Business 3019

Assignment 2

Due Wednesday January 29, 2003, 4:00pm (beginning of class)

1. A firm's short-run revenue is given by $R(e) = 10e - e^2$, where e is the level of effort by a typical manager (all managers are assumed to be identical). A manager chooses his level of effort to maximize his wage net of effort $w - e$ (the per-unit cost of effort is assumed to be 1). Determine the level of effort and the level of profit (revenue less wage paid) for each of the following wage arrangements. Explain why these different principal-agent relationships generate different outcomes.

   (a) $w = 2$ for $e \geq 1$; otherwise $w = 0$.

   Answer: The manager's payoff with effort level e is given by

   $$\pi_m = \begin{cases} 2 - e & \text{if } e \geq 1, \\ -e & \text{if } e < 1. \end{cases}$$

   Note that an effort level $e > 1$ is only more costly to the manager than $e = 1$ since he gets paid 2 anyway. Hence $e = 1$ is superior to any $e > 1$. Working $e < 1$ si also inferior to $e = 1$ since this provides the manager with a payoff of at most zero, whereas working $e = 1$ provides him with a payoff of one. The manager will therefore choose $e = 1$, and the firm's (or principal's) profit will be

   $$\pi_p = R(1) - w = 10 \times 1 - 1^2 - 2 = 7.$$

   (b) $w = R(e)/2$.

   Answer: In the case, the manager is solving the following problem:

   $$\max_e \left\{ \frac{R(e)}{2} - e \right\} = \max_e \left\{ \frac{10e - e^2}{2} - e \right\} = \max_e \left\{ 4e - \frac{e^2}{2} \right\}.$$

The manager's choice of effort, e^*, is then such that

$$4 - e^* = 0 \quad \Rightarrow \quad e^* = 4.$$

This gives the manager a payoff of

$$\pi_m = \frac{R(e^*)}{2} - e^* = \frac{10 \times 4 - 4^2}{2} - 4 = 8,$$

and the firm's profit is

$$\pi_p = R(e^*) - \frac{R(e^*)}{2} = \frac{1}{2}\left(10 \times 4 - 4^2\right) = 12.$$

(c) $w = R(e) - 12.5$.

The manager's problem in this case is

$$\max_e \{ R(e) - 12.5 - e \} = \max_e \left\{ 9e - e^2 - 12.5 \right\}.$$

The manager's choice of effort, e^*, is then such that

$$9 - 2e^* = 0 \quad \Rightarrow \quad e^* = 4.5 .$$

This gives the manager a payoff of

$$\pi_m = R(e^*) - 12.5 - e^* = 9 \times 4 - 4^2 - 12.5 = 7.5$$

and the firm's profit is

$$\pi_p = R(e^*) - (R(e^*) - 12.5) = 12.5 .$$

2. Consider a firm whose total revenue depends on the effort provided by each of its five managers in the following manner:

$$R(e_1, e_2, e_3, e_4, e_5) = \sum_{n=1}^{5} e_n,$$

where e_n denotes Manager n's effort level and $R(\cdot)$ denotes the firm's total revenue. The dollar cost of effort to a manager is given by the function

$$c(e) = \frac{e^2}{100} .$$

(a) Suppose that effort can be perfectly monitored and that each manager is paid according to his performance. That is, Manager n's pay if he works e_n is given by we_n, where w is the wage rate. Calculate the wage rate that maximizes the firm's profits and the effort level, given this wage rate, that maximizes each manager's payoff. What are then the firm's profits?

Answer: Each manager is solving the following problem:

$$\max_{e_n} \left\{ we_n - \frac{e_n^2}{100} \right\}.$$

The first derivative of this function evaluated at the effort level that maximizes the manager's payoff, e_n^*, should be equal to zero. That is,

$$w - \frac{2e_n^*}{100} = 0 \quad \Rightarrow \quad e_n^* = 50w.$$

The chosen effort level being a function of the wage rate, we will use $e_n(w)$ to represent e_n^*.

Knowing $e_n(w) = 50w$, $n = 1, \ldots, 5$, the principal chooses the wage rate that maximizes her payoff, i.e. w solves the following problem:

$$\max_{w} \left\{ R(e_1(w), \ldots, e_5(w)) - \sum_{n=1}^{5} we_n(w) \right\} = \max_{w} \left\{ 250w - 250w^2 \right\}.$$

Thus the wage rate chosen by the prinicipal, w^*, is such that

$$250\,(1 - 2w^*) = 0 \quad \Rightarrow \quad w^* = \frac{1}{2}\,.$$

This gives a payoff of

$$\pi_m^n = w^* \times e_n(w^*) - \frac{e_n(w^*)^2}{100} = \frac{1}{2} \times \frac{50}{2} - \frac{(50/2)^2}{100} = 6.25$$

for manager $n = 1, \ldots, 5$, and the principal's payoff is

$$\pi_p = \sum_{n=1}^{5} e_n(w^*) - \sum_{n=1}^{5} w^* e_n(w^*) = 5 \times 25 - 5 \times 12.5 = 62.5\,.$$

(b) Suppose now that the firm cannot monitor the effort of all of its managers. That is, the firm can only monitor the effort of one of its managers and each manager has an equal chance of being monitored during the period. Suppose also that each manager can earn a wage of \$6.25 without effort outside the firm if they get fired. What wage must the firm pay to induce an effort level of $e^* = 25$ from each of its managers? What are then the firm's profits? Answer these questions with $e^* = 20$ and $e^* = 10$.

Answer: The minimum wage the principal has to pay her managers to induce them to work $e^* = 25$ is

$$W_{25} \;=\; 6.25 \;+\; \frac{c(25)}{1/5} \;=\; 6.25 \;+\; \frac{(25)^2/100}{1/5} \;=\; 37.5\ .$$

Note that this is the total payment to a manager, it is not a rate. Thus the principal has to pay \$37.50 to each of her managers to prevent them from shirking. Since each manager provides the principal with a revenue of \$25, paying them \$37.5 cannot be profitable. More specifically, the principal's payoff with $e^* = 25$ and $W_{25} = 37.5$ is

$$5 \times 25 \;-\; 5 \times 37.5 \;=\; -\,62.5$$

If the effort level demanded is $e^* = 20$, then

$$W_{20} \;=\; 6.25 \;+\; \frac{c(20)}{1/5} \;=\; 26.25\ .$$

The principal is still losing money here since each agent brings a revenue of \$20 while costing \$26.25.

If the effort level demanded is $e^* = 10$, then

$$W_{20} \;=\; 6.25 \;+\; \frac{c(10)}{1/5} \;=\; 11.25\ .$$

Again, this cannot be profitable.

(c) Suppose now that effort levels cannot be monitored at all. To solve this problem, the firm puts in place a revenue-sharing plan that gives each manager one tenth of the firm's total revenue at the end of the period (and no other compensation). What will be the effort provided by each manager under this plan? What are then the firm's profits? (Assume for this question that managers have no outside option.)

Answer: Each manager here is solving

$$\max_{e_n} \ \frac{R(\ldots, e_n, \ldots)}{10} \ - \ c(e_n) \ .$$

For example, Manager 1's problem is

$$\max_{e_1} \ \frac{e_1 + e_2 + e_3 + e_4 + e_5}{10} \ - \ \frac{e_1^2}{100} \ ,$$

and thus his choice of effort, e_1^*, is such that

$$\frac{1}{10} \ - \ \frac{e_1^*}{50} \ = \ 0 \qquad \Rightarrow \qquad e_1^* \ = \ 5.$$

Each manager faces the same problem and $e_n^* = 5$ for all n. The principal's payoff is then

$$\pi_p \ = \ R(5,5,5,5,5) \ - \ \frac{5}{10} \times R(5,5,5,5,5) \ = \ \frac{1}{2} \times 5 \times 5 \ = \ 12.5.$$

Note that this is significantly lower than the principal's payoff of 62.5 under perfect monitoring.

(d) Suppose again that effort levels cannot be monitored at all but that the firm wants to solve this problem with a forcing contract as follows: If the firm's total revenue is \$125 or more, then each manager will be paid a wage of \$12.5. If the firm's total revenue is less than \$125, then managers don't get paid at all. Compute at least three possible equilibria for this case and calculate the firm's profits in each of them. (Assume for this question that managers have no outside option.)

Answer: In this case, any combination of effort levels $(e_1, \ldots, e_5)$ such that $R(e_1, \ldots, e_5) = 125$ and $\pi_m^n \geq 0$ for all n is an equilibrium, along with $e_n = 0$ for all n.

For example, suppose that $e_n = 25$ for all n. Then, since $R(25, 25, 25, 25, 25) = 125$,

$$\pi_m^n \; = \; 12.5 \; - \; \frac{(25)^2}{100} \; = \; 6.25 \; > \; 0$$

for all n. Given that all other managers provide an effort of 25, a manager has no incentive to work less than 25 since then his payoff is at most 0, while he gets 6.25 with an effort of 25. On the other hand, a manager has no incentive to work more than 25 since this increases his cost of effort without increasing his compensation. Therefore, $e_n = 25$ for all n is an equilibrium.

Suppose now that $e_n = 0$ for all n. In this case, each manager's payoff is zero and the firm's profit is also zero. This is an equilibrium since there is no way a manager alone can work hard enough to increase his payoff above zero. That is, given that all other managers choose not to work, a manager alone would have to provide an effort of at least 125 to get paid, which would give him a payoff of

$$\pi_m^n \; \leq \; 12.5 \; - \; \frac{(125)^2}{100} \; = \; -143.75 \; < \; 0.$$

Hence $e_n = 0$ for all n is an equilibrium.

Here is another equilibrium: $e_n = 31.5$ for $n = 1, 2, 3, 4$ and $e_5 = 0$. That is, managers 1 to 4 do all the work while Manager 5 plays chess on the Internet. Since $4 \times 31.5 = 125$, the target revenue is achieved and all managers get paid. At these effort levels,

$$\pi_m^n \; = \; \begin{cases} 12.5 \; - \; \frac{(31.5)^2}{100} \; = \; 2.73 \; > \; 0 & \text{for } n = 1, 2, 3, 4, \\ \qquad 12.5 \; > \; 0 & \text{for } n = 5. \end{cases}$$

Hence no manager alone has an incentive to choose a different effort level since this would only decrease his payoff.

Note: In each of the above problems, the sum of all payoffs (principal and managers) is equal to

$$\begin{aligned}
\pi_p \;+\; \sum_{n=1}^{5} \pi_m^n \;&=\; R(e_1, \ldots, e_5) \;-\; \sum_{n=1}^{5} W_n \;+\; \sum_{n=1}^{5} (W_n \;-\; c(e_n)) \\
&=\; R(e_1, \ldots, e_5) \;+\; \sum_{n=1}^{5} c(e_n)\ ,
\end{aligned}$$

where W_n is the payment to Manager n.

If it were possible to find the effort levels that maximize the sum of all payoffs, we could then simply divide this "suplus" between the principal and the managers. To maximize the sum of payoffs, each unit of effort has to be used efficiently, and thus the marginal revenue of each manager's effort has to be equal to its marginal cost. That is, Manager n's optimal effort level, e_n^*, is such that

$$\frac{d}{de_n}\, R(\ldots, e_n^*, \ldots) \;=\; \frac{d}{de_n}\, c(e_n^*) \qquad \Rightarrow \qquad 1 \;=\; \frac{e_n^*}{50},$$

and this for all n. This gives us $e_n^* = 50$ for all n, and thus a sum of payoffs equal to

$$R(50, 50, 50, 50, 50) \;+\; \sum_{n=1}^{5} c(50) \;=\; 250 \;-\; 125 \;=\; 125.$$

This sum could be divided as follows: 10 to each manager and 75 to the principal. This would provide each individual, principal and managers alike, with a higher payoff than in any of the above exercises.

The allocation $e_n = 50$ for all n is the *Pareto efficient allocation.* It is the allocation that maximizes the overall welfare. There is no other allocation that can increase one's payoff without decreasing another's payoff.