The two-dimensional tuning
A mathematical model

by Linas Plankis

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Preface

“Pure sine waves sound annoying”. This truth is known since the first generator was tried to be converted to a musical instrument. Nobody can say, that it wasn't done, but the instruments, got this way, lacked something, what makes their music acceptable for virtuosos. Then, many masters of sound synthesis tried to imitate sound of natural music instruments. They succeeded in many aspects, but it wasn't a final success. There were also many people, who wanted to create new musical instruments, and sound synthesis technologies allowed to get fantastic sounds. But anybody, who has dealt with it, knows, how it was difficult to overpass the limit of the annoying sine. It's impossible to describe all technologies here, that seemed to be suitable to achieve this goal. The main thing for us, that among general effects, got using fine tuning, reverberation, regular changes of wave shapes and so on, that were used to reduce the annoying sine, some computer music enthusiasts already tried to apply effects, that had different parameters for each sound of the gamut. And, in fact, it's a beginning of the multidimensional tuning in it. We can say now, that the multidimensional tuning and specifically the two-dimensional tuning is really existing practice, although it hasn't any proven theory and, perhaps as a result, has many approaches.

Knowing this and wanting to make some, although small, input developing the idea of the two-dimensional tuning, I present a mathematical model of the two-dimensional tuning. This model is relatively simple. To understand the material in the abstract , knowledge of the mathematical theory of musics is required and the basic knowledge of mathematics is requested. The model isn't any proven theory too, it's intended to be a patch to such theories rather than to be a such theory itself. Some statements in this abstract also can seem not well grounded or taken “from nowhere”, but actually they are thought-out throughly. Particularly this model intended to be helpful both for composing and for designing new software and instrumental hardware. For composing, this model needs a deeper understanding only, for example, the understanding of methods of gamuts transposing, and in many cases it could be preferred to traditional models as being less defined in purely musical aspect, but having strict mathematical definitions (thus it means, that it doesn't pretend to restrict creative ideas of composers, but it gives quite a strict scheme allowing not to wander vainly in infinite world of sounds). The next goal, to make instrumental software, needs an additional input, in what mathematical ideas should be driven to acoustic formulas, and I suggest no concrete approach for it here. For the model is intended to be a start point for discussions and for this abstract contains just the core part of the model, anybody, who interests in it, is encouraged to write me your questions, ideas and suggestions (my e-mail is l.plankis@freemail.lt).

I'm also afraid, some errors are possible in the text, and I'll be thankful, if anybody e-mails me in a case anyone occurred. It also concerns phrases and sentences in the text, some of them can be slightly misunderstood, re my habits from Eastern-European usage.

Linas Plankis　　　　　　　　　　Vilnius, 10-5-2005

The main statements of the model

1. Any musical sound is tuned, changing two (not one) characteristics. The first characteristic is height and the second is hardness of sound.

2. The height of sound (marked f_h in this abstract) is measured in units of frequency and has nature of frequency. The hardness of sound is also based upon a quantity of nature of frequency. This quantity is called the frequency of hardness, or f_d here.

3. Sound is considered the harder the bigger ratio f_d/f_h is.

4. When we have a gamut with a certain number (what number – 1 is marked m here) of sounds with their heights (pitches) $f_{h0}, f_{h1}, ..., f_{hm}$, where $f_{hm} = f_{h0}$, and $0 \leq n_1 < n_2 \leq$ m implies $f_{hn1} < f_{hn2}$, the hardness of sounds $d_n = f_{dn}/f_{hn}$ ($0 \leq n \leq m$) isn't constant. And we have pairs (f_{h0}, d_0), (f_{h1}, d_1), ... (f_{hm}, d_m), that characterize the sounds of the gamut.

(The quantitative elements of the model)

1(5). The more intuitive initial idea of this model contains a condition, that $f_d < f_h$, but it isn't essential for the mathematical description. It's sufficient, to take $\delta = d/d_s$ (where d_s is a constant value) instead of d. In the model, we can take d_s, to define certain ranges of hardness: $d_s/2 \leq d < d_s$ means soft sound, $d_s \leq d < 2d_s$ middle hard sound, and $2d_s \leq d < 4d_s$ hard sound. Using δ, these ranges are $½ \leq \delta < 1$, $1 \leq \delta < 2$ and $2 \leq \delta < 4$ respectively).

2(6). It's desired, that d_{max}/d_{min} in a gamut set were under condition, that $d_{max}/d_{min} \approx 2$ and not $d_{max}/d_{min} >> 2$.

3(7). The main measure of a transition (or, simply, of a musical interval) (f_{hn-1}, d_{n-1}) → (f_{hn}, d_n) ($0 < n \leq m$, in a gamut (f_{h0}, d_0), (f_{h1}, d_1),... (f_{hm}, d_m)) equals $i_{hn} = f_{hn}/f_{hn-1}$. The analogue ratio $i_{dn} = d_n/d_{n-1}$ is also important. One can also speak about transitional pairs (i_{h1}, i_{d1}), ..., (i_{hm}, i_{dm}) here (see the section "The main law of harmonic tuning" for interdependency between i_h and i_d).

(The points for gamuts with relative frequencies)

1(8). If we have a gamut with relative frequencies h_0 , h_1, ..., h_{m-1}, h_m with $h_0 = 1$ and $h_m = 2$ (where $h_i = f_{hj}/f_{h0}$, $0 \leq j \leq m$), which in its turn has n variations, that are based on set where $f_{h0,i} = 2f_{h0,\,i-1}$ for each i $0 < i \leq n$ (where we actually get a sequence of initial height frequencies for any variation $f_{h0,1}$, $2f_{h0,1}$, ... , $2^{n-2}f_{h0,1}$, $2^{n-1}f_{h0,1}$, in which one can easily recognize initial sounds of octaves), there are two kinds of variations for such gamuts. The key variations have the same set of values f_{d0}, ..., f_{dm} for each of these variations, but the additional variations retain the same set of values d_0, ..., d_m, taken from one of key variations. It also means, that there could be few groups of additional variations, each having its own source of a hardness values set.

The nomenclature

The main measures:

f_h – the frequency of height, generally the same what is commonly called the pitch frequency

f_d – the frequency of hardness, the second dimension of our 2-dimensional tuning.

d – the hardness, another quantity of the second dimension, $d = f_d/f_h$.

d_s – a constant, that defines the middle range of values of the hardness (that is, $d_s \le d < 2d_s$ for the middle range).

δ – a substitute of d, to have concrete numeric values, $\delta = d/d_s$.

The intervals:

i_{hn} – the height (pitch) quantity of an interval n in a gamut, or the ratio between height values of interval sounds ($i_{hn} = f_{hn}/f_{hn-1}$).

i_{dn} – the hardness of an interval, or the ratio between hardness values of interval sounds ($i_{dn} = d_n/d_{n-1}$).

i_{fdn} – another hardness quantity of an interval, or the ratio between hardness frequency values of interval sounds ($i_{fdn} = f_{dn}/f_{dn-1}$).

h – the relative frequency of height, the reference value almost always is f_{h0}, so $h = f_h/f_{h0}$ in these cases.

The logarithmic measures:

F_h – the logarithmic height frequency, $F_h = log_2 f_h$, a purely formal quantity.

F_d – the logarithmic hardness frequency, $F_d = log_2 f_d$, a purely formal quantity.

H – the logarithmic (relative) height frequency, $H = log_2 h$.

D – the logarithmic hardness, $D = log_2 d$.

Δ – the logarithmic (relative) hardness, $\Delta = log_2 \delta$.

I_{hn} – the logarithmic height quantity of an interval n in a gamut, $I_{hn} = log_2 i_{hn}$.

I_{dn} – the logarithmic hardness quantity of an interval, $I_{dn} = log_2 i_{dn}$.

Particular usage of some letters:

m – a number of sounds of a gamut,

p – an exponential scaler in the formulas of the laws of harmonic tuning, to get values of hardness in a proper range.

n – , in f_{hn}, f_{dn}, d_n, i_{hn} etc., a number of a sound in a gamut (interval values have numbers from 1 to m, but sound values have numbers from 0 to m).

The comments for the main statements of the model

1. Any musical sound is tuned, changing two (not one) characteristics. The first characteristic is height and the second is hardness of sound.

The statement, that we'll use two characteristics instead of one is already clear from the title. The height as a music term means pitch in many languages such as Italian, Russian, etc. We use this term here rather than the pitch in order not to confuse readers, who often identify pitch and frequency. The name of the second characteristic is taken from another existing characterization of what we heard as musical sound. This term was particularly used in earlier times, and it gave the abbreviation *dur* for sharp gamuts (which have derived from Latin *durus* meaning 'hard'). Sharpness, which is equivalent of the hardness in English in our case, would be even more confusing than pitch, for sharpness implies a semantic connection with frequency and thus dependence of one characteristic from the other. But, besides, the derivation of the term *hardness* doesn't mean that we develop or continue any medieval musical theories here.

2. The height of sound (marked f_h in this abstract) is measured in units of frequency and has nature of frequency. The hardness of sound is also based upon a quantity of nature of frequency. This quantity is called the frequency of hardness, or f_d here.

It's obvious, that the both characteristics of sound should be independent, or we won't have a 2-dimensional tuning. So, the same frequency measure here shouldn't confuse. The both quantities should depend to two different processes, what implies possibility of such independence. A similar example is known from the FM theory, where we have two independent frequencies the modulation frequency and the career frequency. But it's an example only. We don't want to say, that our processes run according with the known formula of the FM, where $s(t) = cos\ (2\pi f_c t + b\ sin\ 2\pi f_m t)$ (where $s(t)$ is a function of sound amplitude, f_c and f_m are career and modulation frequencies respectively, and b is a constant, known as the modulation index). But, if we took a more abstract formula, $s(t) = g_1(2\pi f_h t + g_2(2\pi f_m t))$, where g_1 and g_2 are some functions, it seems like an acceptable pattern to show what we speak about[1].

3. Sound is considered the harder the bigger ratio f_d/f_h is.

This statement could seem too speculative, but it's based on our everyday experience of hearing. But, as a pity, on all logic of this experience, rather than on a single fact of it. Another way to understand this point is simply to accept it as a conventional statement, that makes with such other assertions the building of our model.

4. When we have a gamut with a certain number (what number – 1 is marked m here) of sounds with their heights (pitches) $f_{h0}, f_{h1}, ..., f_{hm}$, where $f_{hm} = f_{h0}$, and $0 \leq n_1 < n_2 \leq$ m implies $f_{hn1} < f_{hn2}$, the hardness of sounds $d_n = f_{dn}/f_{hn}$ $(0 \leq n \leq m)$ isn't constant. And we have pairs (f_{h0}, d_0), (f_{h1}, d_1), ... (f_{hm}, d_m), that characterize the sounds of the gamut.

This statement means, that the model prefers not to reduce all hardnesses of sounds in a gamut to one value, but to variate them. It's given in such mathematical form more from need to introduce nomenclature, than trying to say anything in a sophisticated way.

Additionally, a model of gamut, concerning frequencies of height is given here. The model is the same, that we know from the standard music theory. Thus, concerning height, we take all other attributes of gamuts, e. g. ranges, names of notes, names of intervals, names of octaves, and so on, as they are in the standard music theory. And, for shortening purposes, these definitions aren't included in the list of the main statements above.

It also should be added, that the both characteristics of musical sound don't have symmetric significance. The first characteristic, the sound height is more perceptible, than the second, the sound hardness. If they were equal in this sense, then nobody doubted, that nature of tuning is 2-dimensional. But a less significance don't mean no significance.

1 It's known thing that many people imagine frequency as sine shaped change of a level. But we should try to repeat the theory here, that says, that any repeated process has frequency. Thus we don't speak about two equal processes, such as two sine waves added together. (Explaining the example, two sine waves added together give two different sounds or even one sound with a harmonic, which fact is well known for people who work with sound). Two different processes with frequency measure are object here. And the author of this abstract strongly believes, that such processes are real. But even they are imaginary only, being indivisible parts of one actual process, the further material could be interesting and perhaps helpful for understanding mathematical nature of music.

(The quantitative elements of the model)

1(5). The more intuitive initial idea of this model contains a condition, that $f_d < f_h$, but it isn't essential for the mathematical description. It's sufficient, to take $\delta = d/d_s$ (where d_s is a constant value) instead of d. In the model, we can take d_s, to define certain ranges of hardness: $d_s/2 \le d < d_s$ means soft sound, $d_s \le d < 2d_s$ middle hard sound, and $2d_s \le d < 4d_s$ hard sound. Using δ, these ranges are $½ \le \delta < 1$, $1 \le \delta < 2$ and $2 \le \delta < 4$ respectively).

In this point, we take as convention, that (1) there are 3 ranges of hardness, and (2) every range is $d_b \le d \le 2d_b$ (in analogy with boundaries of the octave). These are statements, that shape our model essentially, but they also help to introduce auxiliary δ instead of d, whose numeric value isn't defined in this model. We simply say that there's a constant d_s, that helps us to convert d to δ (using $\delta = d/d_s$). But this conversion is no more than helpful formality. Particularly the introducing δ helps us to draw graphs, avoiding suspicion, that we think, that namely $d = f_d/f_h$ equals, for example, 1.5 in the middle range, when actually this value is relative (see fig. 1 – 5 in pp. 8 – 14 for better understanding). But the ratio $\frac{\delta_1}{\delta_2} = \frac{d_1}{d_2}$ is true, when we have pairs δ_1, d_1 and δ_2, d_2.

2(6). It's desired, that $d_{\max}/d_{\min}$ in a gamut set were under condition, that $d_{\max}/d_{\min} \approx 2$ and not $d_{\max}/d_{\min} >> 2$.

The next range is defined here. When in generally $d_{\mathrm{Max}}/d_{\mathrm{Min}} = 4d_s/0.5d_s = 8$, then in a gamut we might have $d_{\max}/d_{\min} = 2$ or so. The ratio 2 here is from analogy with the octave. The statement also suggests, that we may construct gamuts in different ranges of hardness. It also doesn't define the hardness range of a gamut strictly, leaving a place for inessential deviations .

3(7). The main measure of a transition (or, simply, of a musical interval) $(f_{hn\text{-}1}, d_{n\text{-}1}) \rightarrow (f_{hn}, d_n)$ ($0 < n \le m$, in a gamut (f_{h0}, d_0), (f_{h1}, d_1),... (f_{hm}, d_m)) equals $i_{hn} = f_{hn}/f_{hn\text{-}1}$. The analogue ratio $i_{dn} = d_n/d_{n\text{-}1}$ is also important. One can also speak about transitional pairs (i_{h1}, i_{d1}), ..., (i_{hm}, i_{dm}) here (see the section "The main law of harmonic tuning" for interdependency between i_h and i_d).

We call entities, that are called musical intervals otherwise, transitions in this text (meaning however mostly intervals between two neighbour members of a gamut). But it doesn't have any significant difference. In contrary, the ratio $f_{hn}/f_{hn\text{-}1}$ is used here as the main measure of an interval. But this ratio obviously has a pair in our model, namely the ratio $d_n/d_{n\text{-}1}$. Practically we have two ratios, connected with hardness. The other is $f_{dn}/f_{dn\text{-}1}$. Now, if we mark $i_{dn} = d_n/d_{n\text{-}1}$ and $i_{fdn} = f_{dn}/f_{dn\text{-}1}$, we'll get

$$i_{dn} = \frac{i_{fdn}}{i_{hn}}$$

And when we have a pair (i_{hn}, i_{dn}) rather than single i_{dn}, we shouldn't bother yet about i_{fdn}. By the way, the definition of intervals suggests to remember, that respective definitions for f_h (i. e. ranges, names, etc.[2]) aren't included here. As we've already said, these definitions are the same, that standard nomenclature of music provides for pitch (although, speaking strictly, we allow this identity in boundaries of purely mathematical theory only, not concerning acoustics).

(The points for gamuts with relative frequencies)

1(8). If we have a gamut with relative frequencies h_0 , h_1, ..., $h_{m\text{-}1}$, h_m with $h_0 = 1$ and $h_m = 2$ (where $h_i = f_{hj}/f_{h0}$, $0 \le j \le m$), which in its turn has n variations, that are based on set where $f_{h0,i} = 2f_{h0,\,i\text{-}1}$ for

2 In other words, such common facts like that *la* of the first octave has pitch frequency of 440 Hz.

each i $0 < i \leq n$ (where we actually get a sequence of initial height frequencies for any variation $f_{h0,1}$, $2f_{h0,1}$, ... , $2^{n-2}f_{h0,1}$, $2^{n-1}f_{h0,1}$, in which one can easily recognize initial sounds of octaves), there are two kinds of variations for such gamuts. The key variations have the same set of values f_{d0}, ..., f_{dm} for each of these variations, but the additional variations retain the same set of values d_0, ..., d_m, taken from one of key variations. It also means, that there could be few groups of additional variations, each having its own source of a hardness values set.

The definition of the gamut means here a gamut in general, that is, a some sequence like *do, re, mi, ..., si, do*, which can be repeated in different octaves. We don't fix a number of notes (sounds of a particular gamut) or concrete frequency value to make the series of initial values for each variation (nor any other concrete point of reference like *la* = 440 Hz). Thus these variations mean simply the same gamut applied to different octaves here. (We used the word *variation* instead of *octave* here, avoiding misunderstanding, that our gamut should necessary start from the C note. No one should confuse this usage with musical variations or with something else). And now, when the values of height frequency are defined, it transfers us to a question, what respective values of hardness (or, perhaps, of hardness frequency) should be. And the point 1(8) gives a partial answer to this question. We state here, that there must be one or more source or key variations, that have respectively same values of hardness frequency. Now, we can try to convert them to hardnesses. So, if there are $f_{d0,i} = f_{d0,j} = f_{d0,k}$, $f_{d1,i} = f_{d1,j} = f_{d1,k}$, ..., $f_{dm,i} = f_{dm,j} = f_{dm,k}$ (where i, j, k are numbers of key variations), then

$$\delta_{0,1} = \frac{f_{h0,i}}{f_{h0,j}}\delta_{0,j} = \frac{f_{h0,i}}{f_{h0,k}}\delta_{0,k} \quad , \quad \delta_{1,1} = \frac{f_{h1,i}}{f_{h1,j}}\delta_{1,j} = \frac{f_{h1,i}}{f_{h1,k}}\delta_{1,k} \quad , ...,$$

$$\delta_{m,1} = \frac{f_{hm,i}}{f_{hm,j}}\delta_{m,j} = \frac{f_{hm,i}}{f_{hm,k}}\delta_{m,k}$$, or, knowing that $\frac{f_{h0,i}}{f_{h0,j}} = 2^{i-j}$, $\delta_{0,i} = 2^{i-j}\delta_{0j} = 2^{i-k}\delta_{0,k}$, $\delta_{1,i} = 2^{i-j}\delta_{1,j} = 2^{i-k}\delta_{1,k}$, ..., $\delta_{m,i} = 2^{i-j}\delta_{0m,j} = 2^{i-k}\delta_{m,k}$. By the way, taking in account the range of possible relative hardness δ
$\delta_{Max}/\delta_{\mathrm{Min}} = 4/0.5 = 8$, $i-j$ or $i-k$ and, at least, $j-k$ should be $|i-j| \leq 2$, $|i-k| \leq 2$, $|j-k| \leq 2$ (or, in other words, we should ensure, that difference of each pair of key variation numbers doesn't exceed 2 in absolute value). But it means that we are very restricted, grouping possible gamut variations. For instance, a number of key variations can't exceed 3. The case with additional variations is easier. Each additional variation is bound to one of key variations and shares the same values of hardness with it. Or $\delta_{0,i} = \delta_{0,a} = \delta_{0,b}$, $\delta_{1,i} = \delta_{1,a} = \delta_{1,b}$, ..., $\delta_{m,i} = \delta_{m,a} = \delta_{m,b}$, where i is a number of a key variation, but a and b are numbers of additional variations.

And what a basis do this mathematical construction have? The main idea is, that our hearing gives us musical information in a form of $(h_0, h_1, ..., h_m)$ and it suggests the same idea about $(\delta_0, \delta_1, ..., \delta_m)$. But musical instruments, including our voice, operate with $(f_{h0}, f_{h1}, ..., f_{hm})_i$, what suggest analogue idea with $(f_{d0}, f_{d1}, ..., f_{dm})_i$ (i is a number of an octave). Developing these assertions, we can assume, that some operational basis for hardness should be made, that would have few variations with repeated hardnesses. And the only fact, that we can't make more than three key variations in a gamut, restricts us not to apply this rule of equal frequencies to all variations of it. It resembles the situation with musical notation rules, when only 2 keys are used in the common pair of staves, while number of octaves is bigger.

An example for the previous section

Let's take the common gamut C – D – E – F – G – A – B – C. For easier understanding, we could also take the values of height for each note from the twelve-tone equally tempered scale and use logarithmic values instead of the simple. Thus, the nomenclature, that we used before, is changed to:

$$F_h = log_2 f_h,$$

$$F_d = log_2 f_d,$$
$$D = log_2\, d = F_d - F_h,$$
$$\varDelta = log_2\, \delta, \text{ and so on}$$

By the way, there's no need to use the logarithmic expression of height frequency (and, at least, what real entity it represents?), so we can take the logarithmic expression of the relative frequency $H = F_h - F_{hs}$, where F_{hs} is height frequency (the logarithmic expression of) of the initial note in a gamut (of the C note in our case). Now, we can group the initial data in the table:

Note nr	0	1	2	3	4	5	6	7
Note	C	D	E	F	G	A	B	C
$\boldsymbol{H_n}$	0	1/6	1/3	5/12	7/12	9/12	11/12	1

And what $\varDelta$ should we add to each value of H ? Let's use two ideas for a solution. The first idea is to link our hardness with the medieval "hardness" (or sharpness) (what we don't do in general here). So, let's assume, that every sharp sign (the so called *diesis*) gives certain increment of (logarithmic) hardness, and $\varDelta_\# = \varDelta + A$. For better understanding, let's take A constant for all possible $\varDelta$ in our gamut. Now, the next idea is the transposition. We know, that adding sharps (however according a certain sequence) we'll get a new, so called transposed, gamut each time. The idea is, that a transposed gamut should have the same distances between its values of hardness as the initial gamut. We also know, that, transposing gamuts in the proper turn, we add one sharp note and skip one simple note of the initial gamut every time. It's obvious, if we want to retain similarity of gamuts, that the note we've still skipped had minimal hardness in the previous gamut and the note, we've added has the maximal in the new one. Knowing the order of the transposing from textbooks of music, we can simply detect order of hardness of notes.

So, we make G-dur from C-dur in the first step, adding F# and skipping F, and it means that

$$\varDelta_{F\#} > \varDelta_{C, D, E, F, G, A, B}\,,$$
$$\varDelta_F < \varDelta_{C, D, E, G, A, B}\,.$$

We make D-dur from G-dur in the second step, adding C# and skipping C, and it means that

$$\varDelta_{C\#} > \varDelta_{C, D, E, F, G, A, B, F\#}\,,$$
$$\varDelta_C < \varDelta_{D, E, G, A, B}\,.$$

We make A-dur from D-dur in the third step, adding G# and skipping G, and it means that

$$\varDelta_{G\#} > \varDelta_{C, D, E, F, G, A, B, F\#, G\#}\,,$$
$$\varDelta_G < \varDelta_{D, E, A, B}\,.$$

And so on, till we make F#-dur in the sixth step, adding E# and skipping E, that has the result

$$\varDelta_{E\#} > \varDelta_{C, D, E, F, G, A, B, C\#, D\#, F\#, G\#, A\#}\,,$$
$$\varDelta_E < \varDelta_B\,.$$

We can easily found the sequence of growing hardness from the inequalities before:
$\varDelta_F < \varDelta_C < \varDelta_G < \varDelta_D < \varDelta_A < \varDelta_E < \varDelta_B < \varDelta_{F\#} < \varDelta_{C\#} < \varDelta_{G\#} < \varDelta_{D\#} < \varDelta_{A\#} < \varDelta_{E\#}$.
Now, let's show it graphically (see fig. 1).

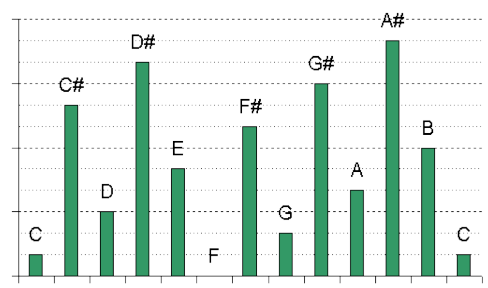

Fig. 1: Relative hardness in the set of 12 equally tempered pitches

The distances between levels of hardness are taken equal in the chart. It among other ensures, that a transposition renders the same differences between respective hardnesses as before. Everyone may try to examine it, remembering only, that a different start point for a new gamut must be taken (that is C for C-dur, D for D-dur and so on). The difference between hardness value of a sharp note and a value of the respective simple note equals to five distances between levels. The E# isn't included in the chart, serving the structure of the piano keyboard, but it should be in the same place as F and have a value one level greater than A# has. Now, let's return to the initial gamut (see fig. 2).

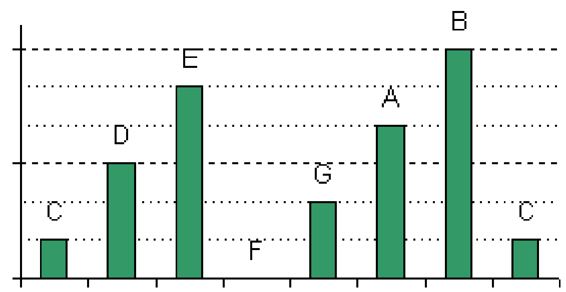

Fig. 2: Relative hardness in the subset C-dur of the set of 12 equally tempered sounds

We see, that note F is drawn to have zero value, what is legal in our model, according with the statement 1(5). But what value of logarithmic hardness the maximal note B should have? This question isn't a simple formality. If we say, that $\Delta_B = 1$, it will mean that note B depend to higher range of hardness, or, saying it in simple words, that it's a sharp note, but if we assume $\Delta_B < 1$ by one level, it will mean, that B is still a normal sound like each one from C to A. By the way, the choice yet affects the number of levels in the range, but it isn't very important in our case. Now, assuming, that $\Delta_B = 1$, we can assign Δ for our gamut.

Note nr	0	1	2	3	4	5	6	7
Note	C	D	E	F	G	A	B	C
H_n	0	1/6	1/3	5/12	7/12	9/12	11/12	1
Δ_n	1/6	1/2	5/6	0	1/3	2/3	1	1/6

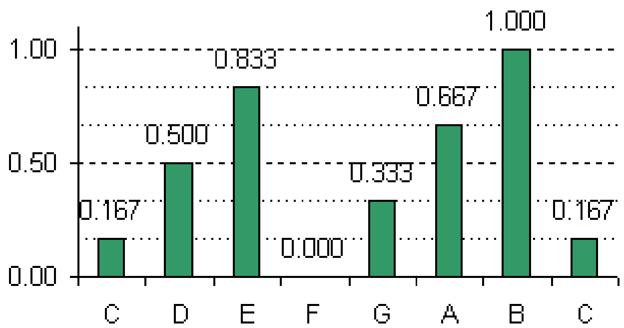

Fig. 3: Relative hardness values of the equally tempered C-dur

We should notice, that anybody can make this table even without knowing the theory above. A making of this table could serve as a good introduction to the theory. But having caught the main idea of the theory one could try to make tables of different gamuts with another number of notes, for example.

For better visuality, we also give another chart here (see fig. 4). In the chart H+2 stands for F_h, but $H + \Delta$ for F_d. It was made, because we don't know the numeral value of F_h and F_d. Knowing, that $F_h = H + F_{h0}$ and $F_d = H + \Delta + D_s$, we could draw a chart, for example, with values of $F_d - D_s$ (what equals to $H + \Delta$) and subsequently with $F_h - D_s = H + (F_{h0} - D_s)$. $F_{h0} - D_s$ is constant for a certain octave, but its value isn't defined in the model, ant it should be chosen arbitrary for a chart. We've chosen $F_{h0} - D_s = 2$ here.

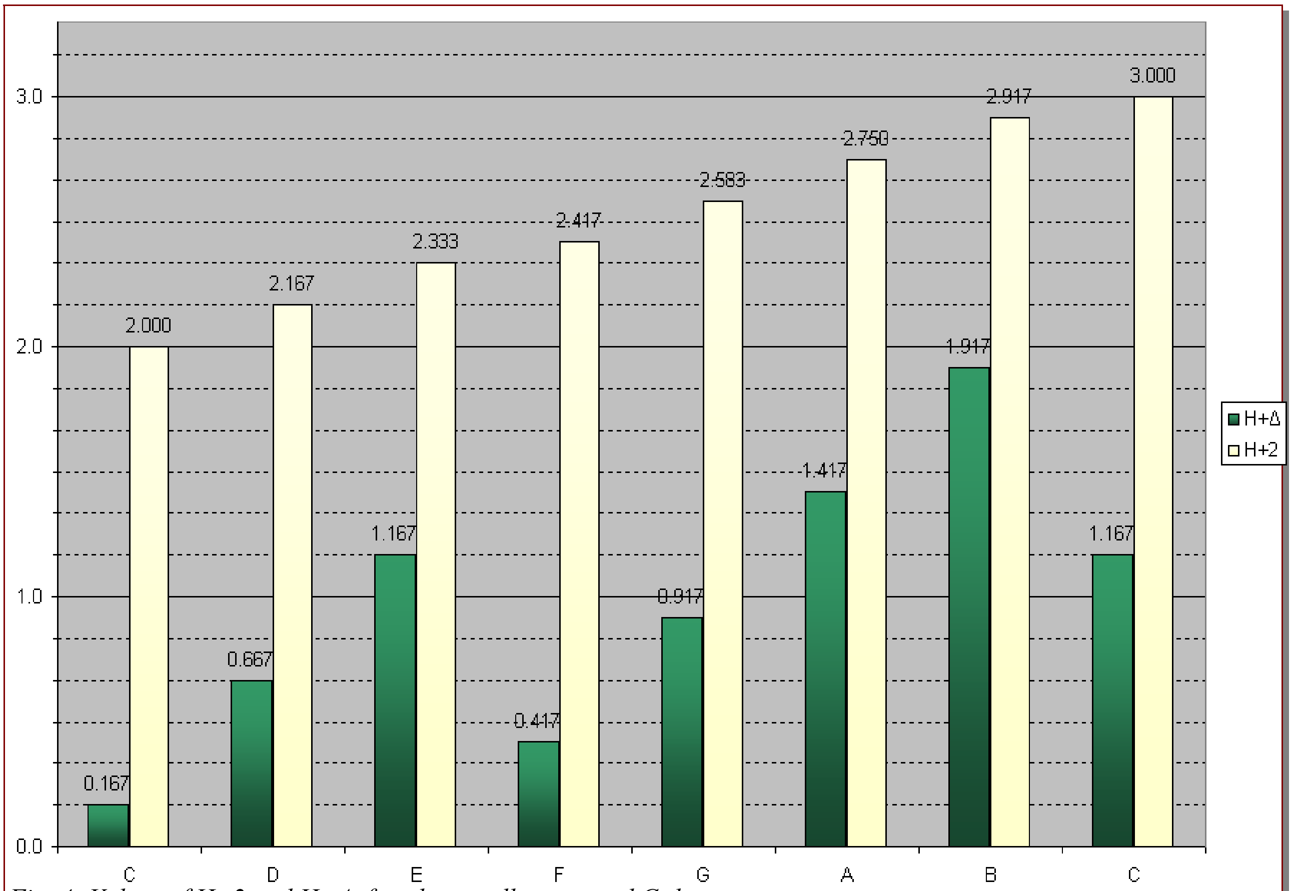

Fig. 4: Values of H+2 and H+Δ for the equally tempered C-dur gamut

The main law of harmonic tuning

The main law of harmonic tuning is based on human understanding and traditions of composing gamuts rather than on mathematical premises. So that one won't find respective ideas among laws, that describe actual acoustic processes. And we could compare this law with laws of perspective in drawings and pictures, which ones are more connected with our inner imagination than with real 3-dimensional world. However in the both cases, both with tuning and with pictures perspective, respective definitions have undoubted regular references with reality, that not allow to keep these laws purely fictional.

The law itself is relatively simple. Let's take a gamut with number of sounds (notes) $m+1$ and values of h (where $h_n = f_{hn}/f_{h0}$) and d $(1, d_0)$, (h_1, d_1), ..., (h_n, d_n), ..., $(2, d_m)$. For easier understanding, we should assume, that $d_m = d_0$ and take another different gamut with $m+1$ members, that has the equally tempered sequence of heights. Using mathematical symbols, this sequence will be $h_{s0} = 1$, $h_{s1} = 2^{1/m}$, ..., $h_{sn} = 2^{n/m}$, ..., $h_{sm} = 2$. Then, using this auxiliary sequence of h_s and having assumed that $d_0 = d_m$, we'll get values of d_n for the main gamut:

$$d_n = d_0 \left(\frac{h_n}{h_{sn}} \right)^p$$

What means, that d_n depends on deviation of h_n from middle values h_{sn}. p is an exponential scaler here, that returns required interval for values of d_n. The more common form of this formula will be

$$d_n = d_0 \left(\frac{h_n}{2^{\frac{n}{m}}} \right)^p$$

This is almost the formula, that we look for. We just should expand it to the case when $d_0 \neq d_n$. It will be done, when we multiply the right part of the formula by geometrically equally distributed increasing of d_m, that is, by $\left(\frac{d_m}{d_0} \right)^{\frac{n}{m}}$. Now, we have the formula of the law:

$$d_n = d_0 \left(\frac{d_m}{d_0} \right)^{\frac{n}{m}} \left(\frac{h_n}{2^{\frac{n}{m}}} \right)^p , p > 0 \quad (1)$$

Besides this, we also have another dependency for tuning, that even has its counterpart in acoustic physics, namely the Pythagorean law for multiple frequencies. And we aren't to negate significance of this Pythagorean law, although it's obvious, that just Pythagorean rules don't make harmony yet. And when we apply Pythagorean rules alone, we get precision rather than harmony, if we haven't cared to add something, we speak about here. Taking in account this, our formula should be called the second law of harmonic tuning rather than the main, leaving the name of the first law for the Pythagorean one. It would be more correct both from logical and from practical and historical point of view. But let's not bother about names here.

The second law of harmonic tuning also has some consequences. The first and perhaps the main one is that any equally tempered gamut will have constant hardness. But such gamuts are illegal in our model, according to the statement 4. Or, speaking in other words, a gamut to be musical mayn't be equally tempered. It should be noticed only, that we mean a musical gamut here and not a series of keys for making several gamuts, which series couldn't have such limitations. By the way, the possibility to make such series of keys, getting a sequence of sounds with equally tempered heights, is also one of consequences of the formula discussed. Look the examples in the next section for how it works in this case.

The second law of harmonic tuning has yet another interpretation, based more on quantities of separate intervals. The main definition in this case consists in that a harmonic division of a gamut is indicated by an existence of a certain real positive number $i_s > 1$, which has to go under certain conditions. In other words, if a certain $i_s > 1$ exists, which fulfills the conditions below, then the gamut is harmonic, otherwise it isn't. The conditions for i_s are these:

Firstly, if we have a gamut, the equality $i_{hn} = i_s$ (where $i_{hn} = f_{hn}/f_{hn\text{-}1}$) has to imply $i_{fdn} = i_{hn}$ (where $i_{fdn} = f_{dn}/f_{dn\text{-}1}$), the inequality $i_{hn} > i_s$ has to imply $i_{fdn} > i_{hn}$, and $i_{hn} < i_s$ has to imply $i_{dhn} < i_{hn}$. We can also change the implied inequalities to their equivalents, for $i_{fdn} = i_{hn}$ means that $i_{dn} = i_{fdn}/i_{hn} = 1$, $i_{fdn} > i_{hn}$ means $i_{dn} > 1$, and $i_{fdn} < i_{hn}$ means $i_{dn} < 1$. But $i_{dn} = 1$, $i_{dn} > 1$ and $i_{dn} < 1$ are in their turn the same as $d_n = d_{n-1}$, $d_n > d_{n-1}$ and $d_n < d_{n-1}$ respectively. So, repeating our first condition in other words, it means that when we have an interval, whose height ratio equals to i_s, the first and the second sounds of the interval must have the same hardness, but when we have an interval, whose height ratio is greater than i_s, the next sound of the interval must be harder than the first, and when we have an interval, whose height ratio is less then i_s, the next sound of the interval must be softer then the first.

The next condition complements the first one. It compares any two transitions (intervals) of the gamut, we've chosen, and it says, that $i_{hn1} < i_{hn2}$ has to imply $i_{fhn1} < i_{fhn2}$ and $i_{hn1} = i_{hn2}$ has to imply $i_{fhn1} = i_{fhn2}$. Just we should notice here, that i_{hn} is always $i_{hn} > 1$, but i_{fhn} is simply a positive value, without such restrictions[3].

The two conditions above describe the harmonic rule in qualitative manner. But, while mathematics prefer quantitative definitions, we need to convert it to a quantitative one. Let's take the simplest formula that fits for our case: $f_{dn}=f_{dn-1}\left(\frac{f_{hn}}{f_{hn-1}}\right)\left(\frac{f_{hn}}{f_{hn-1}i_s}\right)^p$ Or, using *i*-s,

$$i_{fdn}= i_{hn}\left(\frac{i_{hn}}{i_s}\right)^p, p>0 \quad (2)$$

One could examine himself, how this formula fulfills the conditions above. We, in our turn, will add, that it is equivalent to the formula (1) and shares the same value of *p* with it in a case when we deal with the same gamut. i_s in this case equals to $2^{\frac{1}{m}}\left(\frac{d_0}{d_m}\right)^{\frac{1}{pm}}$. If we prefer to get i_{dn} rather than i_{fdn}, the formula is:

$$i_{dn}=\left(\frac{i_{hn}}{i_s}\right)^p \text{, or}$$

$$i_{dn}=\left(\frac{d_m}{d_0}\right)^{\frac{1}{m}}\left(\frac{i_{hn}}{2^{\frac{1}{m}}}\right)^p, p>0 \quad (3)$$

And the formula (3) makes the equivalence of (1), (2) and (3) even more clear.

Now, we still have two problems. We've known nothing about values of *p*, except that *p* is a positive real number and constant for a certain gamut. The next problem is more complicated and consists in what $\frac{d_m}{d_0}$ should be. Starting from this one, $\frac{d_m}{d_0}$ can get values of the range $\approx\frac{1}{2}<\frac{d_m}{d_0}<\approx 2$, if we keep the preset $d_{max}/d_{min} = 2$. For better understanding, what $\frac{d_m}{d_0}$ could be, we also can look how gamuts with different values of $\frac{d_m}{d_0}$ might be applied to real tuning.

Let's take a block of consequent octaves of one gamut, that has just one key octave. The other octaves of this block will be additional. It means that respective values of d_n are equal in all octaves of the block. Now, if we take $d_{m,i} \neq d_{0,i}$ we'll get two pairs, what means two different sounds, for numbers *m* and 0. But it also means a necessity to have two different sounds with one number , for $d_{m,i} = d_{0,i+1}$ and the octave *i* must have the sound with $d_{0,i} = d_{0,i+1} = d_m$, when the gamut has been defined as above. It also means that we deal with two sounds, that have the same frequency of height, but different values of hardness. But it doesn't seem good for tuning. For example, possibilities of transposing gamuts are very restricted in the case when the initial sound of a gamut exclusively has two variants, but other sounds have only one. And this all infers, that $d_m = d_0$ should be very common case. The variant of the formula (3) for this case is:

$$i_{dn}=\left(\frac{i_{hn}}{2^{\frac{1}{m}}}\right)^p, p>0 \quad (3,\ d_m = d_0).$$

3 It's clear, that also $i_{hn} < 2$. i_{fd} has neither this restriction, but we've already said, that d_{max}/d_{min} shouldn't exceed 2 drastically, which restriction consequently is valid for i_{dn} too. The same way $i_{dn} > ½$ is valid, if we not take this ½ strictly. Practically i_{dn} often is much less deviated from 1 than 2 or ½ are.

Unhappily, a finding of p requires a longer explaining, because it's connected with possible variants of i_h sets in different gamuts. So, for this abstract is intended to give as short as possible description, we skip this problem here. However the examples below are designed to cover , among others, this question too.

A few examples for the previous section

1. The common heptatonic scale

We've tried to calculate the common gamut C, D, E,... in the example above. Now, let's try to calculate the same gamut, applying our formula of the second law of harmonic tuning. Here we'll use logarithmic values instead of not logarithmic as we've did in the previous example. The formula (3) in the logarithmic form, assuming $D_0 = D_m$, is:

$$I_{dn} = \left(I_{hn} - \frac{1}{m} \right) p \,,\ p > 0$$

where $I_{dn} = \varDelta_n - \varDelta_{n\text{-}1}$ and $I_{hn} = H_n - H_{n\text{-}1}$. Now, let's take the same table with values of the relative height, which table contains values of respective sounds from equally tuned set with 12 notes.

Note nr	0	1	2	3	4	5	6	7
Note	C	D	E	F	G	A	B	C
$\boldsymbol{H_n}$	0	1/6	1/3	5/12	7/12	9/12	11/12	1

The respective values of I_{hn} are:

Note nr	0	1	2	3	4	5	6	7
Note	C	D	E	F	G	A	B	C
$\boldsymbol{I_{hn}}$		1/6	1/6	1/12	1/6	1/6	1/6	1/12

We have just two different interval values I_h, so the formula gives us two values of I_{dn} . Knowing, that $m = 7$, we get $I_{d1} = (1/6 – 1/7)p = p/42$ and $I_{d3} = (1/12 – 1/7)p = – 5p/84$. $\varDelta_n$ equals $\varDelta_{n\text{-}1} + I_{dn}$, so the series of $\varDelta_n$ values may be calculated, starting from the unknown $\varDelta_0$: $\varDelta_0$, $\varDelta_0+p/42$, $\varDelta_0+p/21$, ... The results are:

Note nr	0	1	2	3	4	5	6	7
Note	C	D	E	F	G	A	B	C
$\boldsymbol{I_{dn}}$		$p/42$	$p/42$	$– 5p/84$	$p/42$	$p/42$	$p/42$	$– 5p/84$
$\boldsymbol{\varDelta_n}$	$\varDelta_0$	$\varDelta_0+p/42$	$\varDelta_0+p/21$	$\varDelta_0–p/84$	$\varDelta_0+p/84$	$\varDelta_0+3p/84$	$\varDelta_0+5p/84$	$\varDelta_0$

Now, we can see from here, that $\varDelta_{\min} = \varDelta_3 = \varDelta_0 – p/84$ and $\varDelta_{\max} = \varDelta_6 = \varDelta_0 + 5p/84$. And from that, assuming, that $\varDelta_{\min} = 0$ and $\varDelta_{\max} = 1$ (see the previous example section for grounding), $\varDelta_0 = 1/6$ and p = 14. The results for $\varDelta_n$ are:

$\boldsymbol{\varDelta_n}$	1/6	1/2	5/6	0	1/3	2/3	1	1/6

Thus we have exactly the same results as in the previous example section.

2. The pentatonic scale

The profit of our formula is, that we can tune not only common scales, whose tunings are well known, but also more exotic scales and gamuts. And it allows us to get data not only from an equally tuned set of relative heights, but also of any set of heights, where $h_0 = 1$, $h_m = 2$ and $h_{n1} > h_{n2}$ with any possible $n_1 > n_2$. Speaking about what, we should remember, that gamuts with Pythagorean tuning or similar well tempered ones are preferred over others.

Now, let's take a gamut with the pentatonic scale. There are few variants to express it in particular notes, and we'll take the one, whose notes (meaning pitches only) coincide with C, D, E, G, A, C of the common scale. And let's use the Pythagorean tuning in this case rather than an equal one. The height data are these:

Note nr	0	1	2	3	4	5
Note	C	D	E	G	A	C
h_n	1	9/8	5/4	3/2	5/3	2
H_n	0	0.1699	0.3219	0.5850	0.7370	1

The logarithmic equivalent for the formula (1) for $d_m = d_0$ is:

$$D_n = D_0 + \left(H_n - \frac{n}{m}\right)p$$

And further:

Note nr	0	1	2	3	4	5
Note	C	D	E	G	A	C
$H_n - n/m$	0	–0.0301	–0.0781	–0.0150	–0.0630	0

Thus, the maximal value of hardness will be obtained by note C, the minimal by note E. And from it $\Delta_0 = 1$ and $\Delta_2 = 0$. Then $p = (\Delta_E - \Delta_C)/(H_E - n/m) = -1 / -0.0781 \approx 12.81$. The further results are:

Note nr	0	1	2	3	4	5
Note	C	D	E	G	A	C
Δ_n	1	0.6148	0	0.8074	0.1926	1
I_{dn}		–0.3852	–0.3192	0.8074	–0.6148	0.8074

Don't you recognize that colored monotonic mood of the Chinese music in this distribution? (fig. 5)

We also should add that this gamut has different character of hardness values distribution than the gamut in the previous example (see it, comparing fig. 3, 4 with fig. 5, 6). The difference can be explained this way. Let's group all intervals of a gamut into small ones, that cause decreasing of hardness, and big ones, that have increasing Δ values[4]. And let the previous assuming, that the

4 We skip the possibility of the middle intervals with equal hardnesses here. It's done for simplification and it doesn't change anything essentially.

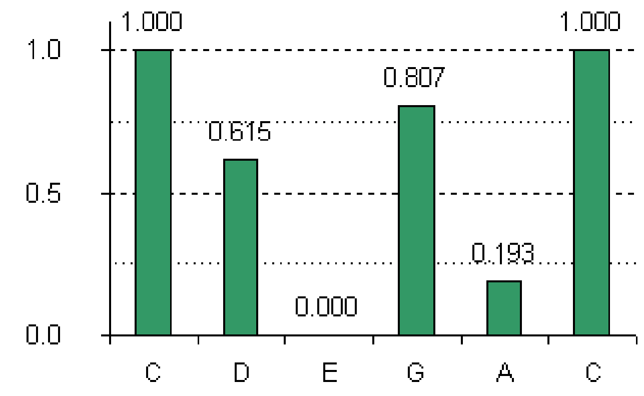

Fig. 6: Relative hardness values for Pythagorean pentatonics

zero sound of the gamut has the same hardness as the last sound, still be valid here. Now, it's clear, that if the number of the bigger intervals exceeds the number of the smaller intervals, then the average decreasing of hardness for an interval is bigger than the average increasing (meaning absolute values of I_d !), and we have a gamut like the one of the previous example. And if the number of the bigger intervals is less than the number of the smaller intervals, we'll get the average decreasing of hardness lesser than the average increasing and our gamut will be of the same kind as in this example. The gamuts like the one in the previous example (the gamuts with the slow increasing of hardness) are more traditional, but the other group (the gamuts with the slow decreasing) raises a question, if they not overpass traditional understanding of what is musical and harmonic. We recognize the "harmonic" in this case, basing on mathematical similarity rather than on traditions or supposed sounding.

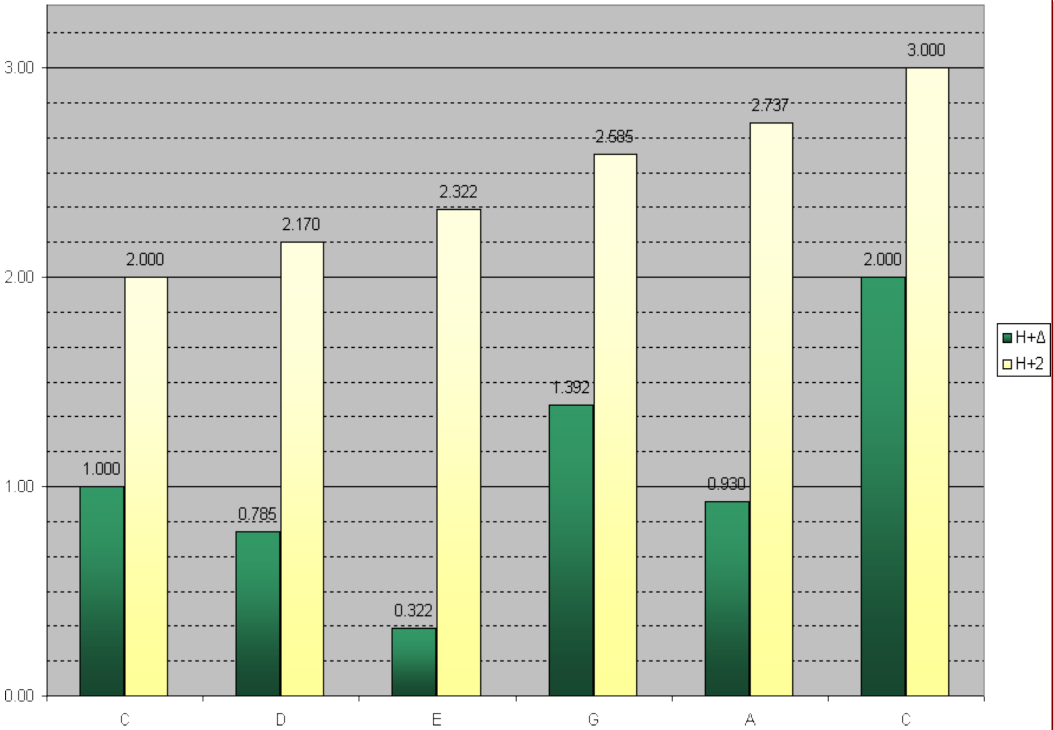

Fig. 5: Values of H+2 and H+Δ for the Pythagorean pentatonic gamut

This also raises a question, how a "positive" pentatonic gamut could be made. It's possible. The gamut must be subdivided into another number of parts than 12. Actually we didn't use an equally tempered subdivision in this example, for we have a well tempered gamut here, but a well tempered gamut with Pythagorean sounds have comparable character of hardnesses with gamuts with 12-parted equally tuned subdivision. So, the pentatonic gamut may be subdivided, for example, into 8 equally tempered parts, to get it "positive" (every bigger interval will encompass two parts of this subdivision and will have $I_h = 1/4$, but every smaller one will have $I_h = 1/8$, encompassing a single part of the subdivision, in this case). One may also try to use fractional values for height (meaning pure height, not the logarithmic) instead of the irrational values, finding the closest variant to this new subdivision. The gamut, that one will get, will be well tempered but not Pythagorean, and it may be “positive” too (what depends on how close are the fractional heights to their counterparts of the equally tempered subdivision).

Literature

The special literature on mathematical theories of music, that was used, was all in Russian or Lithuanian, and is hardly available for readers in the Western countries. An examples of such books is:

S. S. Gazaryan, *V mire muzykalnyh instrumentov*. Moscow, Prosveshcheniye, 1985. (In Russian).

From less special literature, many handbooks on mathematics and (acoustic) physics were used, also in Russian or Lithuanian, which books are like to:

G. Korn, T. Korn, *Mathematical Handbook for scientists and engineers*, 2nd edition. McGraw - Hill Book Company, New York, San Francisco etc., 1968.
(Actually the Russian translation of this book was used).

Also many Internet sites were useful, for finding the newest information, checking English terms as well as getting some useful ideas. The sites are:

http://en.wikipedia.org/wiki/Category:Musical_instruments

http://en.wikipedia.org/wiki/Category:Mathematics_of_music

http://en.wikipedia.org/wiki/Category:Musical_terminology

http://tonalsoft.com/enc/encyclopedia.aspx

http://www.harmonics.com/lucy/tuning.html

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http://ccrma.stanford.edu/marl/

http://ccrma.stanford.edu/software/clm/

http://ccrma.stanford.edu/software/stk/

Thanks to all promoters and contributors of the sites above!