Ï ðèëîæåíèå À: Âû÷èñëåíèå ãàì ìà-ôóíêöèè

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$$\Gamma(a)=\int_0^{\infty} t^{a-1}e^{-t}dt,$$

âõîäèò â íîðìèðóþùèé ìíîæèòåëü ïðàêòè÷åñêè âî âñå ôóíêöèè ðàñïðåäåëåíèÿ. Ï îýòîì ó, åñòåñòâåííî, âàæíî óì åòü åå âû÷èñëÿòü. Òðóäíîñòü, îáùåê , â òîì , ÷òî ýòà ôóíêöèÿ î÷åíü áûñòðî ðàñòåò – âåäü äëÿ íåå âûïîëíåíî ñîîòíîøåíèå $\Gamma(a+1)=a\times\Gamma(a)$, ò.å. äëÿ öåëûõ àðãóì åíòîâ îíà ðàñòåò êàê ôàêòîðèàë, è íåì åäëåíííå äëÿ äðîáíûõ.

Áûñòðûé ðîñò ÷èñåë, ñ êîòîðûì è ïðèõîäèòñÿ îïåðèðîâàòü, íåóäîáåí î ó÷èòûâàòü è ïðè âû÷èñëåíèè ñîáñòâåííî ôóíêöèé ðàñïðåäåëåíèÿ. Ñêàæåì , âû÷èñëÿÿ ãàì ìà-ðàñïðåäåëåíèå, ìû äîëæíû ïîìíèòü î òîì , ÷òî íåïîëíàÿ ãàì ìà-ôóíêöèÿ, íî ðèðóþùèì ìíîæèòåëåì äëÿ êîòîðîé ÿâëÿåòñÿ $\Gamma(a)$, ïðè áîëüøîì àðãóìåíòå ñòàíîâèòñÿ áëèçêîé ê $\Gamma(a)$. ò.å. òîæå áîëüøîé. Ï îýòîì ó, íå óäàäèòñÿ ðàçäåëèòü âû÷èñëåíèÿ – ñíà÷àëà âû÷èñëèòü íåïîëíóþ ãàì ìà-ôóíêöèþ , ïîòîì ïîëíóþ , à ïîòîì ïîäåëèòü îäíó íà äðóãóþ . Ï ðèõîäèòñÿ èçîáðåòàòü óëîâêè, ïîçâîëÿþùèå ñ ñèòóàöèåé.

Ñòàíäàðòíàÿ «âîëøáà» â ïîäîáíûõ ñëó÷àÿõ – ðàçäåëèòü âû÷èñëÿåìóþ ôóíêöèþ íà ïðîèçâåäåíèå «òðóäíîé» è ëåãêîé ÷àñòåé, ãäå «òðóäíàÿ» ÷àñòü ÿâëÿåòñÿ ïðîèçâåäåíèåì äâóõ èëè áîëüøå ÷èñëà áûñòðî ðàñòóù èõ ñîìíîæèòåëåé, è âû÷èñëÿòü íå ñàì ó «òðóäíóþ» ÷àñòü, à ëîãàðèôì îòíåñ.

Âîòïðèìåðû.

Êîãäà $x < a+1$, äëÿ âû÷èñëåíèÿ ôóíêöèè ãàì ìà-ðàñïðåäåëåíèÿ $\Gamma(x\mid a)$ ìû èñïîëüçóåì ðàçëîæåíèå

$$\Gamma(x|a)=\frac{x^a e^{-x}}{\Gamma(a)}\sum_{n\ge 0}\frac{x^n}{a(a+1)\ldots(a+n)},$$

ðàñïàäàþùåãîñÿ íà «ïðîñòóþ» ÷àñòü $\sum_{n\ge 0}\frac{x^n}{a(a+1)\ldots(a+n)}$, ïðè ðàáîòå ñ êîòîðîé ìû íå ñòàëêèâàåìñÿ ñ ñëèøêîì áîëüøèì è ÷èñëàìè (ôàêòîðèàë ðàñòåò áûñòðåå ñòåïåíè), è «òðóäíóþ», êîòîðóþ ìû âû÷èñëÿåì êàê $\exp(-x+a\times\ln(x)-\ln(\Gamma(a)))$. Ï ðè $x > a+1$ äëÿ âû÷èñëåíèÿ ãàì ìà-ðàñïðåäåëåíèÿ èñïîëüçóåòñÿ ðàçëîæåíèå â öåïíóþ äðîáü, êóäà âõîäèò àíàëîãè÷íûé «òðóäíûé» ñìûñëîâèòñÿ.

Âû÷èñëÿÿ áåòà-ðàñïðåäåëåíèå $B(x\mid a,b)$ ïðè íåáîëüøèõ x, ìû èñïîëüçóåì öåïíóþ äðîáü

$$B(x|a,b)=\frac{x^a(1-x)^b}{aB(a,b)}\left[\frac{1}{1+}\frac{d_1}{1+}\frac{d_2}{1+}\ldots\right].$$

È ñíîâà ðàçëîæåíèå ðàñïàäàåòñÿ, è «òðóäíóþ» ÷àñòü ìû âû÷èñëÿåì êàê $\exp(a\times\ln(x)+b\times\ln(1-x)-\ln(a)-\ln(B(a,b)))$. È ñíîâà íóæíî óìåòü âû÷èñëÿòü ëîãàðèôì ãàììà-ôóíêöèè, ïîñêîëüêó $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

Â íèæåñëåäóþùèõ êîäàõ èñïîëüçóåòñÿ àñèìïòîòè÷åñêîå ðàçëîæåíèå, êîòîðîå (ñ íåáîëüøèìè, â îñíîâíîì, ïîïðàâêàìè) ìîæíî íàéòè â ãîðÿ÷î ðåêîìåíäóåìîì ñïðàâî÷íèêå ïî ñïåöèàëüíûì ôóíêöèÿì Ì.Àáðàìîâèöà è È.Ñòèãàíà:

$$\ln\Gamma(z)\sim\left(z-\frac{1}{2}\right)\ln(z)-z+\frac{1}{2}\ln(2\pi)+\frac{1}{12z}-\frac{1}{360z^3}+\frac{1}{1260z^5}-\frac{1}{1680z^7}+\ldots$$

Êîãäà àðãóìåíò z äîñòàòî÷íî âåëèê, åãî ïåðâûå 20 ÷ëåíîâ äàþò òî÷íîñòü 50 âåðíûõ çíàêîâ, åñëè íå ó÷èòûâàòü îøèáêè, âûçûâàåìûå êîíå÷íîé ðàçðÿäíîñòüþ. Îòðàæåíèåì àñèìïòîòè÷åñêîãî õàðàêòåðà ðÿäà â êîäàõ ÿâëÿåòñÿ óâåëè÷åíèå ñëèøêîì ìàëîãî àðãóìåíòà: åñëè z ìåíüøå íåêîòîðîé ãðàíèöû (â äàííûõ êîäàõ 7), îí óâåëè÷èâàåòñÿ, ïðè÷åì äëÿ êîððåêòèðîâêè èñïîëüçóåòñÿ óïîìÿíóòîå âûøå ñîîòíîøåíèå $\Gamma(a+1)=a\times\Gamma(a)$.

Ôàéë loggamma.h

```
#ifndef __LOGGAMMA_H__        /* To prevent redefinition*/

#define ENTRY   extern
#define   LOCAL  static

ENTRY double logGamma(double x);

#define __LOGGAMMA_H__        /* Prevents redefinition   */
#endif                  /* Ends #ifndef__LOGGAMMA_H__    */
```

Ôàéë loggamma.cpp

```
#include <math.h>
#include <float.h>            /* LDBL_MIN_EXP */

#include "logGamma.h"

/************************************************************/
/*                           logGamma                       */
/************************************************************/

#define LGM_LIM         7
/* Implementation dependent const used to increase
 * convergence in logGamma and gammaDF.
 *      May be changed when porting functions to
 *      computers with different float/double lengths.
 */

ENTRY double
```

```
logGamma(double x)
/*
 * Compute natural logarithm of Gamma(x)
 *      using the asymptotic Sterling's expansion.
 * See Abramowitz & Stegun,
 *      Handbook of Mathematical Functions, 1964 [6.1.41]
 * The first 20 terms give the result with 50 digits.
 * If x <= 0, HUGE_VAL is returned to indicate error.
 */
{
   long double static c[20] =
   {
      /* Asymptotic expansion coefficients             */
      1.0 / 12.0, -1.0 / 360.0, 1.0 / 1260.0,
     -1.0 / 1680.0, 1.0 / 1188.0,
     -691.0 / 360360.0, 1.0 / 156.0,
     -3617.0 / 122400.0, 43867.0 / 244188.0,
     -174611.0 / 125400.0, 77683.0 / 5796.0,
     -236364091.0 / 1506960.0,
      657931.0 / 300.0, -3392780147.0 / 93960.0,
      1723168255201.0 / 2492028.0,
      -7709321041217.0 / 505920.0, 151628697551.0 / 396.0,
      -26315271553053477373.0 / 2418179400.0,
      154210205991661.0 / 444.0,
      - 261082718496449122051.0 / 21106800.0
   };


   double x2, presum, sum, den, z;
   int  i;

   if (x <= 0)
      return HUGE_VAL;        /* Error!                              */
   else if (x == 1 || x == 2)
      return 0;

   for (z = 0; x < LGM_LIM; x += 1)  /* Increase argument if necessary.
          */
      z += log(x);

   den = x;
   x2 = x * x;          /* Compute the asymptotic expansion      */
   presum = (x - 0.5) * log(x) - x + 0.91893853320467274178032973640;
   for (i = 0; i < 20; i++) {
      sum = presum + c[i] / den;
      if (sum == presum)
           break;
      den = den * x2;
      presum = sum;
   }
   return sum - z;      /* Fit the increased argument if any    */

}/* logGamma */
```