# Security Analysis

## 1 Dividend Discount Models

Let $V_0$ denote the intrinsic value of the stock, let $k$ denote the discount rate, let $D_t$ denote the dividend at time $t$ and let $P_t$ denote the price at time $t$. Then

$$V_0 = \frac{E[D_1] + E[P_1]}{1+k}$$

Suppose $V_0 = P_0$, the initial price of the stock. Then

$$P_0 = \frac{E[D_1] + E[P_1]}{1+k} \qquad E[P_1] = \frac{E[D_2] + E[P_2]}{1+k}$$

$$= \frac{E[D_1] + \frac{E[D_2]+E[P_2]}{1+k}}{1+k}$$

$$= \frac{E[D_1]}{1+k} + \frac{E[D_2]}{(1+k)^2} + \frac{E[P_2]}{(1+k)^2}$$

$$= \frac{E[D_1]}{1+k} + \frac{E[D_2]}{(1+k)^2} + \frac{E[D_3]}{(1+k)^3} + \frac{E[P_3]}{(1+k)^3}$$

$$\vdots$$

$$= \frac{E[D_1]}{1+k} + \frac{E[D_2]}{(1+k)^2} + \dots + \frac{E[D_T]}{(1+k)^T} + \frac{E[P_T]}{(1+k)^T}$$

$$= \lim_{T\to\infty} \left( \sum_{t=1}^{T} \frac{E[D_t]}{(1+k)^t} + \frac{E[P_T]}{(1+k)^T} \right)$$

Suppose

$$\lim_{T \to \infty} \frac{E[P_T]}{(1+k)^T} \;=\; 0,$$

which can be called the "no-bubble condition". Then

$$P_0 \;=\; \sum_{t=1}^{\infty} \frac{E[D_t]}{(1+k)^t} \;.$$

For notational simplicity, the expectation term $E[\cdot]$ will be dropped hereafter.

## 1.1 Constant-Growth DDM

Suppose

$$\begin{aligned}
D_1 &= D_1 \\
D_2 &= (1+g)D_1 \\
D_3 &= (1+g)D_2 \;=\; (1+g)^2 D_1 \\
D_4 &= (1+g)D_3 \;=\; (1+g)^2 D_2 \;=\; (1+g)^3 D_1 \\
&\;\vdots \\
D_t &= (1+g)^{t-1} D_1.
\end{aligned}$$

**Note:** Let

$$S \;=\; 1 \;+\; q \;+\; q^2 \;+\; q^3 \;+\; \ldots \;+\; q^{T-1}.$$

Then

$$qS \;=\; q \;+\; q^2 \;+\; q^3 \;+\; \ldots \;+\; q^{T-1} \;+\; q^T,$$

and thus

$$S \;-\; qS \;=\; (1-q)S \;=\; 1 \;-\; q^T \qquad \Rightarrow \qquad S \;=\; \frac{1-q^T}{1-q} \;.$$

If $0 \leq q < 1$, then

$$\sum_{t=1}^{\infty} q^{t-1} \;=\; \lim_{T \to \infty} \frac{1-q^T}{1-q} \;=\; \frac{1}{1-q} \;,$$

since $\lim_{T \to \infty} q^T = 0$ when $0 \leq q < 1$.

We can use this result to derive the stock price:

$$
\begin{aligned}
P_0 &= \sum_{t=1}^{\infty} \frac{D_t}{(1+k)^t} \\
&= \sum_{t=1}^{\infty} \frac{(1+g)^{t-1} D_1}{(1+k)^t} \\
&= \frac{D_1}{1+k} \sum_{t=1}^{\infty} \frac{(1+g)^{t-1}}{(1+k)^{t-1}} \\
&= \frac{D_1}{1+k} \left( 1 \;+\; \frac{1+g}{1+k} \;+\; \left(\frac{1+g}{1+k}\right)^2 \;+\; \left(\frac{1+g}{1+k}\right)^3 \;+\; \ldots \right) \\
&= \frac{D_1}{1+k} \times \frac{1}{1-\frac{1+g}{1+k}} \qquad\qquad \text{(as long as } g < k\text{)} \\
&= \frac{D_1}{1+k-(1+g)} \\
&= \frac{D_1}{k-g} \;.
\end{aligned}
$$

Rearranging, we obtain

$$
\begin{array}{ccccc}
k & = & \dfrac{D_1}{P_0} & + & g \\
\text{required return} & = & \text{current dividend yield} & + & \text{growth rate in price}
\end{array}
$$

Note that

$$
\begin{aligned}
P_1 &= \frac{D_2}{k-g} = \frac{(1+g)D_1}{k-g} = (1+g)P_0 \\
P_2 &= \frac{D_3}{k-g} = \frac{(1+g)D_2}{k-g} = (1+g)P_1 = (1+g)^2 P_0 \\
&\vdots \\
P_t &= \frac{D_{t+1}}{k-g} = \frac{(1+g)D_t}{k-g} = (1+g)P_{t-1} = (1+g)^t P_0.
\end{aligned}
$$

The constant growth model arises from the following assumptions: The firm maintains a constant dividend policy (a constant retention ratio) and earns a stable return on any new equity investment over time.

A firm is referred as a "cash cow" if it pays all of its earnings as dividends. Investment is zero in each period and thus earnings are constant in this case. The price of such a firm is then

$$P_0 = \frac{E_1}{k} .$$

Otherwise, if some earnings are reinvested in the firm, then

$$P_0 = \frac{E_1}{k} + \text{PVGO},$$

where PVGO stands for *present value of growth opportunities.*

Let $b$ denote the firm's retention ratio, i.e. the fraction of net earnings retained within the firm, let $r$ denote the rate of return the firm will earn on any new investment, let $I_t$ denote investment at time $t$ and let $E_t$ denote earnings at time $t$. Note that $r$ can be approximated by the return on equity (ROE), as this is the return stockholders get from the firm's activities. Then

$$E_t = E_{t-1} + rI_{t-1} = E_{t-1} + rbE_{t-1} = (1+rb)E_{t-1} .$$

Dividends being a constant fraction of earnings, both grow at the same rate:

$$\begin{aligned} g &= \frac{D_t - D_{t-1}}{D_{t-1}} \\ &= \frac{(1-b)E_t - (1-b)E_{t-1}}{(1-b)E_{t-1}} \\ &= \frac{E_t - E_{t-1}}{E_{t-1}} \\ &= rb \ . \end{aligned}$$

Hence $rb$ is also the growth rate in the stock price.

Let $r = ck$. Then

$$\begin{aligned} k &= \frac{D_1}{P_0} + g \\ &= \frac{(1-b)E_1}{P_0} + rb \\ &= \frac{(1-b)E_1}{P_0} + ckb \quad \Rightarrow \quad k = \frac{(1-b)E_1}{(1-cb)P_0} \ . \end{aligned}$$

The last equation can be rewritten as

$$\frac{P_0}{E_1} = \frac{1-b}{(1-cb)k} \quad \text{or} \quad P_0 = \frac{(1-b)E_1}{(1-cb)k} \ .$$

That is,

$$P_0 = \frac{(1-b)E_1}{(1-cb)k} \begin{cases} > \dfrac{E_1}{k} & \text{if } c > 1, \\ = \dfrac{E_1}{k} & \text{if } c = 1, \\ < \dfrac{E_1}{k} & \text{if } c < 1, \end{cases}$$

If the firm has no extraordinary return on its investments (c=1), then its stock price is simply $\frac{E_1}{k}$, the stock price if earnings were all paid as dividends, regardless of the plowback ratio.

If, on the other hand, $c > 1$, then $P_0 = \frac{(1-b)E_1}{(1-cb)k} > \frac{E_1}{k}$, as long as $b > 0$. Note that the stock price in this case increases with $b$, i.e. the more the firm reinvests in itself, the greater the stock price.

Finally, if $c < 1$, then $P_0 < \frac{E_1}{k}$ whenever $b > 0$, and the stock price inthis case decreases with $b$, i.e. any additional earnings retained by the firm reduces the stock price.

# 2 Multistage Dividend Growth Models

Suppose the growth rate of the firm's dividends is $g_1$ for the next $T$ years and $g_2$ thereafter. That is,

$$\begin{aligned}
D_2 &= (1+g_1)D_1 \\
D_3 &= (1+g_1)D_2 = (1+g_1)^2 D_1 \\
D_4 &= (1+g_1)D_3 = (1+g_1)^3 D_1 \\
&\vdots \\
D_T &= (1+g_1)D_{T-1} = (1+g_1)^{T-1} D_1 \\
D_{T+1} &= (1+g_2)D_T = (1+g_2)(1+g_1)^{T-1} D_1 \\
D_{T+2} &= (1+g_2)D_{T+1} = (1+g_2)^2(1+g_1)^{T-1} D_1 \\
&\vdots
\end{aligned}$$

Then

$$
\begin{aligned}
P_0 &= \sum_{t=1}^{\infty} \frac{D_t}{(1+k)^t} \\
&= \sum_{t=1}^{T} \frac{D_t}{(1+k)^t} + \sum_{t=T+1}^{\infty} \frac{D_t}{(1+k)^t} \\
&= \sum_{t=1}^{T} \frac{D_t}{(1+k)^t} + \frac{1}{(1+k)^T} \underbrace{\sum_{s=1}^{\infty} \frac{D_{T+s}}{(1+k)^s}}_{P_T} \\
&= \sum_{t=1}^{T} \frac{(1+g_1)^{t-1} D_1}{(1+k)^t} + \frac{1}{(1+k)^T} \sum_{s=1}^{\infty} \frac{(1+g_2)^{s-1} D_{T+1}}{(1+k)^s} \\
&= \frac{D_1}{1+k} \sum_{t=1}^{T} \frac{(1+g_1)^{t-1}}{(1+k)^{t-1}} + \frac{D_{T+1}}{(1+k)^{T+1}} \sum_{s=1}^{\infty} \frac{(1+g_2)^{s-1}}{(1+k)^{s-1}} \\
&= \frac{D_1}{1+k} \sum_{t=1}^{T} \left( \frac{1+g_1}{1+k} \right)^{t-1} + \frac{D_{T+1}}{(1+k)^{T+1}} \sum_{s=1}^{\infty} \left( \frac{1+g_2}{1+k} \right)^{s-1} \\
&= \frac{D_1}{1+k} \times \frac{1 - \left(\frac{1+g_1}{1+k}\right)^T}{1 - \frac{1+g_1}{1+k}} + \frac{1}{(1+k)^T} \times \frac{D_{T+1}}{1+k} \times \frac{1}{1 - \frac{1+g_2}{1+k}} \\
&= \frac{D_1}{k-g_1} \left( 1 - \left( \frac{1+g_1}{1+k} \right)^T \right) + \frac{1}{(1+k)^T} \times \underbrace{\frac{D_{T+1}}{k-g_2}}_{P_T} \\
&= \frac{D_1}{k-g_1} \left( 1 - \left( \frac{1+g_1}{1+k} \right)^T \right) + \frac{(1+g_2)(1+g_1)^{T-1} D_1}{(k-g_2)(1+k)^T} .
\end{aligned}
$$

# 3 Bubbles

Suppose

$$P_t \ = \ \sum_{s=1}^{\infty} \frac{E[D_{t+s}]}{(1+k)^s} \ + \ (1+k)^t b \ .$$