The Principal-Agent Problem

Class Notes

A principal (she) hires an agent (he) or more than one agent for one period. Agents' effort levels provide a revenue to the principal, who pays a wage to each agent. We want to find the contractual arrangements that is best for the principal in terms of her profit.

Suppose the principal hires one agent. The agent's effort, denoted e, provides a revenue $R(e)$ to the principal, who pays a wage W to the agent. The agent's cost of effort is given by the function $c(e)$. So the principal's payoff when the agent works e is given by

$$\pi_p \ = \ R(e) \ - \ W,$$

and the agent's payoff is

$$\pi_a \ = \ W \ - \ c(e).$$

To be consistent with reality, we want the functions $R(\cdot)$ and $c(\cdot)$ to be increasing. We may also impose concavity on R and convexity on c, and that $c(0) = 0$.

Suppose n agents are hired by the principal, where e_i denotes Agent i's effort, $i = 1, \ldots, n$. As before the cost of effort to Agent i is given by $c(e_i)$ but the principal's revenue is now given by $R(e_1, \ldots, e_n)$. Let W_i denote the wage paid to Agent i. The principal's payoff in this case is

$$\pi_p \ = \ R(e_1, \ldots, e_n) \ - \ \sum_{i=1}^{n} W_i,$$

and Agent i's payoff when working e_i is

$$\pi_a \ = \ W_i \ - \ c(e_i).$$

We first look at the case where agents can be perfectly monitored to then look at cases where agents cannot be perfectly monitored.

1 Agents can be Perfectly Monitored

The principal in this case will be able to pay each agent according to his effort level, i.e. an agent's overal wage will be given by

$$W_i = we_i,$$

w is a *wage rate*. Agent i's problem is then

$$\max_{e_i} \; we_i - c(e_i) \; .$$

The solution to this problem, denoted e_i^*, si such that

$$\frac{d}{de_i}\left(we_i - c(e_i)\right) = 0 \quad \Rightarrow \quad w = c'(e_i^*),$$

where $c'(e_i) = \frac{dc(e_i)}{de_i}$. Since e_i^* depends on the wage rate w, we will write $e_i^* = e_i(w)$.

The principal's objective is to choose a wage that solves

$$\max_{w} \; R(e_1(w), e_2(w), \ldots, e_n(w)) - \sum_{i=1}^{n} we_i(w) \; .$$

The wage rate that maximizes the principal's payoff, denoted w^*, is then such that

$$\frac{d}{dw}R(e_1(w^*), e_2(w^*), \ldots, e_n(w^*)) = \sum_{i=1}^{n}\left(e_i(w^*) + w^*\frac{de_i(w^*)}{dw} \right)$$

Example 1 *Let $n = 1$, $R(e) = e$, $c(e) = \frac{e^2}{100}$, and suppose that the principal is perfectly able to monitor the agent's effort. Then, given a wage rate w, the agent's effort, e^*, is such that*

$$w = c'(e^*) = \frac{2e^*}{100} = \frac{e^*}{50},$$

which gives us $e^* = e(w) = 50w$. The principal's problem is then

$$\max_{w} \; e(w) - we(w) \quad \equiv \quad \max_{w} \; 50w - 50w^2,$$

and the wage rate that maximizes her payoff is such that

$$50 - 100w^* = 0 \quad \Rightarrow \quad w^* = \frac{1}{2} \; .$$

Hence for this problem, the agent's effort level is

$$e(w^*) \;=\; 50 \times \frac{1}{2} \;=\; 25,$$

the overall payment to the agent is

$$w^* \times e(w^*) \;=\; \frac{1}{2} \times 25 \;=\; 12.5$$

and the principal's payoff is

$$R(e(w^*)) \;-\; w^* e(w^*) \;=\; 25 \;-\; 12.5 \;=\; 12.5 \;.$$

The agent's payoff in this case is

$$w^* e(w^*) \;-\; c(e(w^*)) \;=\; 12.5 \;-\; \frac{25^2}{100} \;=\; 6.25 \;.$$

2 The Pareto Efficient Allocation

Suppose that the principal works for herself, and let e_i denote the effort level she devotes to task i, say. Her problem is then

$$\max_{e_1,\dots,e_n} \; R(e_1,\dots,e_n) \;-\; \sum_{i=1}^{n} c(e_i) \;,$$

and thus the effort levels that maximize her payoff are such that

$$\frac{d}{de_i} R(e_1^*,\dots,e_n^*) \;=\; c'(e_i^*)$$

for all i. This allocation is called the Pareto efficient allocation, it is the allocation that maximize the overall wealth (the sum of all payoffs).

Example 2 *Let's find the Pareto efficient allocation for the problem described in Example 1. Here we have*

$$R'(e^*) \;=\; c'(e^*) \quad\Rightarrow\quad 1 \;=\; \frac{e^*}{50} \quad\Rightarrow\quad e^* \;=\; 50.$$

The firm's revenue is then 50, the cost of effort is $\frac{50^2}{100} = 25$ *and the firm's profit is 25, which is the overall wealth, as it can be divided between the principal and the agent. The overall wealth here is greater than in the previous example since then the sum of both payoffs was* $12.5 + 6.25 = 18.75$.

3 A Revenue Sharing Plan

Suppose that agents cannot be perfectly monitored and that the principal wants to motivate them by sharing her revenue with them. That is Agent i's compensation consists of a share s_i of the total firm's revenue, where $(s_1, s_2, \ldots, s_n)$ is such that $s_i \geq 0$ for all i and $\sum_i s_i \leq 1$. Agent i's problem is now

$$\max_{e_i} \; s_i R(e_1, \ldots, e_n) \; - \; c(e_i),$$

and the Agent i's choice of effort, e_i^*, is such that

$$\frac{d}{de_i}\left(s_i R(e_1, \ldots, e_n) \; - \; c(e_i)\right) \; = \; 0 \qquad \Rightarrow \qquad s_i \times \frac{d}{de_i} r(e_1, \ldots, e_n) \; = \; c'(e_i^*).$$

Example 3 *Consider a situation as in Example 1 but instead of paying a wage rate, the principal gives a fraction* s *if her revenue as a wage to the agent. The agent's choice of effort,* e^**, is such that*

$$sR'(e^*) \; = \; c'(e^*) \qquad \Rightarrow \qquad s \; = \; \frac{e^*}{50} \qquad \Rightarrow \qquad e^* \; = \; 50s.$$

The principal's payoff is simply the amount of revenue left, i.e. $(1 - s)R(e^*)$.

If $s = \frac{1}{5}$*, for instance, the agent's effort is 10 and the principal's revenue is* $\frac{4}{5} \times 10 = 8$. *Note that this is much lower than the principal's payoff when she can perfectly monitor effort. For the agent to work as much as in the perfect monitoring case, i.e.* $e = 25$*, the share of revenue that the principal must give to the agent is* $\frac{1}{2}$.

4 The Shirking Model

Suppose that the principal wants each of her agents to provide a level of effort equal to $\hat{e}$, where $\hat{e}$ is some arbitrary number. The principal cannot monitor perfectly, but she can monitor. That is, each employee's effort may be monitored with probability p. If an employee is caught working less than what is demanded, $\hat{e}$, he will be fired. The wage paid by the principal is denoted $\widehat{W}$. In this case, we assume that the agent has an outside option, which is a payment of $\overline{W}$. More specifically, if an agent is fired by the principal, then he receives a payment of $\overline{W}$. The agent's expected payoff is then

$$\pi_a = \begin{cases} \widehat{W} - c(e) & \text{if } e \geq \hat{e}, \\ p\overline{W} + (1-p)\widehat{W} - c(e) & \text{if } e < \hat{e}, \end{cases}$$

From this payoff function, we can see that the agent will either work $\hat{e}$ or $e = 0$. So in order for the principal to be able to hire some employees, the wage $\widehat{W}$ has to be such that

$$\widehat{W} - c(\hat{e}) \geq p\overline{W} + (1-p)\widehat{W} \quad \Rightarrow \quad \widehat{W} \geq \overline{W} + \frac{c(\hat{e})}{p}.$$

Note that the agent's expected payoff when shirking is $p\overline{W} + (1-p)\widehat{W}$ since we have assumed that $c(0) = 0$.

Example 4 *Consider a situation as in Example 1, and assume that $\overline{W} = 6.25$, the agent's payoff effort can be perfectly monitored. If $p = \frac{1}{2}$, that must $\widehat{W}$ be to induce the agent to work $\hat{e} = 25$?*

$$\widehat{W} \geq \overline{W} + \frac{c(\hat{e})}{p}. = 6.25 + \frac{(25)^2/100}{.5} = \$18.75 \ .$$

Note that the principal has to pay a significant premium to obtain this effort level.

5 A Forcing Contract

Suppose the principal tells her employees: "Either revenue is at least R^*, in which case you get paid a wage W, or either revenue is below R^* and you don't get paid at all. What is then the equilibrium? I'll leave this one to you.