Option Pricing Theory

1 Types of Options

1.1 Calls

A call option gives the right to buy a fixed number of shares, usually 100, at a fixed price, the exercise (or strike) price, before or at some fixed date, the expiration date. A European option can only be exercised at the expiration date, whereas an American option can be exercised any time before or at the expiration date. An option is a contract between two traders, the writer and the holder. In the case of a call option, the option writer must sell the prespecified number of shares to the option holder at the strike price if the latter decides to exercise her option.

Throughout this document, let

$$
\begin{aligned}
S_0 &\equiv \text{current stock price} \\
S_T &\equiv \text{stock price at the option's expiration date} \\
X &\equiv \text{exercise price} \\
C &\equiv \text{price of a call option} \\
P &\equiv \text{price of a put option}
\end{aligned}
$$

Note that even though an American option can be exercised before the expiration date, we will see that it is not advantageous to do so when the stock does not pay any dividends during this period.

A call option is said to be "in the money" if $S_T > X$. If $S_T < X$, then it is said to be "out of the money". An option is "at the money" if $S_T = X$.

The profit to the holder of a call option, π_h, is given by

$$\pi_h = \max\{ S_T - X , 0 \} - C.$$

The profit to the writer of the option, π_w, assuming that the option is exercised whenever it is in the money, is given by

$$\pi_w = C - \max\{ S_T - X , 0 \}.$$

These profit functions are represented in Figure 1.

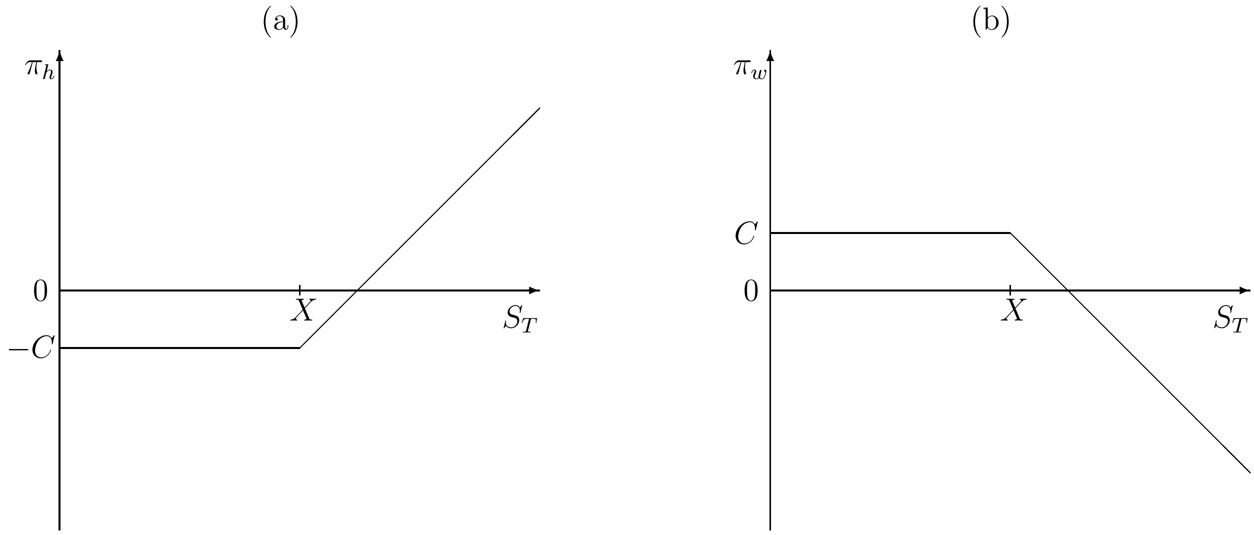

Figure 1: Profit to the holder (a) and the writer (b) of a call option.

1.2 Puts

A put option gives its holder the right to sell a prespecified number of shares at a prespecified price before or at a prespecified date. In this case, the option is in the money if the strike price is above the stock price, and it is out of the money if the strike price is below the stock price. The profit to the holder of a put option is given by

$$\pi_h = \max\{ X - S_T , 0 \} - P,$$

and the profit to the writer of a put is

$$\pi_w = P - \max\{ X - S_T , 0 \}.$$

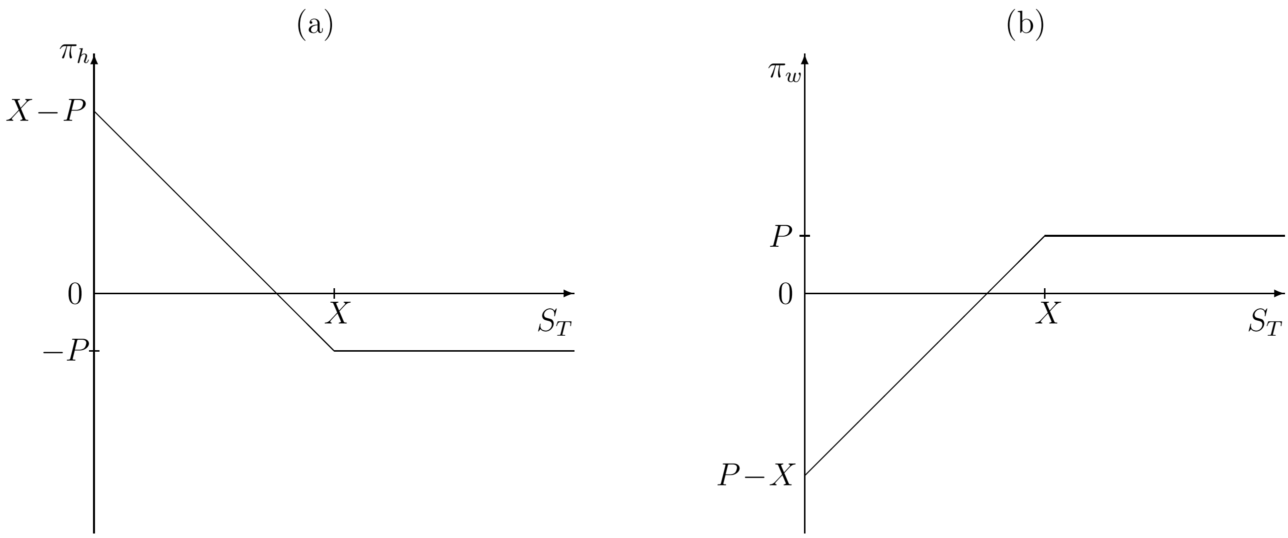

Figure 2: Profit to the holder (a) and the writer (b) of a put option.

1.3 Warrants

A warrant is call option issued by the corporation itself. Hence when warrants are issued, the company receives the money. The firm also receives the money (the exercise price) when these options are exercised, and this increases the number of shares outstanding.

2 Combinations

2.1 Straddle

A straddle consists of one put and one call with the same exercise price. The buyer of a straddle expects a significant change in the stock price, either negative or positive. This could be due, for example, to an expected court ruling that can be very good or very bad for the company.

The profit to the holder of a straddle is given by

$$\pi_h = \begin{cases} S_T - X - (C+P) & \text{if } S_T \geq X, \\ X - S_T - (C+P) & \text{if } S_T < X, \end{cases}$$

The profit to the writer of a straddle is, on the other hand,

$$\pi_w = \begin{cases} C + P - (S_T - X) & \text{if } S_T \geq X, \\ C + P - (X - S_T) & \text{if } S_T < X, \end{cases}$$

These profit functions are depicted in Figure 3.

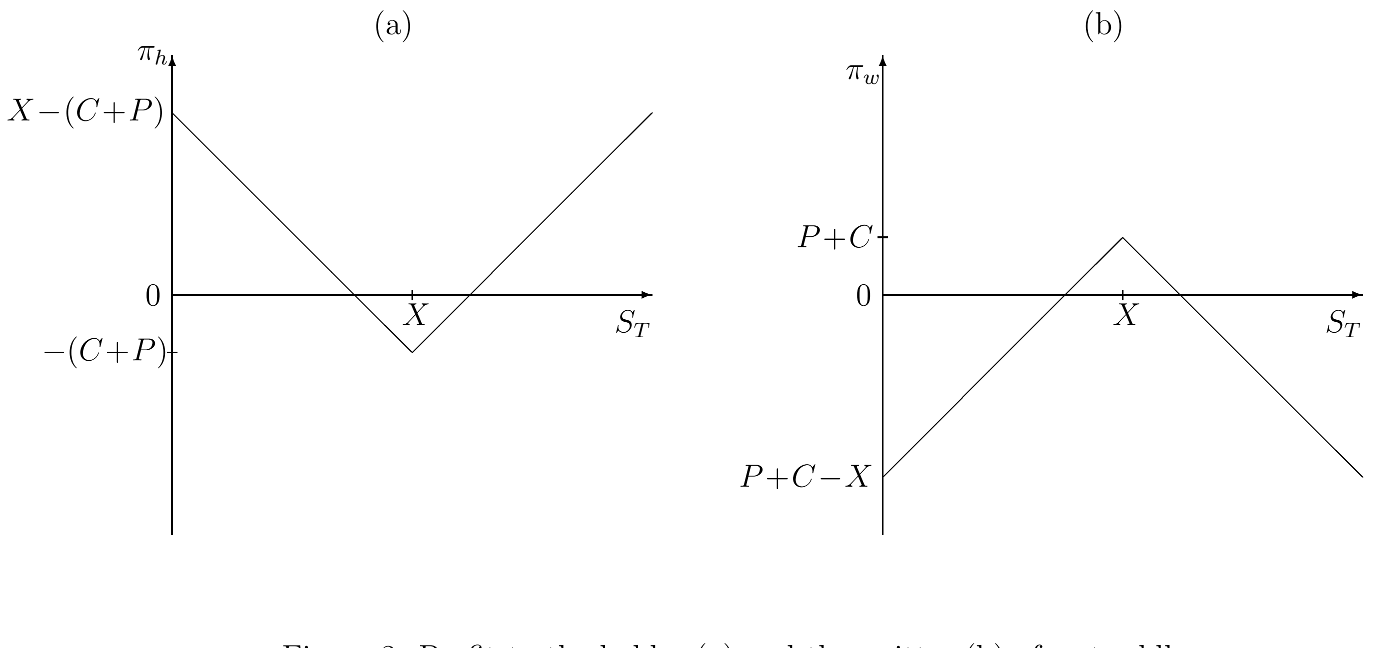

Figure 3: Profit to the holder (a) and the writter (b) of a straddle.

2.2 Covered Call

A covered call denotes a situation where a trader who already owns a stock writes a call on it. The profit of such a trading strategy is (note that we use π_w since we refer to the option writer)

$$\pi_w = \begin{cases} C + S_T - S_0 & \text{if } S_T \leq X, \\ C + X - S_0 & \text{if } S_T > X. \end{cases}$$

If $S_0 = X$, then

$$\pi_w = \begin{cases} C + S_T - X & \text{if } S_T \leq X, \\ C & \text{if } S_T > X. \end{cases}$$

The last profit function is depicted in Figure 4. Note that the profit function has a shape similar to that of a put writer, with C as the option price instead of P.

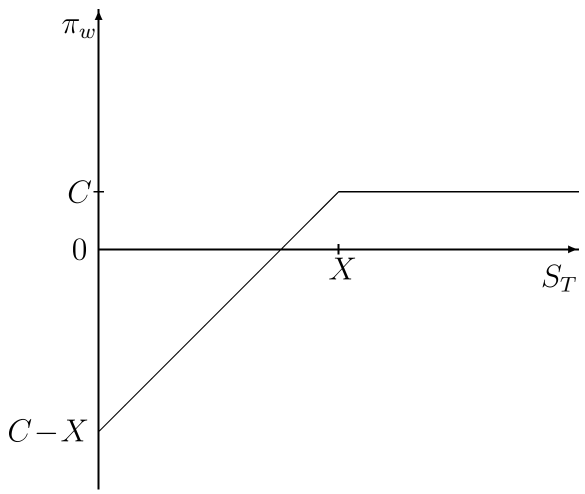

Figure 4: Profit to the writer of a covered call with $X = S_0$.

2.3 Protective Put

We say that a trader buys a protective put if he buys a put option on a stock he already owns. The profit to such a strategy is as follows (here we refer to the option holder, so we use π_h):

$$\pi_h = \begin{cases} S_T - S_0 - P & \text{if } S_T \geq X, \\ X - S_0 - P & \text{if } S_T < X, \end{cases}$$

If $S_0 = X$, then we have

$$\pi_h = \begin{cases} S_T - X - P & \text{if } S_T \geq X, \\ -P & \text{if } S_T < X. \end{cases}$$

The last profit function is depicted in Figure 5. Note that the shape of this profit function is similar to that of a call option, with P instead of C as the option price.

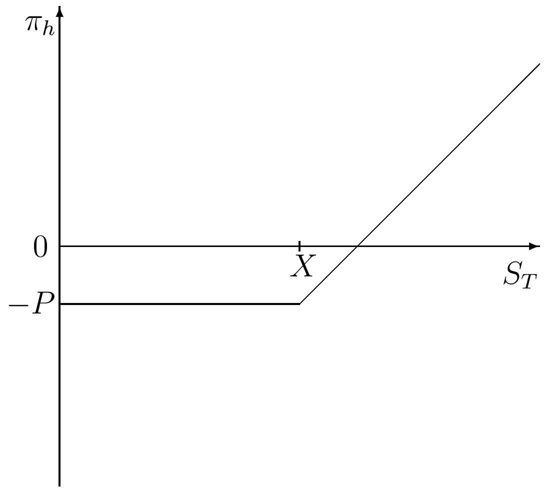

Figure 5: Profit to the holder of a protective put with $X = S_0$.

2.4 Strips and Straps

A strip is a combination of 2 puts and one call. The profit to the holder of such a combination is

$$\pi_h = \begin{cases} 2(X - S_T) - (2P + C) & \text{if } S_T \leq X, \\ S_T - X - (2P + C) & \text{if } S_T > X, \end{cases}$$

A strap, on the other hand, consists of one put and two calls. Its holder's profit is

$$\pi_h = \begin{cases} X - S_T - (P+2C) & \text{if } S_T \le X, \\ 2(S_T - X) - (P+2C) & \text{if } S_T > X, \end{cases}$$

2.5 Spreads

A money spread is the simultaneous purchase and sale of options with different strike prices. A time spread is the simultaneous purchase and sale of options with different expiration dates.

2.5.1 Bullish Spreads

Suppose a trader buys a call option worth C_1 with strike price X_1 and writes a call option worth C_2 with strike price X_2. Let $X_2 > X_1$, which implies that $C_1 > C_2$ (the lower the strike price, the more valuable the option). The profit from such a strategy is

$$\pi = \begin{cases} -(C_1 - C_2) & \text{if } S_T \le X_1, \\ S_T - X_1 - (C_1 - C_2) & \text{if } X_1 < S_T \le X_2, \\ X_2 - X_1 - (C_1 - C_2) & \text{if } S_T > X_2. \end{cases}$$

Note that if $S_T > X_2$, both options are exercised and the profit from the spread is $S_T - X_1 - (S_T - X_2) - (C_1 - C_2) = X_2 - X_1 - (C_1 - C_2)$. This payoff function is depicted in Figure 6.

The trader using such a strategy believes the stock price is likely to increase, i.e. he is bullish about the stock, and this strategy is called a *bullish spread*. Were the trader only buying a call option with strike price X_1, his loss if $S_T \le X_1$ would be C_1, which is greater than $C_1 - C_2$. Hence the goal of this spread is to reduce the loss incurred in the eventuality that the option is not exercised. The cost of doing so is the reduction in profit when $S_T > X_2$.

2.6 Collar

A collar limits gains and losses from holding a stock within a given range. That is, suppose a trader who currently owns a stock simultaneously buys a put and writes a call on the stock,

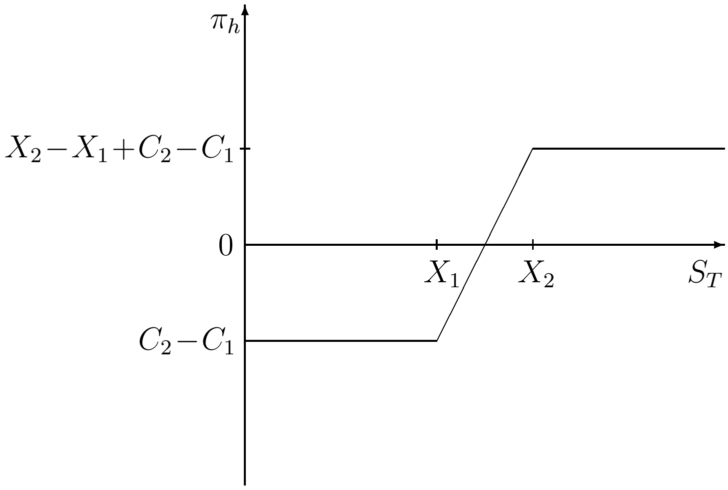

Figure 6: Profit from a spread.

with strike prices such that $C = P$. That is, the purchase of the put option is financed by the sale of the call option. Let X_1 denote the put's strike price and let X_2 denote the call's strike price. For both options to have the same price, we must have $X_1 < S_0 < X_2$. The profit from such a strategy is then

$$\pi = \begin{cases} X_1 - S_0 & \text{if } S_T < X_1, \\ S_T - S_0 & \text{if } X_1 \le S_T < X_2, \\ X_2 - S_0 & \text{if } S_T \ge X_2. \end{cases}$$

The shape of the profit function from a collar is similar to that of a spread.

3 Basic Characteristics of Option Values

3.1 Relative Prices of Calls with Alternative Characteristics

1. An American call option cannot be worth less than a European call option on the same stock, with the same strike price and the same expiration date.

2. Consider two American call options written on the same stock with the same strike price. Then the option with a shorter life cannot be worth more than the other.

3. Consider two call options with the same expiration date written on the same stock. Then the option with a higher strike price cannot be worth more than the option with a lower strike price.

3.2 Minimum Value of a European Call

Consider a European call option with an exercise price X and an expiration date T. Let S_0 denote the current stock price, let S_T denote the stock price a time T and let C denote the price of the option.

Suppose there exists a bond that pays X at time T and let r denote the interest rate between now and T. The current price of this bond today is then $\frac{X}{1+r}$. Let portfolio A represent the purchase of this bond and the call option described above. The price of this portfolio is $C + \frac{X}{1+r}$ and its payoff at time T is

$$\begin{array}{ccccccl} S_T & - & X & + & X & = & S_T \quad \text{if } S_T \geq X, \\ & 0 & & + & X & = & X \quad \text{if } S_T < X. \end{array}$$

Let portfolio B denote the purchase of the stock. The cost of B is then S_0 and its payoff at time T is S_T. If $S_T < X$, portfolio A never pays less than B at time T and strictly pays more when $S_T < X$. Thus the price of A has to be higher than that of B, otherwise nobody would buy the stock. That is,

$$C + \frac{X}{1+r} \geq S_0 \quad \Rightarrow \quad C \geq S_0 - \frac{X}{1+r} .$$

Hence a European call cannot sell for less than the current stock price minus the present value of the exercise price.

3.3 Early Exercise of an American Call

Does it pay to exercise an American call option before the expiration date on a stock that doesn't pay dividends or whose exercise price is adjusted for dividend payments?

We have seen earlier that an American call cannot sell for less than a European call written on the same stock with the same expiration date and the same strike price. We have also seen that the price of a European call is no less than $S_0 - \frac{X}{1+r}$, where S_0, X and r are as defined above. Hence the price of an American call is no less than $S_0 - \frac{X}{1+r}$. Since $r > 0$, this means that

$$C \geq S_0 - \frac{X}{1+r} > S_0 - X,$$

where $S_0 - X$ is the payoff from exercising the option today. The option holder is then better off selling the option than exercing it, so the option should not be exercised before maturity.

3.4 Put Call Parity

A put and its underlying stock can be combined in such a way that the payoff pattern of the combination is the same as that of a call. We will derive this result for a European-type option, assuming that the stock won't pay any dividend before the option expires. Let

$$\begin{aligned} r_b &\equiv \text{borrowing rate from time 0 to } T \\ r_l &\equiv \text{lending rate from time 0 to } T \end{aligned}$$

Let combination A be one share of stock, a put on the stock and a loan for an amount $\frac{X}{1+r_b}$. Since $\frac{X}{1+r_b}$ is borrowed, $(1+r_b) \times \frac{X}{1+r_b} = X$ has to be repaid at time T. The cost of this combination is $S_0 + P - \frac{X}{1+r_b}$, and its payoff at time T is

$$\begin{array}{lllllll} S_T & + & X - S_T & - & X & = & 0 & \text{if } S_T \leq X, \\ S_T & + & 0 & - & X & = & S_T - X & \text{if } S_T > X. \end{array}$$

If, instead, an investor purchases a call option, then her cost of doing so is C and her payoff is

$$\begin{array}{ll} 0 & \text{if } S_T \leq X, \\ S_T - X & \text{if } S_T > X. \end{array}$$

Suppose portfolio A is less expensive than the call option. Then an investor can make a sure profit by writing a call and buying portfolio A with the proceeds. If this arbitrage opportunity prevails, an arbitrarily large number of calls will be written and arbitrarily large amounts of money will be invested into portfolio A. This will result in a decrease the price of the call and increase the price of portfolio A until

$$C \;\leq\; S_0 \;+\; P \;-\; \frac{X}{1+r_b} \;, \tag{1}$$

which is sustainable.

Consider now “selling” portfolio A, which means selling one unit of stock, writing one put option and lending $\frac{X}{1+r_l}$. Since $\frac{X}{1+r_l}$ is lent, $(1+r_l) \times \frac{X}{1+r_l} = X$ is repaid at time T. The proceeds from selling A is $S_0 + P - \frac{X}{1+r_l}$, and the payoff at time T is

$$\begin{array}{ccccccccl} -S_T & + & S_T - X & + & X & = & 0 & & \text{if } S_T \leq X, \\ -S_T & + & 0 & + & X & = & X - S_T & & \text{if } S_T > X, \end{array}$$

which offsets the payoff to purchasing a call option. If $S_0 + P - \frac{X}{1+r_l} > C$, then an investor can make an arbitrarily large profit by selling A and buying call options with the proceeds, a risk-free arbitrage opportunity. This should drive the price of call options up and drive S_0 and P down until

$$C \;\geq\; S_0 \;+\; P \;-\; \frac{X}{1+r_l} \;. \tag{2}$$

Combining (1) and (2) gives us

$$S_0 \;+\; P \;-\; \frac{X}{1+r_l} \;\leq\; C \;\leq\; S_0 \;+\; P \;-\; \frac{X}{1+r_b} \;.$$

If $r_b = r$, then

$$C \;=\; S_0 \;+\; P \;-\; \frac{X}{1+r} \;,$$

which is called the put-call parity.