ respectively H_ψ. Moreover, since

$$\varphi = \varphi_r \circ \pi_r = \varphi_r \circ \pi_{pr} \circ \pi_p$$

and

$$\psi = \psi_q \circ \pi_q = \psi_q \circ \pi_{pq} \circ \pi_p$$

φ_p and ψ_p are representations of A_p associated with φ respectively ψ. From

$$U\varphi_p(\pi_p(a)) = U\varphi(a) = \psi(a)\, U = \psi_p(\pi_p(a))U$$

for all $a \in A$, we conclude that φ_p and ψ_p are unitarily equivalent. ■

References for Section 7.1: **[7]**.

7.2 Induced representations

In this Section, by analogy with the case of C^* -algebras, we introduce the notion of induced representation of a locally C^*-algebra and we present some properties of the induced representations of locally C^*-algebras.

Let B be a locally C^*-algebra, let E be a Hilbert B -module and let φ be a non-degenerate representation of B on a Hilbert space H. Then the interior tensor product $E \otimes_\varphi H$ of E and H using φ is a Hilbert space and the map φ_* from $L_B(E)$ to $L\left(E \otimes_\varphi H\right)$ defined by $\varphi_*(T)\left(\xi \otimes_\varphi h\right) = T\xi \otimes_\varphi h$ is a representation of $L_B(E)$ on $E \otimes_\varphi H$. Moreover, φ_* is non-degenerate.

Let A be a locally C^*-algebra and let Φ be a non-degenerate representation of A on E. Then $\varphi_* \circ \Phi$ is a non-degenerate representation of A on $E \otimes_\varphi H$.

Definition 7.2.1 *The representation $\varphi_* \circ \Phi$ of A on $E \otimes_\varphi H$ constructed above is called the Rieffel-induced representation from B to A via E, and it will be denoted by E -$\mathrm{Ind}_B^A \varphi$.*

Remark 7.2.2 *If the representation φ of B is faithful, then by Proposition 4.3.4, φ_* is a faithful representation of $L_B(E)$ on $E \otimes_\varphi H$. Therefore, if φ and Φ are faithful, then the Rieffel-induced representation E -Ind$_B^A\varphi$ is faithful.*

Proposition 7.2.3 *Let φ_1 and φ_2 be two non-degenerate representations of B on H_1 respectively H_2. If the representations φ_1 and φ_2 of B are unitarily equivalent, then the representations E -Ind$_B^A\varphi_1$ and E -Ind$_B^A\varphi_2$ of A are unitarily equivalent.*

Proof. Let U be a unitary operator from H_1 to H_2 such that $U\varphi_1(b) = \varphi_2(b)U$ for all $b \in B$. Define a linear map V from $E \otimes_{\text{alg}} H_1$ to $E \otimes_{\text{alg}} H_2$ by

$$V\left(\xi \otimes h\right) = \xi \otimes Uh.$$

Since

$$\begin{aligned}\left\langle V\left(\xi \otimes h\right), V\left(\xi \otimes h\right)\right\rangle_{\varphi_2}^0 &= \left\langle Uh, \varphi_2\left(\langle \xi, \xi\rangle\right) Uh\right\rangle = \left\langle Uh, U\varphi_1\left(\langle \xi, \xi\rangle\right) h\right\rangle \\ &= \left\langle h, \varphi_1\left(\langle \xi, \xi\rangle\right) h\right\rangle = \left\langle V\xi \otimes h, \xi \otimes h\right\rangle_{\varphi_1}^0\end{aligned}$$

for all $\xi \in E$ and for all $h \in H_1$, and since $V\left(E \otimes_{\text{alg}} H_1\right) = E \otimes_{\text{alg}} H_2$, V can be extended to an isometric, surjective, linear map V from $E \otimes_{\varphi_1} H_1$ onto $E \otimes_{\varphi_2} H_2$ such that

$$V\left(\xi \otimes_{\varphi_1} h\right) = \xi \otimes_{\varphi_2} Uh.$$

Therefore V is a unitary operator from $E \otimes_{\varphi_1} H_1$ onto $E \otimes_{\varphi_2} H_2$, and since

$$\begin{aligned}\left(V\left(E\text{-Ind}_B^A\varphi_1\right)(a)\right)\left(\xi \otimes_{\varphi_1} h\right) &= V\left(\Phi\left(a\right)\xi \otimes_{\varphi_1} h\right) = \Phi\left(a\right)\xi \otimes_{\varphi_2} Uh \\ &= \left(E\text{-Ind}_B^A\varphi_2\right)(a)\left(\xi \otimes_{\varphi_2} Uh\right) \\ &= \left(\left(E\text{-Ind}_B^A\varphi_2\right)(a)\, V\right)\left(\xi \otimes_{\varphi_1} h\right)\end{aligned}$$

for all $\xi \in E$, for all $h \in H$ and for all $a \in A$, the representations E -Ind$_B^A\varphi_1$ and E -Ind$_B^A\varphi_2$ of A are unitarily equivalent. ■

Proposition 7.2.4 *Let F be a Hilbert B -module which is isomorphic with E. If U is a unitary operator in $L_B(E, F)$, then the map Ψ from A to $L_B(E)$ defined* *by* $\Psi(a) = U\Phi(a)U^*$ *is a non-degenerate representation of* A *on* F *and the representations* E *-Ind*$_B^A\varphi$ *and* F *-Ind*$_B^A\varphi$ *of* A *are unitarily equivalent.*

Proof. It is not difficult to check that Ψ is a non-degenerate representation of A on F. Consider the linear map W from $E \otimes_{\text{alg}} H$ onto $F \otimes_{\text{alg}} H$ defined by

$$W(\xi \otimes h) = U\xi \otimes h.$$

Since

$$\begin{aligned}\langle W(\xi \otimes h), W(\xi \otimes h)\rangle_\varphi^0 &= \langle h, \varphi(\langle U\xi, U\xi\rangle) h\rangle = \langle h, \varphi(\langle \xi, \xi\rangle) h\rangle \\ &= \langle \xi \otimes h, \xi \otimes h\rangle_\varphi^0\end{aligned}$$

for all $\xi \in E$ and for all $h \in H$, and since $W(E \otimes_{\text{alg}} H) = F \otimes_{\text{alg}} H$, W can be extended to an isometric, surjective linear map W from $E \otimes_\varphi H$ onto $F \otimes_\varphi H$ such that

$$W(\xi \otimes_\varphi h) = U\xi \otimes_\varphi h$$

for all $\xi \in E$ and for all $h \in H$. Therefore W is a unitary operator from $E \otimes_\varphi H$ onto $F \otimes_\varphi H$, and since

$$\begin{aligned}\left(\left(F\text{-Ind}_B^A\varphi\right)(a)\right) W\left(\xi \otimes_\varphi h\right) &= \left(F\text{-Ind}_B^A\varphi\right)(a)\left(U\xi \otimes_\varphi h\right) = \Psi(a)U\xi \otimes_\varphi h \\ &= U\Phi(a)\xi \otimes_\varphi h = W\left(\Phi(a)\xi \otimes_\varphi h\right) \\ &= \left(W\left(E\text{-Ind}_B^A\varphi\right)(a)\right)\left(\xi \otimes_\varphi h\right)\end{aligned}$$

for all $\xi \in E$, for all $h \in H$ and for all $a \in A$, the representations E -Ind$_B^A\varphi$ and F -Ind$_B^A\varphi$ of A are unitarily equivalent. ■

Lemma 7.2.5 *Let* φ *be a non-degenerate representation of* B *on the Hilbert space* H*. If* φ_q *is a non-degenerate representation of* B_q *associated to* φ*, then there is* $p \in S(A)$ *such that* A_p *acts non-degenerately on* E_q *and the representations* E*-Ind*$_B^A\varphi$ *and* $\left(E_q\text{-}Ind_{B_q}^{A_p}\varphi_q\right) \circ \pi_p$ *of* A *are unitarily equivalent.*

Proof. Define a linear map U from $E \otimes_{\text{alg}} H$ into $E_q \otimes_{\text{alg}} H$ by

$$U(\xi \otimes h) = \sigma_q^E(\xi) \otimes h.$$

Since

$$\begin{aligned}\langle U(\xi\otimes h), U(\xi\otimes h)\rangle^0_{\varphi_q} &= \left\langle h, \varphi_q\left(\left\langle \sigma^E_q(\xi), \sigma E_q(\xi)\right\rangle\right) h\right\rangle \\ &= \left\langle h, \left(\varphi_q\circ\pi_q\right)\left(\langle\xi,\xi\rangle_E\right) h\right\rangle \\ &= \langle \xi\otimes h, \xi\otimes h\rangle^0_{\varphi}\end{aligned}$$

for all $\xi\in E$ and $h\in H$, and since $U(E\otimes_{\text{alg}} H) = E_q\otimes_{\text{alg}} H$, U can be extended to an isometric, surjective, linear map U from $E\otimes_\varphi H$ onto $E_q\otimes_{\varphi_q} H$ such that

$$U\left(\xi\otimes_\varphi h\right) = \sigma^E_q(\xi)\otimes_{\varphi_q} h$$

for all $\xi\in E$ and $h\in H$. Therefore U is a unitary operator from $E\otimes_\varphi H$ onto $E_q\otimes_{\varphi_q} H$. Moreover, $U\varphi_*(T) = \left(\varphi_q\right)_*\left((\pi_q)_*(T)\right)U$ for all $T\in L_B(E)$, since

$$\begin{aligned}(U\varphi_*(T))\left(\xi\otimes_\varphi h\right) &= U\left(T\xi\otimes_\varphi h\right) = \sigma^E_q(T\xi)\otimes_{\varphi_q} h \\ &= (\pi_q)_*(T)\sigma^E_q(\xi)\otimes_{\varphi_q} h \\ &= \left(\varphi_q\right)_*\left((\pi_q)_*(T)\right)\left(\sigma^E_q(\xi)\otimes_{\varphi_q} h\right) \\ &= \left(\left(\varphi_q\right)_*\left((\pi_q)_*(T)\right)U\right)\left(\xi\otimes_\varphi h\right)\end{aligned}$$

for all $\xi\in E$ and $h\in H$.

The continuity of Φ implies that there is $p\in S(A)$ such that $\widetilde{q}(\Phi(a))\leq p(a)$ for all $a\in A$. Therefore there is a morphism of C^* -algebras Φ_p from A_p to $L_{B_q}(E_q)$ such that $\Phi_p\circ\pi_p = (\pi_q)_*\circ\Phi$. Moreover, since $\Phi_p(A_p)E_q = (\pi_q)_*(\Phi(A))\,\sigma^E_q(E) = \sigma^E_q(\Phi(A)E)$, Φ_p is a non-degenerate representation of A_p on E_q. From

$$\begin{aligned}U\left(E\text{-Ind}^A_B\varphi\right)(a) &= U\varphi_*(\Phi(a)) = \left(\varphi_q\right)_*\left((\pi_q)_*(\Phi(a))\right)U \\ &= \left(\varphi_q\right)_*(\Phi_p(\pi_p(a)))\,U \\ &= \left(\left(E_q\text{-Ind}^{A_p}_{B_q}\varphi_q\right)\circ\pi_p\right)(a)\,U\end{aligned}$$

for all $a\in A$, we conclude that the representations $E\text{-Ind}^A_B\varphi$ and $\left(E_q\text{-Ind}^{A_p}_{B_q}\varphi_q\right)\circ\pi_p$ of A are unitarily equivalent and the lemma is proved ■

Proposition 7.2.6 *Let φ be a non-degenerate representation of B on a Hilbert space H such that $\varphi = \bigoplus_{i\in I} \varphi_i$, where φ_i is a non-degenerate representation of B on the Hilbert space H_i, for each $i \in I$. Then the representations E-Ind$_B^A\varphi$ and $\bigoplus_{i\in I} \left(E\text{-}Ind_B^A\varphi_i\right)$ of A are unitarily equivalent.*

Proof. Let φ_q be a representation of B_q associated to φ. Then, for each $i \in I$, there is a representation φ_{iq} of B_q on the Hilbert space H_i such that $\varphi_{iq} \circ \pi_q = \varphi_i$. Moreover, $\varphi_q = \bigoplus_{i\in I} \varphi_{iq}$. By Lemma 7.2.5, there is $p \in S(A)$ such that the representations $E\text{-Ind}_B^A\varphi$ and $\left(E_q\text{-Ind}_{B_q}^{A_p}\varphi_q\right) \circ \pi_p$ of A are unitarily equivalent as well as the representations $E\text{-Ind}_B^A\varphi_i$ and $\left(E_q\text{-Ind}_{B_q}^{A_p}\varphi_{iq}\right) \circ \pi_p$ for all $i \in I$.

On the other hand, the representations $E_q\text{-Ind}_{B_q}^{A_p}\varphi_q$ and $\bigoplus_{i\in I} \left(E_q\text{-Ind}_{B_q}^{A_p}\varphi_{iq}\right)$of A_p are unitarily equivalent, Corollary 5.4, [**41**]. This implies that the representations $\left(E_q\text{-Ind}_{B_q}^{A_p}\varphi_q\right) \circ \pi_p$ and $\bigoplus_{i\in I} \left(E_q\text{-Ind}_{B_q}^{A_p}\varphi_{iq}\right) \circ \pi_p$ of A are unitarily equivalent.

From these facts, we conclude that the representations $E\text{-Ind}_B^A\varphi$ and $\bigoplus_{i\in I} \left(E\text{-Ind}_B^A\varphi_i\right)$ of A are unitarily equivalent and the proposition is proved. ∎

Let A, B and C be three locally C^*-algebras, let Φ_1 be a non-degenerate representation of A on a Hilbert B -module E and let Φ_2 be a non-degenerate representation of B on a Hilbert C-module F. The inner tensor product $E \otimes_{\Phi_2} F$ of E and F using Φ_2 is isomorphic with the Hilbert C-module $\lim\limits_{\substack{\leftarrow \\ r\in S(C)}} E \otimes_{\Phi_{2r}} F_r$, where $\Phi_{2r} = (\pi_r)_* \circ \Phi_2$, and the locally C^*-algebras $L_C(E \otimes_{\Phi_2} F)$ and $\lim\limits_{\substack{\leftarrow \\ r\in S(C)}} L_{C_r}(E \otimes_{\Phi_{2r}} F_r)$ are isomorphic as well as $K_C(E \otimes_{\Phi_2} F)$ and $\lim\limits_{\substack{\leftarrow \\ r\in S(C)}} K_{C_r}(E \otimes_{\Phi_{2r}} F_r)$, Proposition 4.3.3. Moreover, the continuous $*$ -morphism $(\Phi_2)_*$ from $L_B(E)$ to $L_C(E \otimes_{\Phi_2} F)$ defined by $(\Phi_2)_* (T) (\xi \otimes_{\Phi_2} \eta) = T\xi \otimes_{\Phi_2} \eta$ is a non-degenerate representation of $L_B(E)$ on $E \otimes_{\Phi_2} F$. Let $\Phi = (\Phi_2)_* \circ \Phi_1$. Then Φ is a non-degenerate representation of A on $E \otimes_{\Phi_2} F$.

Let φ be a non-degenerate representation of C on a Hilbert space H. Then φ induces a non-degenerate representation of A via $E \otimes_{\Phi_2} F$ and a non-degenerate representation of B via F which induces a non-degenerate representation of A

via E. As in the case of induced representations of C^*-algebras, we will show that these representations of A are unitarily equivalent.

Theorem 7.2.7 *Let A, B, C, E, F, Φ_1 and Φ_2 be as above. If φ is a non-degenerate representation of C, then the representations $(E \otimes_{\Phi_2} F)$-$\mathrm{Ind}_C^A\varphi$ and E-$\mathrm{Ind}_B^A(F$-$\mathrm{Ind}_C^B\varphi)$ of A are unitarily equivalent.*

Proof. Let φ_r be a non-degenerate representation of C_r associated to φ. Then there is $q \in S(B)$ and a non-degenerate representation Ψ_{2q} of B_q on F_r such that $\Psi_{2q} \circ \pi_q = (\pi_r)_* \circ \Phi_2$ and there is $p \in S(A)$ and two non-degenerate representations Ψ_{1p} and Φ_p of A_p on E_q respectively $(E \otimes_{\Phi_2} F)_r$ such that $\Psi_{1p} \circ \pi_p = (\pi_q)_* \circ \Phi_1$ respectively $\Phi_p \circ \pi_p = (\pi_r)_* \circ \Phi$. According to Lemma 7.2.5, the representations $(E \otimes_{\Phi_2} F)$-$\mathrm{Ind}_C^A\varphi$ and $\left((E \otimes_{\Phi_2} F)_r\text{-}\mathrm{Ind}_{C_r}^{A_p}\varphi_r\right) \circ \pi_p$ of A are unitarily equivalent as well as the representations F-$\mathrm{Ind}_C^B\varphi$ and $\left(F_r\text{-}\mathrm{Ind}_{C_r}^{B_q}\varphi_r\right) \circ \pi_q$ of B. Since the representations F-$\mathrm{Ind}_C^B\varphi$ and $\left(F_r\text{-}\mathrm{Ind}_{C_r}^{B_q}\varphi_r\right) \circ \pi_q$ of B are unitarilly equivalent, from Lemma 7.2.5 and Proposition 7.2.3, we conclude that the representations E-$\mathrm{Ind}_B^A\left(F\text{-}\mathrm{Ind}_C^B\varphi\right)$ and $\left(E_q\text{-}\mathrm{Ind}_B^A\left(F_r\text{-}\mathrm{Ind}_{C_r}^{B_q}\varphi_r\right)\right) \circ \pi_p$ of A are unitarily equivalent.

To prove the theorem it is sufficient to show that the representations $(E\otimes_{\Phi_2} F)_r$-$\mathrm{Ind}_{C_r}^{A_p}\varphi_r$ and E_q-$\mathrm{Ind}_B^A\left(F_r\text{-}\mathrm{Ind}_{C_r}^{B_q}\varphi_r\right)$ of A_p are unitarily equivalent. But, according to Theorem 5.9, [**41**], the representations X_r-$\mathrm{Ind}_{C_r}^{A_p}\varphi_r$, where $X_r = E_q \otimes_{\Psi_{2q}} F_r$, and E_q-$\mathrm{Ind}_B^A\left(F_r\text{-}\mathrm{Ind}_{C_r}^{B_q}\varphi_r\right)$ of A_p are unitarily equivalent. Thus, taking into account Proposition 7.2.4, it is enough to show that there is a unitary operator U in $L_{C_r}(\,(E \otimes_{\Phi_2} F)_r\,, X_r)$ such that $\Phi_p(\pi_p(a)) = U^*\left((\Psi_{2q})_* \circ \Psi_{1p}\right)(\pi_p(a)\,U$ for all $a \in A$. Define a linear map U from $E \otimes_{\mathrm{alg}} F_r$ to $E_q \otimes_{\mathrm{alg}} F_r$ by

$$U\left(\xi \otimes \eta_r\right) = \sigma_q^E\left(\xi\right) \otimes \eta_r.$$

Since

$$U\left(\left(\xi \otimes \eta_r\right) c_r\right) = U\left(\xi \otimes \eta_r c_r\right) = \sigma_q^E\left(\xi\right) \otimes \eta_r c_r = U\left(\xi \otimes \eta_r\right) c_r$$

and

$$
\begin{aligned}
\langle U(\xi \otimes \eta_r), U(\xi \otimes \eta_r)\rangle^0_{\Psi_{2q}} &= \left\langle \eta_r, \Psi_{2q}\left(\left\langle \sigma_q^E(\xi), \sigma_q^E(\xi)\right\rangle\right)\eta_r\right\rangle \\
&= \langle \eta_r, (\Psi_{2q} \circ \pi_q)(\langle \xi, \xi\rangle)\eta_r\rangle \\
&= \langle \eta_r, \Phi_{2r}(\langle \xi, \xi\rangle)\eta_r\rangle \\
&= \langle \xi \otimes \eta_r, \xi \otimes \eta_r\rangle^0_{\Phi_{2r}}
\end{aligned}
$$

for all $\xi \in E$, for all $\eta_r \in F_r$ and for all $c_r \in C_r$, and since $U(E \otimes_{\text{alg}} F_r) = E_q \otimes_{\text{alg}} F_r$, the linear map U can be extended to an isometric, surjective C_r -linear map U from $E \otimes_{\Phi_{2r}} F_r$ onto $E \otimes_{\Psi_{2q}} F_r$ such that

$$
U(\xi \otimes_{\Phi_{2r}} \eta_r) = \sigma_q^E(\xi) \otimes_{\Psi_{2q}} \eta_r
$$

for all $\xi \in E$ and for all $\eta_r \in F_r$, which, according to Theorem 3.5 in [**29**], U is a unitary operator in $L_{C_r}(E \otimes_{\Phi_{2r}} F_r, E \otimes_{\Psi_{2q}} F_r)$. Let $a \in A$. Then , since

$$
\begin{aligned}
\left(U\Phi_p(\pi_p(a))\right)\left(\xi \otimes_{\Phi_{2r}} \sigma_r^F(\eta)\right) &= \left(U(\pi_r)_*\left((\Phi_2)_*(\Phi_1(a))\right)\right)\left(\xi \otimes_{\Phi_{2r}} \sigma_r^F(\eta)\right) \\
&= U\left(\sigma_r^{E\otimes_{\Phi_2}F}\left((\Phi_2)_*(\Phi_1(a))(\xi \otimes_{\Phi_2} \eta)\right)\right) \\
&= U\left(\sigma_r^{E\otimes_{\Phi_2}F}(\Phi_1(a)\xi \otimes_{\Phi_2} \eta)\right) \\
&= U\left(\Phi_1(a)\xi \otimes_{\Phi_{2r}} \sigma_r^F(\eta)\right) \\
&= \sigma_q^E(\Phi_1(a)\xi) \otimes_{\Psi_{2q}} \sigma_r^F(\eta) \\
&= (\pi_q)_*(\Phi_1(a))\left(\sigma_q^E(\xi)\right) \otimes_{\Psi_{2q}} \sigma_r^F(\eta) \\
&= \Psi_{1p}(\pi_p(a))\left(\sigma_q^E(\xi)\right) \otimes_{\Psi_{2q}} \sigma_r^F(\eta) \\
&= (\Psi_{2q})_*(\Psi_{1p}(\pi_p(a)))\left(\sigma_q^E(\xi) \otimes_{\Psi_{2q}} \sigma_r^F(\eta)\right) \\
&= \left((\Psi_{2q})_* \circ \Psi_{1p}\right)(\pi_p(a))\, U\left(\xi \otimes_{\Phi_{2r}} \sigma_r^F(\eta)\right)
\end{aligned}
$$

for all $\xi \in E$ and for all $\eta \in F$, $\Phi_p(\pi_p(a)) = U^*\left((\Psi_{2q})_* \circ \Psi_{1p}\right)(\pi_p(a))\, U$ and the theorem is proved. ■

References for Section 7.2: **[22], [29], [30], [41].**

7.3 The imprimitivity theorem

In this Section we prove an imprimitivity theorem for induced representations of locally C^*-algebras.

Let A and B be two strong Morita equivalent locally C^*-algebras, and let E be a Hilbert A -module which gives the strong Morita equivalence between A and B. Let $\widetilde{E} = K_A(E, A)$. Then $\widetilde{E}$ can be regarded as Hilbert B-module and it gives the strong Morita equivalence between B and A, Proposition 5.3.4. Moreover, the Hilbert A_p -modules $\widetilde{E_p}$ and $\left(\widetilde{E}\right)_p$ are isomorphic for each $p \in S(A)$.

Lemma 7.3.1 *Let A and B be two locally C^*-algebras. If $A \sim_M B$, then for each $p \in S(A)$ there is $q_p \in S(B)$ such that $A_p \sim_M B_{q_p}$. Moreover, the set $\{q_p \in S(B); p \in S(A)$ and $A_p \sim_M B_{q_p}\}$ is a cofinal subset of $S(B)$.*

Proof. Let E be a Hilbert A -module which gives the strong Morita equivalence between A and B, and let $p \in S(A)$. If Φ is an isomorphism of locally C^*-algebras from B onto $K_A(E)$, then the map $\widetilde{p} \circ \Phi$, denoted by q_p, is a continuous C^*-seminorm on B. Since $\ker \pi_{q_p} = \ker\left((\pi_p)_* \circ \Phi\right)$, there is a unique continuous $*$ -morphism Φ_{q_p} from B_{q_p} onto $K_{A_p}(E_p)$ such that $\Phi_{q_p} \circ \pi_{q_p} = (\pi_p)_* \circ \Phi$. Moreover, Φ_{q_p} is an isomorphism of C^*-algebras, and since E_p is a full Hilbert A_p-module (Remark 5.1.6), we conclude that $A_p \sim_M B_{p_q}$.

To show that $\{q_p \in S(B); p \in S(A)$ and $A_p \sim_M B_{q_p}\}$ is a cofinal subset of $S(B)$, let $q \in S(B)$. Then, since Φ is an isomorphism of locally C^*-algebras, there is $p_0 \in S(A)$ such that

$$q\left(\Phi^{-1}\left(\Phi\left(b\right)\right)\right) \leq \widetilde{p_0}\left(\Phi\left(b\right)\right)$$

for all $b \in B$. But $q\left(\Phi^{-1}\left(\Phi\left(b\right)\right)\right) = q(b)$ and $\widetilde{p_0}\left(\Phi\left(b\right)\right) = q_{p_0}(b)$, and then $q \leq q_{p_0}$. Thus, we shoved that for any $q \in S(B)$ there is $q_{p_0} \in \{q_p \in S(B); p \in S(A)$ and $A_p \sim_M B_{q_p}\}$ such that $q \leq q_{p_0}$ and the lemma is proved. ■

Remark 7.3.2 *If E is a Hilbert A-module which gives the strong Morita equivalence between the locally C^*-algebras A and B, then E_p gives the strong Morita equivalence between the C^*-algebras A_p and B_{q_p}.*

Theorem 7.3.3 *Let A and B be two strong Morita equivalent locally C^*-algebras, and let φ be a non-degenerate representation of A on a Hilbert space H. Then the representations φ and $\widetilde{E}$-$Ind_B^A\left(E\text{-}Ind_A^B\varphi\right)$ of A, where E is a Hilbert A -module which gives the strong Morita equivalence between A and B, are unitarily equivalent.*

Proof. Let φ_p be a non-degenerate representation of A_p associated to φ. By Lemma 7.2.1, there is $q \in S(B)$ such that $A_p \sim_M B_q$. Moreover, the Hilbert A_p-module E_p gives the strong Morita equivalence between A_p and B_q (Remark 7.2.2). Then the representations φ_p and $\widetilde{E_p}\text{-}\mathrm{Ind}_{B_q}^{A_p}\left(E_q\text{-}\mathrm{Ind}_{A_p}^{B_q}\varphi\right)$ of A_p are unitarily equivalent, Theorem 6.23, [**41**]. But, according to Theorem 7.2.7, the representations $\widetilde{E_p}\text{-}\mathrm{Ind}_{B_q}^{A_p}\left(E_q\text{-}\mathrm{Ind}_{A_p}^{B_q}\varphi\right)$ and $\widetilde{E}_p\text{-}\mathrm{Ind}_{B_q}^{A_p}\left(E_q\text{-}\mathrm{Ind}_{A_p}^{B_q}\varphi\right)$ of A_p are unitarily equivalent. Therefore the representations φ and $\left(\widetilde{E}_p\text{-}\mathrm{Ind}_{B_q}^{A_p}\left(E_q\text{-}\mathrm{Ind}_{A_p}^{B_q}\varphi\right)\right)\circ\pi_p$ of A are unitarily equivalent.

On the other hand, according to Lemma 7.2.5, the representations $E\text{-}\mathrm{Ind}_B^A\varphi$ and $\left(E_p\text{-}\mathrm{Ind}_{A_p}^{B_q}\varphi\right)\circ\pi_q$ of B are unitarily equivalent. From this and Proposition 7.2.3 and Lemma 7.2.5, we deduce that the representations $\widetilde{E}\text{-}\mathrm{Ind}_B^A\left(E\text{-}\mathrm{Ind}_A^B\varphi\right)$ and $\left(\widetilde{E}_q\text{-}\mathrm{Ind}_{B_q}^{A_p}\left(E_p\text{-}\mathrm{Ind}_{A_p}^{B_q}\varphi\right)\right)\circ\pi_p$ of A are unitarily equivalent. Therefore the representations φ and $\widetilde{E}\text{-}\mathrm{Ind}_B^A\left(E\text{-}\mathrm{Ind}_A^B\varphi\right)$of A are unitarily equivalent. ∎

Theorem 7.3.4 *Let A and B be two strong Morita equivalent locally C^*-algebras. Then there is a bijective correspondence between equivalence classes of non-degenerate representations of A and B which preserves direct sums and irreducibility.*

Proof. Let E be a Hilbert A -module which gives the strong Morita equivalence between A and B. By Theorem 7.3.3 and Proposition 7.2.3, the map $\varphi \to E\text{-}\mathrm{Ind}_A^B\varphi$ from the set of all non-degenerate representations of A to the set of all non-degenerate representations of B induces a bijective correspondence between equivalence classes of non-degenerate representations of A respectively B. Moreover, this correspondence preserves direct sums, Proposition 7.2.6.

To show this correspondence preserves irreducibility, let φ be an irreducible, non-degenerate representation of A. Suppose that $E\text{-Ind}_A^B\varphi$ is not irreducible. Then $E\text{-Ind}_A^B\varphi = \psi_1 \oplus \psi_2$ and by Proposition 7.2.6 and Theorem 7.3.3, the representations $\left(\widetilde{E}\text{-Ind}_B^A\psi_1\right) \oplus \left(\widetilde{E}\text{-Ind}_B^A\psi_2\right)$ and φ of A are unitarily equivalent, a contradiction. So the bijective correspondence defined above preserves irreducibility. ■

References for Section 7.3: **[22]**, **[30]**, **[41]**.

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