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Estimation Under Unknown Correlation: Covariance Intersection Revisited

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***Abstract*—This note addresses the problem of obtaining a consistent estimate (or upper bound) of the covariance matrix when combining two quantities with unknown correlation. The combination is defined linearly with two gains. When the gains are chosen *a priori*, a family of consistent estimates is presented in the note. The member in this family having minimal trace is said to be "family-optimal." When the gains are to be optimized in order to achieve minimal trace of the family-optimal estimate of the covariance matrix, it is proved that the global optimal solution is actually given by the Covariance Intersection algorithm, which conducts the search only along a one-dimensional curve in the n-squared-dimensional space of combination gains.**

***Index Terms*—Consistent estimation, Covariance Intersection, data fusion, filtering, Kalman filter, unknown correlation.**

I. Introduction

The Kalman filter has become one of the cornerstones of modern technology. At each recursive step, it provides a convenient way to combine a projected estimate of some state with information provided by a measurement on this state in order to obtain a new estimate together with its accuracy (covariance). In deriving the covariance matrix of the new estimation error and the optimal Kalman gain that minimizes its trace (so that the estimation is optimal in the least-square sense), it is assumed that the prior estimation error and the new measurement error are uncorrelated. Although this assumption is often only an approximation to the reality, in many situations it suffices for the problems being considered, and Kalman filter has been successfully applied in a wide spectrum of fields.

However, there are situations in which the assumption of independence may lead to serious problems for estimation. For example, in a distributed network, when a node A receives a piece of information from a node B, the topology of the network may be such that B is passing along the information it originally received from A, not "new" information. Thus, if A were to "combine" this information with the old one using Kalman filter update-type equations under the independence assumption, then the covariance matrix (as an indicator of the uncertainty about this information) would be reduced, when in fact it should remain at the same level. Clearly there is a need to combine two quantities in the presence of unknown correlation, and to provide an appropriate estimate of the resulting covariance matrix.

In the seminal papers [1] and [2], the Covariance Intersection (CI) algorithm was proposed to deal with this problem. The objective is to obtain a *consistent* estimate of the covariance matrix when two random variables are linearly combined. By "consistent," we mean that the estimated covariance is always an "upper-bound" (in the positive–definite sense; see Section II) of the true covariance, even when the correlation is unknown. Thus, in the example above, after node A combines the information, the covariance matrix will remain approximately the same, rather than incorrectly reduced. Judiciously combined with Kalman filter and prior knowledge about the systems, the CI algorithm has found wide applications, particularly in the area of distributed estimation [3]–[10].

Yet, there are questions that are not answered by the CI algorithm. When the variables being combined are n-dimensional vectors, a combination gain is a matrix with n^2 elements, thus, it can be chosen from an n^2-dimensional space. However, the variable ω in the CI algorithm parameterizes only a one dimensional curve in this space. In order to get a complete picture, we pose two separate problems in this note. The first problem is to obtain a consistent estimate of the covariance matrix when *fixed* combination gains are used. We solve this problem by presenting a family of such estimates. The member in this family having the minimal trace can be determined analytically, and is referred to as the "family-optimal" estimate. The second problem is to find the *best* pair of gains that minimizes the trace of the above family-optimal estimate. In general this is an optimization problem in an n^2 dimensional space of combination gains. However, we prove that the global optimal solution is actually given by the CI algorithm, even though it conducts the search only along a one dimensional curve.

The note is organized as follows. First, a statement of the problem is given in Section II. Following this, the CI algorithm is reviewed in Section III. The main results of this note are presented in Sections IV–V. Finally, some conclusions are drawn in Section VI.

II. Problem Statement

To highlight the essence of the results, no dynamics are considered in this note, and the problem is simply stated as combining two estimates of the mean value of a random variable when the correlation between the estimation errors is unknown. The basic notations in [1] are followed here, but for simplicity no distinction is made between a random variable and its observation. More specifically, let c^* be the mean value of some random variable to be estimated. Two sources of information are available: estimate a and estimate b. Define their estimation errors as

$$\tilde{a} = a - c^* \quad \tilde{b} = b - c^*$$

and assume that

$$\begin{aligned} E\{\tilde{a}\} &= 0 \quad E\{\tilde{a}\tilde{a}^T\} = \tilde{P}_{aa} \\ E\{\tilde{b}\} &= 0 \quad E\{\tilde{b}\tilde{b}^T\} = \tilde{P}_{bb}. \end{aligned}$$

The true values of $\tilde{P}_{aa}$ and $\tilde{P}_{bb}$ may not be known, but some consistent estimates are known

$$P_{aa} \geq \tilde{P}_{aa} \quad P_{bb} \geq \tilde{P}_{bb}. \tag{1}$$

Here, inequality is in the sense of matrix positive semidefiniteness, i.e., $A \geq B$ if and only if $A - B$ is positive semidefinite. The correlation between the two estimation errors $E\{\tilde{a}\tilde{b}^T\} = \tilde{P}_{ab}$ is also unknown.

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Our objective is to construct a *linear unbiased* estimator c that combines a and b

$$c = K_1 a + K_2 b \tag{2}$$

where $a, b \in \mathbb{R}^n$ and $K_1, K_2 \in \mathbb{R}^{n\times n}$.

Define $\tilde{c} = c - c^*$. It follows that $E\{\tilde{c}\} = 0$ if and only if

$$K_1 + K_2 = I. \tag{3}$$

The covariance $E\{\tilde{c}\tilde{c}^T\} = \tilde{P}_{cc}$ may not be known, but we want to find a consistent estimate P_{cc}

$$P_{cc} \geq \tilde{P}_{cc}. \tag{4}$$

Note that

$$\tilde{P}_{cc} = K_1\tilde{P}_{aa}K_1^T + K_2\tilde{P}_{bb}K_2^T + K_1\tilde{P}_{ab}K_2^T + K_2\tilde{P}_{ab}^T K_1^T. \tag{5}$$

We formulate the following two problems.

Problem 1: Determine a consistent estimate (upper bound) P_{cc} for $\tilde{P}_{cc}$ in (5) for any *given* pair of K_1 and K_2.

Problem 2: Find the pair of K_1 and K_2 such that the upper bound P_{cc} is optimal in some sense, e.g., minimal trace or determinant.

If $\tilde{P}_{ab} = 0$, then Problem 1 is easily solved by noting that the estimate

$$P_{cc} = K_1 P_{aa} K_1^T + K_2 P_{bb} K_2^T$$

is consistent as a direct consequence of (1). For Problem 2, the trace of the aforementioned P_{cc} is minimized by

$$\begin{aligned} P_{cc} &= \left(P_{aa}^{-1} + P_{bb}^{-1}\right)^{-1} \\ K_1 \triangleq P_{cc}P_{aa}^{-1} &= P_{bb}(P_{aa}+P_{bb})^{-1} \\ K_2 \triangleq P_{cc}P_{bb}^{-1} &= P_{aa}(P_{aa}+P_{bb})^{-1}. \end{aligned}$$

This corresponds to the derivation of the Kalman gain in the Kalman filter.

III. The Covariance Intersection Algorithm

If $\tilde{P}_{ab} \neq 0$ but is *known*, then P_{cc} can be given by

$$P_{cc} = [K_1, K_2]\begin{bmatrix} P_{aa} & \tilde{P}_{ab} \\ \tilde{P}_{ab}^T & P_{bb}\end{bmatrix}\begin{bmatrix} K_1^T \\ K_2^T\end{bmatrix}.$$

The best choice of K_1 and K_2 that minimizes the trace of P_{cc} can be obtained by solving the following constrained optimization problem:

$$\min_K \operatorname{tr}\{KPK^T\} \text{ subject to } K\begin{bmatrix} I \\ I\end{bmatrix} = I$$

where

$$K \triangleq [K_1, K_2] \quad P \triangleq \begin{bmatrix} P_{aa} & \tilde{P}_{ab} \\ \tilde{P}_{ab}^T & P_{bb}\end{bmatrix}. \tag{6}$$

The optimal solution of K_1 and K_2 yields a P_{cc} in the following form:

$$\begin{aligned} P_{cc}^{-1} &= [I \quad I]\, P^{-1}\begin{bmatrix} I \\ I\end{bmatrix} \\ &= P_{aa}^{-1} + \left(P_{aa}^{-1}\tilde{P}_{ab} - I\right) \\ &\quad \times \left(P_{bb} - \tilde{P}_{ab}^T P_{aa}^{-1}\tilde{P}_{ab}\right)^{-1}\left(\tilde{P}_{ab}^T P_{aa}^{-1} - I\right). \end{aligned} \tag{7}$$

Covariance ellipses, as defined in the following, provide a convenient way of visualizing the relative size of covariance matrices. For a positive–definite matrix Q, we define

$$\mathcal{B}_Q(l) \triangleq \{x : x^T Q^{-1} x < l\}. \tag{8}$$

A covariance ellipse at level l is the boundary of $\mathcal{B}_Q(l)$. (We will omit l in the following discussions). Thus, if $Q_1 < Q_2$, then $\mathcal{B}_{Q_1} \subset \mathcal{B}_{Q_2}$. Now, we show the following.

1) For a *given* $\tilde{P}_{ab}$ and, hence, the optimal P_{cc} in (7), we have

$$\mathcal{B}_{P_{cc}} \subset \mathcal{B}_{P_{aa}} \cap \mathcal{B}_{P_{bb}}.$$

Proof: Since P in (6) is positive definite, the second term in (7) is positive definite. Thus, $x^T P_{cc}^{-1} x < l$ implies $x^T P_{aa}^{-1} x < l$, i.e., $\mathcal{B}_{P_{cc}} \subset \mathcal{B}_{P_{aa}}$. Similarly, $\mathcal{B}_{P_{cc}} \subset \mathcal{B}_{P_{bb}}$. ∎

2) For any point $x \in \mathcal{B}_{P_{aa}} \cap \mathcal{B}_{P_{bb}}$, there exists a correlation matrix $\tilde{P}_{ab}$ such that: i) P as given in (6) is positive definite, and ii) $x \in \mathcal{B}_{P_{cc}}$ where P_{cc} is given by (7).

Proof: First, we assume that

$$l > x^T P_{aa}^{-1} x > x^T P_{bb}^{-1} x > 0.$$

Define

$$\lambda^2 \triangleq \frac{x^T P_{bb}^{-1} x}{x^T P_{aa}^{-1} x}.$$

It follows that $\lambda < 1$. Since the vector $\lambda P_{aa}^{-(1/2)} x$ and $P_{bb}^{-(1/2)} x$ have the same length, one can be rotated to another by a unitary matrix U, i.e.,

$$\lambda P_{aa}^{-(1/2)} x = U P_{bb}^{-(1/2)} x \quad UU^T = U^T U = I.$$

The matrix

$$\tilde{P}_{ab} \triangleq \lambda P_{aa}^{1/2} U P_{bb}^{1/2}$$

satisfies our requirements as follows.

i) P as given in (6) is positive definite

$$P_{aa} > 0, \qquad P_{bb} - P_{ab}^T P_{aa}^{-1} P_{ab} = (1-\lambda^2)P_{bb} > 0.$$

ii) For P_{cc} as given in (7), we have

$$\begin{aligned} x^T P_{cc}^{-1} x &= x^T P_{aa}^{-1} x \\ &\quad + x^T\left(P_{aa}^{-1}\tilde{P}_{ab} - I\right)\left(P_{bb} - \tilde{P}_{ab}^T P_{aa}^{-1}\tilde{P}_{ab}\right)^{-1} \\ &\quad \times \left(\tilde{P}_{ab}^T P_{aa}^{-1} - I\right)x \\ &= x^T P_{aa}^{-1} x < l \end{aligned}$$

and, therefore, $x \in \mathcal{B}_{P_{cc}}$. Other cases can be similarly proved by symmetry or by continuity argument [since we use strict inequality in (8)]. ∎

Based on the aforementioned observation, when $\tilde{P}_{ab}$ is not known, a consistent estimate of P_{cc} should be such that $\mathcal{B}_{P_{cc}} \supset \mathcal{B}_{P_{aa}} \cap \mathcal{B}_{P_{bb}}$, or loosely speaking, P_{cc} should include the intersection of P_{aa} and P_{bb}. This motivated the CI algorithm [1], [2]

$$P_{cc}^{-1} = \omega P_{aa}^{-1} + (1-\omega)P_{bb}^{-1} \tag{9}$$

$$K_1 = \omega P_{cc}P_{aa}^{-1}, K_2 = (1-\omega)P_{cc}P_{bb}^{-1} \tag{10}$$

where $\omega \in [0, 1]$ is a parameter.

The CI algorithm requires ω to be optimized at every step, for example by minimizing the trace or the determinant of P_{cc}. Since the CI algorithm computes the gains K_1 and K_2, it does not provide a complete solution to Problem 1, where the gains are fixed *a priori*. For Problem 2, we show in Section V that the CI algorithm does provide the global optimal solution, even though it searches only along a one-dimensional curve as shown by (10), while the gains in the general case can be chosen from $\mathbb{R}^{n\times n}$.

An illustration of the previous discussion on intersection is shown in Figs. 1 and 2. In the former figure, three different known matrices $\tilde{P}_{ab}$

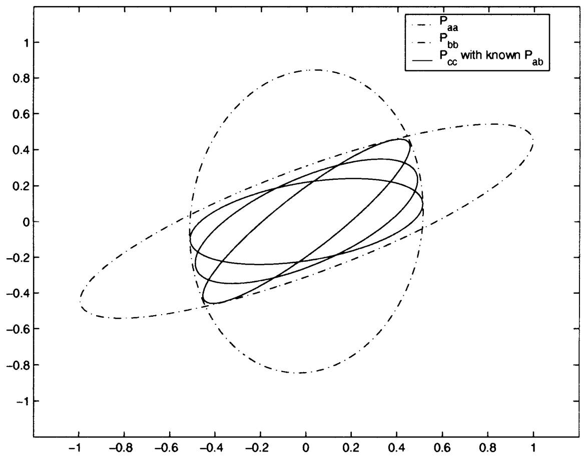

Fig. 1. Solid: Three examples of P_{cc} with $\tilde{P}_{ab}$ known. Dashed: P_{aa} and P_{bb}.

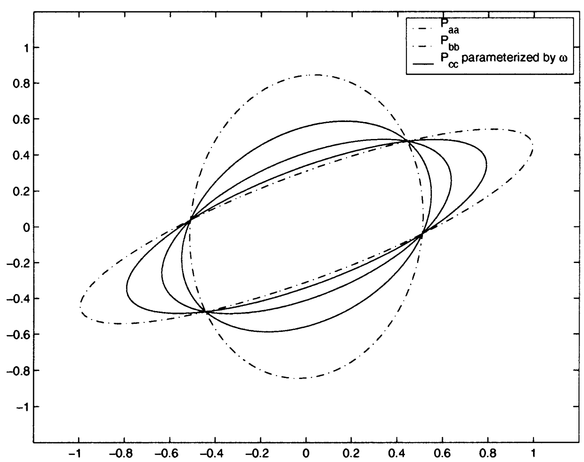

Fig. 2. Solid: Three examples of P_{cc} containing the intersection. Dashed: P_{aa} and P_{bb}.

are chosen, and the corresponding optimal covariance matrices P_{cc} are obtained, whose covariance ellipses at level 1 are shown in solid lines, while those for P_{aa} and P_{bb} are shown in dashed lines. In the latter figure, three different values of ω are chosen, and the corresponding covariance matrices are shown in the same fashion.

IV. A Solution to Problem 1

In order to obtain an upper bound for

$$\tilde{P}_{cc} = K_1\tilde{P}_{aa}K_1^T + K_2\tilde{P}_{bb}K_2^T + K_1\tilde{P}_{ab}K_2^T + K_2\tilde{P}_{ab}^TK_1^T$$

when the correlation $\tilde{P}_{ab}$ is unknown, the following inequality is utilized:

$$E\left\{\left(\sqrt{\gamma}K_1\tilde{a} - \frac{1}{\sqrt{\gamma}}K_2\tilde{b}\right)\left(\sqrt{\gamma}K_1\tilde{a} - \frac{1}{\sqrt{\gamma}}K_2\tilde{b}\right)^T\right\} \geq 0 \quad (11)$$

where $\gamma > 0$ is a scalar parameter. It follows that

$$\gamma K_1\tilde{P}_{aa}K_1^T + \frac{1}{\gamma}K_2\tilde{P}_{bb}K_2^T \geq K_1\tilde{P}_{ab}K_2^T + K_2\tilde{P}_{ab}^TK_1^T. \quad (12)$$

Therefore, from (1) and (12), a consistent estimate P_{cc} of $\tilde{P}_{cc}$ is characterized by the family

$$P_{cc} = (1+\gamma)K_1P_{aa}K_1^T + \left(1+\frac{1}{\gamma}\right)K_2P_{bb}K_2^T, \qquad \gamma > 0. \quad (13)$$

It should be noted that this family of upper bounds is tight only for certain pairs of P_{aa} and P_{bb}, and is not tight in general. As an example, when

$$K_1 = K_2 = \frac{1}{2}I \quad P_{aa} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad P_{bb} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$$

there is no γ for which the bound is tight.

Values of γ can be chosen to optimize various performance criteria, and we will refer to this type of optimality as "family-optimal." To minimize the trace of P_{cc}, note that

$$\begin{aligned}\mathrm{tr}\{P_{cc}\} &= t_1 + t_2 + \gamma t_1 + \frac{1}{\gamma}t_2\\ &\geq t_1 + t_2 + 2\sqrt{t_1t_2}\\ &= \left(\sqrt{t_1} + \sqrt{t_2}\right)^2\end{aligned} \quad (14)$$

where

$$t_1 \triangleq \mathrm{tr}\left\{K_1P_{aa}K_1^T\right\} \quad t_2 \triangleq \mathrm{tr}\left\{K_2P_{bb}K_2^T\right\}$$

and the equality holds when

$$\gamma = \sqrt{\frac{t_2}{t_1}} = \sqrt{\frac{\mathrm{tr}\left\{K_2P_{bb}K_2^T\right\}}{\mathrm{tr}\left\{K_1P_{aa}K_1^T\right\}}}. \quad (15)$$

Therefore, we have the following.

Theorem 1—Fixed Gains: For any given K_1 and K_2, a family of consistent estimates of the covariance matrix is given by (13). When γ is chosen by (15), the trace of the corresponding estimate is minimized and its value is given by (14).

Minimizing the trace of the covariance matrix is convenient when the CI algorithm is used in combination with the Kalman filter in a distributed estimation scheme. If each node in the network updates its own estimates using estimates from other nodes as well as measurements from its own set of sensors, a possible estimation scheme is as follows. The CI algorithm is used to update the current estimate when an estimate from a different node arrives, since the two may be correlated and the correlation is often unknown. Kalman filter update equations are used to update the current estimate when measurements from its own sensors become available, if the measurements are known to be uncorrelated. Within this framework, and in view of the fact that the Kalman filter minimizes the trace of the covariance matrix, it is important to have the CI algorithm minimize the trace as well.

V. Solution to Problem 2

A general solution to this problem takes the form

$$K_1, K_2 = \arg\min_{K_1,K_2}\min_{\gamma} J(P_{cc}(K_1, K_2, \gamma)), \qquad K_1 + K_2 = I$$

where J represents a performance criteria such as trace or determinant. In the case of trace minimization, we show that the above optimal solution is given by the CI algorithm (with trace minimization).

Theorem 2—Optimal Gains: There exists $\omega^* \in [0, 1]$ such that

$$h \triangleq \sqrt{\mathrm{tr}\left\{K_1P_{aa}K_1^T\right\}} + \sqrt{\mathrm{tr}\left\{K_2P_{bb}K_2^T\right\}}, \qquad K_1 + K_2 = I \quad (16)$$

is minimized by

$$\begin{aligned}P_{cc} &= \left(\omega^*P_{aa}^{-1} + (1-\omega^*)P_{bb}^{-1}\right)^{-1}\\ K_1 &= \omega^*P_{cc}P_{aa}^{-1} \quad K_2 = (1-\omega^*)P_{cc}P_{bb}^{-1}.\end{aligned}$$

Proof:: For the case when h in (16) is minimized by either $K_1 = 0$ or $K_2 = 0$, ω^* can be chosen as 0 or 1. In the following, we will assume that $K_1 \neq 0$ and $K_2 \neq 0$.

Define the following Lagrange function:

$$L = \sqrt{\mathrm{tr}\left\{K_1 P_{aa} K_1^T\right\}} + \sqrt{\mathrm{tr}\left\{K_2 P_{bb} K_2^T\right\}} - \sum_{i,j} \left(\Lambda \cdot (K_1 + K_2 - I)\right)_{i,j}$$

where $\sum_{i,j} M_{i,j}$ is the summation of all the elements of the matrix M, the operator "$\cdot$" denotes elementwise product of two matrices, and Λ is a matrix of Lagrange multipliers. Using the identity

$$\frac{\partial \mathrm{tr}\{XPX^T\}}{\partial X} = 2XP, \qquad P = P^T$$

the stationary points are given by the following equations:

$$\frac{\partial L}{\partial K_1} = \frac{K_1 P_{aa}}{\sqrt{\mathrm{tr}\left\{K_1 P_{aa} K_1^T\right\}}} - \Lambda = 0 \tag{17}$$

$$\frac{\partial L}{\partial K_2} = \frac{K_2 P_{bb}}{\sqrt{\mathrm{tr}\left\{K_2 P_{bb} K_2^T\right\}}} - \Lambda = 0 \tag{18}$$

$$\frac{\partial L}{\partial \Lambda} = K_1 + K_2 - I = 0. \tag{19}$$

Let

$$\alpha \triangleq \sqrt{\mathrm{tr}\left\{K_1 P_{aa} K_1^T\right\}} \quad \beta \triangleq \sqrt{\mathrm{tr}\left\{K_2 P_{bb} K_2^T\right\}}.$$

From (17), we have $K_1 = \alpha \Lambda P_{aa}^{-1}$. Similarly, $K_2 = \beta \Lambda P_{bb}^{-1}$. Substituting into (19), we have

$$\Lambda = \left(\alpha P_{aa}^{-1} + \beta P_{bb}^{-1}\right)^{-1}. \tag{20}$$

Note that $\Lambda^T = \Lambda$. Now, the definition of α and β yields

$$\alpha = \sqrt{\mathrm{tr}\left\{\alpha \Lambda P_{aa}^{-1} P_{aa} P_{aa}^{-1} \Lambda \alpha\right\}} = \alpha \sqrt{\mathrm{tr}\left\{\Lambda P_{aa}^{-1} \Lambda\right\}}$$

or

$$\mathrm{tr}\left\{\Lambda P_{aa}^{-1} \Lambda\right\} = 1. \tag{21}$$

Similarly

$$\mathrm{tr}\left\{\Lambda P_{bb}^{-1} \Lambda\right\} = 1. \tag{22}$$

Equations (20), (21), and (22) lead to polynomial equations of order $2n$ in the variables α and β. Our objective here is to parameterize the solutions using a one-dimensional variable. Recall that the minimum trace h^2 (where h is defined in the theorem) is achieved by the choice of (15) in the family given by (13). The parameter γ now becomes

$$\gamma = \sqrt{\frac{\mathrm{tr}\left\{K_2 P_{bb} K_2^T\right\}}{\mathrm{tr}\left\{K_1 P_{aa} K_1^T\right\}}} = \sqrt{\frac{\beta^2 \mathrm{tr}\left\{\Lambda P_{bb}^{-1} \Lambda\right\}}{\alpha^2 \mathrm{tr}\left\{\Lambda P_{aa}^{-1} \Lambda\right\}}} = \frac{\beta}{\alpha}.$$

Thus, the family-optimal covariance matrix is

$$\begin{aligned} P_{cc} &= \left(1 + \frac{\beta}{\alpha}\right) K_1 P_{aa} K_1^T + \left(1 + \frac{\alpha}{\beta}\right) K_2 P_{bb} K_2^T \\ &= (\alpha + \beta)\alpha \Lambda P_{aa}^{-1} \Lambda + (\alpha + \beta)\beta \Lambda P_{bb}^{-1} \Lambda \\ &= (\alpha + \beta)\Lambda \left(\alpha P_{aa}^{-1} + \beta P_{bb}^{-1}\right) \Lambda \\ &= (\alpha + \beta)\Lambda \\ &= \left(\frac{\alpha}{\alpha + \beta} P_{aa}^{-1} + \frac{\beta}{\alpha + \beta} P_{bb}^{-1}\right)^{-1} \end{aligned}$$

and the gains are

$$K_1 = \frac{\alpha}{\alpha + \beta} P_{cc} P_{aa}^{-1} \quad K_2 = \frac{\beta}{\alpha + \beta} P_{cc} P_{bb}^{-1}.$$

The theorem is proved by setting $\omega^* = \alpha/(\alpha + \beta)$. ■

This theorem reveals the nature of the optimality of the best ω in CI algorithm. According to the theorem, the n^2 dimensional optimization problem can be reduced to a one-dimensional one.

VI. Conclusion

The CI algorithm is reexamined in this note, in the general framework of obtaining a consistent estimate of the covariance matrix when combining two quantities with unknown correlation. For the case when the gains are chosen, a family of consistent estimates is given. For the case when optimal gains are to be found in order to minimize the trace of the estimated covariance, it is proved that the solution is given by the CI algorithm, which conducts the search on a one-dimensional curve rather than in the whole parameter space and, thus, the optimization problem can be solved very efficiently. The results reported in this note can be extended to the case with dynamical equations in a straightforward fashion. It can also be extended to the case of combining more than two variables, and to the case of partial observations where only $y = Hx$ is available, x being the quantity of interest. It is the authors' belief that with the newly gained understanding, the CI algorithm will find more applications in the areas of distributed filtering and estimation and data fusion.

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