Lecture 31
INTEGRATION

Substitution.

Example 1. $\int \frac{x^2\ dx}{\sqrt{x^3+5}}$ (let $u = x^3+5 \ \therefore \dfrac{du}{dx} = 3x^2 \ \therefore du = 3x^2\ dx$)

$= \frac{1}{3}\int \frac{3x^2\ dx}{\sqrt{x^3+5}}$

$= \frac{1}{3}\int \frac{du}{\sqrt{u}}$

$= \frac{1}{3}\int u^{-\frac{1}{2}} du$

$= \frac{1}{3} 2u^{\frac{1}{2}} + C$

$= \frac{2}{3}\sqrt{u} + C$

$= \frac{2}{3}\sqrt{x^3+5} + C. \quad \square$

Example 2. $\int \frac{dx}{3x+5}$ (let $u = 3x+5 \therefore \frac{du}{dx} = 3 \therefore du = 3dx.$)

$= \frac{1}{3}\int \frac{3\ dx}{3x+5}$

$= \frac{1}{3}\int \frac{du}{u}$

$= \frac{1}{3}\ln u + C$

$= \frac{1}{3}\ln(3x+5) + C. \quad \square$

Example 3. $\int x\sqrt{1+x^2}\ dx$ (let $u = 1+x^2 \therefore \ \frac{du}{dx} = 2x \ \therefore \ du = 2x\ dx$)

$= \frac{1}{2}\int \sqrt{1+x^2}\ 2x\ dx$

$= \frac{1}{2}\int \sqrt{u}\ du$

$= \frac{1}{2}\int u^{\frac{1}{2}}\ du$

$= \frac{1}{2} 2u^{\frac{3}{2}} + C$

$= \frac{u^{\frac{3}{2}}}{3} + C$

$= \frac{(1+x^2)\sqrt{1+x^2}}{3} + C. \quad \square$

Lecture 32

Integration : Substitution (cont'd).

Example 1. $\int \frac{\cos x \ dx}{(1+\sin x)^2}$ (let $u = 1 + \sin x \therefore \ \frac{du}{dx} = \cos x \ \therefore \ du = \cos x \ dx$)

$= \int \frac{du}{u^2}$

$= \int u^{-2} \ du$

$= -u^{-1} + C$

$= -\frac{1}{u} + C$

$= \frac{-1}{1+\sin x} + C. \quad \square$

Example 2. $\int \frac{dx}{x \log_e x}$ (let $u = \log_e x \ \therefore \ \frac{du}{dx} = \frac{1}{x} \ \therefore \ du = \frac{dx}{x}$)

$= \int \frac{1}{\log_e x} \ \frac{dx}{x}$

$= \int \frac{1}{u} \ du$

$= \log_e u + C$

$= \log_e(\log_e x) + C. \quad \square$

Lecture 33

Partial Fractions.

Example 1. Express $\frac{2x-1}{(x-2)(x-3)}$ in the form $\frac{A}{x-2} + \frac{B}{x-3}$.

$$\begin{aligned}\frac{2x-1}{(x-2)(x-3)} &= \frac{A}{x-2} + \frac{B}{x-3} \\ &= \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}\end{aligned}$$

$\therefore 2x - 1 = A(x-3) + B(x-2)$
Substitute $x = 3$. $\therefore B = 5$. Substitute $x = 2 \therefore -A = 3 \ \therefore \ A = -3$.

Alternatively, $(A+B)x - (3A+2B) = 2x - 1 \Rightarrow A + B = 2$ & $3A + 2B = 1$ $\Rightarrow 3(2-B) + 2B = 1 \ \therefore \ B = 5 \ \therefore \ A = 2 - -5 = -3$.
$\therefore \ \frac{2x-1}{(x-2)(x-3)} = \frac{-3}{x-2} + \frac{5}{x-3}$. $\square$

Note: The degree of numerators is less than the degree of the demominators for breaking up of partial fractions.

Example 2. Decompose $\frac{2x+3}{(x^2+2)(x-2)}$ into partial fractions.

$\frac{2x+3}{(x^2+2)(x-2)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-2}$ (since the degree of the numerators are at most, one less than the degree of the denominators.)

$$= \frac{(Ax+B)(x-2)+C(x^2+2)}{(x^2+2)(x-2)}$$

$\therefore 2x + 3 = (Ax+B)(x-2) + C(x^2+2)$

$\therefore 7 = 6C \Rightarrow C = \frac{7}{6}$

When $x = 0, \ 3 = -2B + 2C \therefore 3 = -2B + \frac{14}{6} \therefore B = -\frac{1}{3}$.
When $x = 1, 5 = -(A+B) + c(3) = -(A - \frac{1}{3}) + \frac{7}{6}(3) \therefore A = -\frac{7}{6}$
$\therefore \ \frac{2x+3}{(x^2+2)(x-2)} = \frac{-\frac{7}{6}x - \frac{1}{3}}{x^2+2} + \frac{\frac{7}{6}}{x-2} = \frac{7}{6x-12} - \frac{7x+2}{6x^2+12}$. $\square$

Example 3. Decompose $\frac{1}{(x-2)(x+1)^2}$.

$\frac{1}{(x-2)(x+1)^2} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$$1 = A(x+1)^2 + B(x-2)(x+1) + C(x-2)$$

When $x = 2, 9A = 1 \therefore A = 9$.
When $x = -1, -3C = 1 \therefore C = -\frac{1}{3}$.
When $x = 0, 1 = A - 2B - 2C, \therefore 2B = A - 2C - 1 = \frac{1}{2} + \frac{2}{3} - 1 \therefore B = -\frac{1}{9}$.
$\therefore \ \frac{1}{(x-2)(x+1)^2} = \frac{\frac{1}{9}}{x-2} + \frac{-\frac{1}{9}}{x+1} + \frac{-\frac{1}{3}}{(x+1)^2} = \frac{1}{9x-18} - \frac{1}{9x+9} - \frac{1}{3(x+1)^2}$ $\square$.

Lecture 34

Integration with partial fractions.

Example. Find $\int \frac{dx}{(x-1)(x-2)}$.

$\frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \therefore 1 = A(x-2) + B(x-1)$. If $x = 2, B = 1$. If $x = 1, -A = 1 \therefore$ $A = -1$. Hence $\frac{1}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{1}{x-2}$.

$$\begin{aligned}\therefore \int \frac{dx}{(x-1)(x-2)} &= \int \left(\frac{-1}{x-1} + \frac{1}{x-2}\right) dx \\ &= -\ln|x-1| + \ln|x-2| + C \\ &= \ln\left|\frac{x-2}{x-1}\right| + C \quad \square\end{aligned}$$

Lecture 35

Example. Find $\int \frac{3x^2+5x+7}{(x-1)(x-2)}\,dx$.

First, $(x-1)(x-2) = x^2-3x+2$, which has the same degree as the numerator. So divide:

$$\begin{array}{r|l} & 3 \\ \hline x^2-3x+2 & 3x^2+5x+7 \\ & 3x^2-9x+6 \\ \hline & 14x+1 \end{array}$$

Now use partial fractions:

$$\begin{aligned} \frac{14x+1}{(x-1)(x-2)} &= \frac{A}{x-1} + \frac{B}{x-2} \\ &= \frac{A(x-2)+B(x-1)}{(x-1)(x-2)} \\ \therefore 14x+1 &= A(x-2)+B(x-1) \end{aligned}$$

$\therefore -A = 15 \therefore A = -15$ & $B = 29$.

So,

$$\begin{aligned} \int \frac{3x^2+5x+7}{(x-1)(x-2)}\,dx &= \int \left(3 + \frac{14+1}{(x-1)(x-2)}\right) dx \\ &= \int \left(3 - \frac{15}{x-1} + \frac{29}{x-2}\right) dx \\ &= 3x - 15\ln|x-1| + 29\ln|x-2| + C \\ &= 3x + \ln\left|\frac{(x-2)^{29}}{(x-1)^{15}}\right| + C \quad \square \end{aligned}$$

Lecture 36

Integration by Parts.

$\frac{d}{dx}(uv) = u\frac{d}{dx}v + v\frac{d}{dx}u$

$\therefore u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$.

$\therefore \int u\frac{dv}{dx} \cdot dx = uv - \int v\frac{du}{dx} \cdot dx$

i.e., $\int u\ dv = uv - \int v\ du$.

Example 1. Find $\int xe^x\ dx$.

Let $u = x \therefore \frac{du}{dx} = 1, v = e^x$

$$\begin{aligned} \therefore \int xe^x\ dx &= xe^x - \int e^x\ dx \\ &= xe^x - e^x + C \\ &= e^x(x-1) + C \quad \square \end{aligned}$$

Example 2. Find $\int x\cos x\ dx$.

$$\begin{aligned} \int x\cos x\ dx &= \int x\frac{d}{dx}\sin x\ dx \\ &= x\sin x - \int \sin x\frac{d}{dx}x\ dx \\ &= x\sin x - \int \sin x\ dx \\ &= x\sin x + \cos x + C \quad \square \end{aligned}$$

Lecture 37

Integration by Parts (cont'd)

Example 1. Find $\int \log_e x \, dx$.

Let $u = \ln x, v = x$

$$\begin{aligned} \therefore \int \log_e x \, dx &= \int 1 \cdot \log_e x \, dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + C \quad \square \end{aligned}$$

Example 2. Find $\int \sin^{-1} x \, dx$.

Let $u = \sin^{-1} x, v = x$.

$$\begin{aligned} \therefore \int \sin^{-1} x \, dx &= \int 1 \cdot \sin^{-1} x \, dx \\ &= x \sin^{-1} x - \int \tfrac{x}{\sqrt{1-x^2}} \, dx \text{ let } w = 1 - x^2 \therefore dx = \tfrac{-dw}{2x} \\ &= x \sin^{-1} x - \int \tfrac{x}{\sqrt{w}} \left(\tfrac{-dw}{2x}\right) \\ &= x \sin^{-1} x + \tfrac{1}{2} \int w^{-\frac{1}{2}} \, dw \\ &= x \sin^{-1} x + w^{\frac{1}{2}} + C \\ &= x \sin^{-1} x + (1 - x^2)^{\frac{1}{2}} + C \quad \square \end{aligned}$$

Lecture 38

Trigonometric Integrals.

Powers of sin x

Example 1. $\int \sin x \; dx = -\cos x + C \quad \square$

Example 2.
$$\begin{aligned} \int \sin^2 x \; dx &= \tfrac{1}{2} \int (1 - \cos 2x) \; dx \\ &= \tfrac{1}{2}(x - \tfrac{1}{2} \sin 2x) + C \quad \square \end{aligned}$$

Example 3.
$$\begin{aligned} \int \sin^3 x \; dx &= \int \sin^2 x \sin x \; dx \\ &= \int (1 - \cos^2 x) \sin x \; dx \text{ (let } u = \cos x \therefore du = -\sin x \; dx) \\ &= -\int (1 - u^2) \; du \\ &= -(u - \tfrac{1}{3} u^3) + C \\ &= \tfrac{1}{3} \cos^3 x - \cos x + C \quad \square \end{aligned}$$

Example 4.
$$\begin{aligned} \int \sin^4 x \; dx &= \int (\sin^2 x)^2 \; dx \\ &= \int (\tfrac{1}{2}(1 - \cos 2x))^2 \; dx \\ &= \tfrac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) \; dx \\ &= \tfrac{1}{4} \int (1 - 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)) \; dx \\ &= \tfrac{1}{4} x - \sin 2x + \tfrac{1}{2}(x + \tfrac{1}{4} \sin 4x)) + C \\ &= \tfrac{3}{8} x - \tfrac{1}{4} \sin 2x + \tfrac{1}{32} \sin 4x + C \quad \square \end{aligned}$$

Example 5.
$$\begin{aligned} \int \sin^5 x \; dx &= \int \sin^4 x \sin x \; dx \\ &= \int (1 - \cos^2 x)^2 \sin x \; dx \text{ (let } u = \cos x \therefore du = -\sin x \; dx) \\ &= -\int (1 - u^2)^2 du \\ &= -\int (1 - 2u^2 + u^4) \; du \\ &= -(u + \tfrac{2}{3} u^3 + \tfrac{u^5}{5}) + C \\ &= \tfrac{2}{3} \cos^3 x - \tfrac{\cos^5 x}{5} - \cos x + C \quad \square \end{aligned}$$

For $\sin^n x$, for odd powers, use the substitution method $u = \cos x$.
For even powers, use the fact that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Lecture 39

$\int \tan^n x \, dx$ for $n = 1 \in \mathbb{Z}^+$

Example 1.
$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\ &= -\ln \cos x + C \\ &= \ln \frac{1}{\cos x} + C \\ &= \ln \sec x + C \quad \square\end{aligned}$$

Example 2.
$$\begin{aligned}\int \tan^2 x \, dx &= \int (\sec^2 x - 1) \, dx \\ &= \tan x - x + C \quad \square\end{aligned}$$

Example 3.
$$\begin{aligned}\int \tan^3 x \, dx &= \int \tan x (\sec^2 x - 1) \, dx \\ &= \int \tan x \sec^2 x \, dx - \int \tan x \, dx \\ &= \int \tan x \, d(\tan x) - \int \tan x \, dx \\ &= \frac{1}{2} \tan^2 x + \ln \cos x + C \quad \square\end{aligned}$$

Example 4.
$$\begin{aligned}\int \tan^4 x \, dx &= \int \tan^2 x (\sec^2 x - 1) \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\ &= \int \tan^2 x \, d(\tan x) - \int \tan^2 x \, dx \\ &= \frac{1}{3} \tan^3 x - (\tan x - x) + C \\ &= x + \frac{1}{3} \tan^3 x - \tan x + C \quad \square\end{aligned}$$

Example 5.
$$\begin{aligned}\int \tan^5 x \, dx &= \int \tan^3 x (\sec^2 x - 1) \, dx \\ &= \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx \\ &= \int \tan^3 x \, d(\tan x) - \int \tan^3 x \, dx \\ &= \frac{1}{4} \tan^4 x - (\frac{1}{2} \tan^2 x + \ln \cos x) + C \\ &= \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \ln \cos x + C \quad \square\end{aligned}$$

Lecture 40

Trigonometric integrals

Example 1. $\int \sin x \cos x \; dx \; (u = \cos x, du = -\sin x \; dx)$
$= -\int u \; du$
$= -\frac{u^2}{2} + C$
$= -\frac{\cos^2 x}{2} + C$

Alternatively,
$\int \sin x \cos x \; dx \; (v = \sin x, dv = \cos x \; dx)$
$= \int v \; dv$
$= \frac{v^2}{2} + K$
$= \frac{\sin^2 x}{2} + K$

Note: $\frac{\sin^2 x}{2}$ and $-\frac{\cos^2 x}{2}$ differ only by a constant, so although the answers look different, they are really the same, only expressed differently, because C and K are *arbitrary constants.*

Example 2. $\int \sin^{1/2} x \cos x \; dx \; (u = \sin x \therefore du = \cos x \; dx)$
$= \int u^{1/2} \; du$
$= \frac{2u^{3/2}}{3} + C$
$= \frac{2}{3} \sin^{3/2} x + C$

Example 3. $\int \cos^7 x \sin^4 x \; dx$
$= \int \cos^6 x \sin^4 x \cos x \; dx$ (Split the odd power)
$= \int (1 - \sin^2 x)^3 \sin^4 \cos x \; dx \; (u = \sin x \therefore du = \cos x \; dx)$
$= \int (1 - u^2)^3 u^4 \; du$
$= \int (1 - 3u^2 + 3u^4 - u^6) u^4 \; du$
$= \int (u^4 - 3u^6 + 3u^8 - u^{10}) du$
$= \frac{u^5}{5} - \frac{3u^7}{7} + \frac{3u^9}{9} - \frac{u^{11}}{11} + C$
$= \frac{1}{5} \sin^5 x - \frac{3}{7} \sin^7 x + \frac{1}{3} \sin^9 x - \frac{1}{11} \sin^{11} x + C$

Example 4. $\int \frac{\cos^7 x}{\sin^4 x}\ dx\ (u = \sin x \therefore u = \cos x\ dx)$

$$= \int \frac{\cos^6 x \cos x\ \ dx}{\sin^4 x}$$

$$= \int \frac{(1-\sin^2 x)^3 \cos x\ \ dx}{\sin^4 x}$$

$$= \int \frac{(1-u^2)^3 du}{u^4}$$

$$= \int \frac{1-3u^2+3u^4-u^6}{u^4}\ du$$

$$= \int (u^{-4} - 3u^{-2} + 3 - u^2) du$$

$$= -\frac{u^{-3}}{3} + 3u^{-1} + 3u - \frac{1}{3}u^3 + C$$

$$= -\frac{1}{3}(\sin x)^{-3} + 3(\sin x)^{-1} + 3\sin x - \frac{1}{3}\sin^3 x + C$$

Lecture 41

Integration using tangents of half angles

Let $t = \tan\frac{x}{2}$. Then $\sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ \tan x = \frac{2t}{1-t^2}$.
$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$
$\frac{dt}{dx} = \frac{1}{2}(1+\tan^2\frac{x}{2})$
$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$
$\therefore \frac{dx}{dt} = \frac{2}{1+t^2}$
$\therefore dx = \frac{2\ dt}{1+t^2}$

Example 1.
$$\begin{aligned}\int \frac{dx}{3+2\cos x} &= \int \frac{\left(\frac{2\ dt}{1+t^2}\right)}{3+2\left(\frac{1-t^2}{1+t^2}\right)}\\ &= \int \frac{\left(\frac{2\ dt}{1+t^2}\right)}{\left(\frac{3(1+t^2)+2(1-t^2)}{1+t^2}\right)}\\ &= \int \frac{2\ dt}{3+3t^2+2-2t^2}\\ &= \int \frac{2\ dt}{5+t^2}\\ &= \frac{2}{\sqrt{5}}\tan^{-1}\frac{t}{\sqrt{5}} + C\\ &= \frac{2}{\sqrt{5}}\tan^{-1}\frac{\tan\frac{x}{2}}{\sqrt{5}} + C\end{aligned}$$

Example 2.
$$\begin{aligned}\int \frac{\cos x\ dx}{3+2\cos x} &= \int \frac{\frac{1}{2}(3+2\cos x)-\frac{3}{2}}{3+2\cos x}\ dx\\ &= \int \frac{1}{2}\ dx - \frac{3}{2}\int \frac{dx}{3+2\cos x}\\ &= \frac{1}{2}x - \frac{3}{2}\cdot\frac{2}{\sqrt{5}}\tan^{-1}\frac{\tan\frac{x}{2}}{\sqrt{5}} + C\\ &= \frac{1}{2}x - \frac{3}{\sqrt{5}}\tan^{-1}\frac{\tan\frac{x}{2}}{\sqrt{5}} + C\end{aligned}$$

Lecture 42

Integrals with quadratic denominators

Example 1. $\int \frac{dx}{9+4x^2} = \int \frac{dx}{4(\frac{9}{4}+x^2)}$

$= \frac{1}{4}\int \frac{dx}{\frac{9}{4}+x^2}$

$= \frac{1}{4}(\frac{2}{3})\tan^{-1}\frac{2x}{3} + C$

$= \frac{1}{6}\tan^{-1}\frac{2x}{3} + C$

Example 2. $\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2-\frac{9}{4}}}$

$= \frac{1}{2}\ln\left(x + \sqrt{x^2 - \frac{9}{4}}\right) + C$

Example 3. $\int \frac{dx}{x^2+4x+1} = \int \frac{dx}{(x+2)^2-3}$ (Let $u = x+2 \therefore\ du = dx$)

$= \int \frac{du}{u^2-3}$

$= \frac{1}{2\sqrt{3}}\ln\frac{u-\sqrt{3}}{u+\sqrt{3}} + C$

$= \frac{1}{2\sqrt{3}}\ln\frac{x+2-\sqrt{3}}{x+2+\sqrt{3}} + C$

Example 4. $\int \frac{dx}{3x^2+2x+1} = \frac{1}{3}\int \frac{dx}{x^2+\frac{2}{3}x+\frac{1}{3}}$

$= \frac{1}{3}\int \frac{dx}{x^2+\frac{2}{3}x+\frac{1}{9}+\frac{1}{3}-\frac{1}{9}}$

$= \frac{1}{3}\int \frac{dx}{(x+\frac{1}{3})^2+\frac{2}{9}}$ Let $u = x+\frac{1}{3} \therefore du = dx$

$= \frac{1}{3}\int \frac{du}{u^2+\frac{2}{9}}$

$= \frac{1}{3}\left(\frac{3}{\sqrt{2}}\right)\tan^{-1}\frac{3u}{\sqrt{2}} + C$

$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{3(x+\frac{1}{2})}{\sqrt{2}} + C$

$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{3x+1}{\sqrt{2}} + C$

Example 5. $\int \frac{dx}{1+4x-x^2} = \int \frac{dx}{1-(x^2-4x)}$

$= \int \frac{dx}{5-(x^2-4x+4)}$

$= \int \frac{dx}{5-(x-2)^2}$ (Let $u = x-2 \therefore du = dx$)

$= \int \frac{du}{5u^2}$

$= \frac{1}{2\sqrt{5}}\ln\frac{\sqrt{5}+u}{\sqrt{5}-u} + C$

$= \frac{1}{2\sqrt{5}}\ln\frac{\sqrt{5}-2+x}{\sqrt{5}+2-x} + C$

Example 6.

$$
\begin{aligned}
\int \frac{x\ dx}{x^2+6x+1} &= \frac{1}{2}\int \frac{(2x+6)-6}{x^2+6x+1}\ dx \\
&= \frac{1}{2}\int \frac{2x+6}{x^2+6x+1}\ dx - \frac{1}{2}\int \frac{6}{x^2+6x+1}\ dx \\
&= \frac{1}{2}\ln(x+6x+1) - 3\int \frac{dx}{x^2+6x+1} \\
&= \frac{1}{2}\ln(x^2+6x+1) - 3\int \frac{dx}{x^2+6x+9-8} \\
&= \frac{1}{2}\ln(x^2+6x+1) - 3\int \frac{dx}{(x+3)^2-8} \qquad \text{Let } u = x+3 \therefore du = dx \\
&= \frac{1}{2}\ln(x^2+6x+1) - 3\int \frac{du}{u^2-8} \\
&= \frac{1}{2}\ln(x^2+6x+1) - \frac{3}{2\sqrt{8}}\ln\left(\frac{u-\sqrt{8}}{u+\sqrt{8}}\right) + C \\
&= \frac{1}{2}\ln(x^2+6x+1) - \frac{3}{2\sqrt{8}}\ln\frac{x+3-\sqrt{8}}{x+3+\sqrt{8}} + C
\end{aligned}
$$

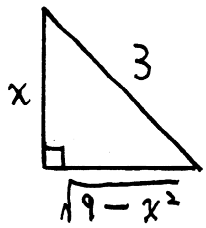
Lecture 43

Substitutions using trigonometry.

- for $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$, use substitutions $x = a\sin\theta, x = a\tan\theta$ and $x = a\sec\theta$ respectively.

Example . $\int \frac{x^2\ dx}{\sqrt{9-x^2}}$ (Let $x = 3\sin\theta \therefore dx = 3\cos\theta\ d\theta$)

$$= \int \frac{9\sin^2\theta 3\cos\theta\ d\theta}{\sqrt{9-9\sin^2\theta}}$$
$$= \int \frac{27\sin^2\theta\cos\theta\ d\theta}{3\cos\theta}$$
$$= 9\int \sin^2\theta\ d\theta$$
$$= \frac{9}{2}\int(1-\cos 2\theta)\ d\theta$$
$$= \frac{9}{2}[\theta - \frac{1}{2}\sin 2\theta] + C \quad \text{where } x = 3\sin\theta, \sin\theta = \frac{x}{3} \therefore \theta = \sin^{-1}\frac{x}{3}$$
$$= \frac{9}{2}(\theta - \frac{1}{2}\cdot 2\sin\theta\cos\theta) + C$$
$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C$$
$$= \frac{9}{2}(\sin^{-1}\frac{x}{3} - \frac{x}{3}\sqrt{\frac{9-x^2}{3}}) + C$$
$$= \frac{9}{2}(\sin^{-1}\frac{x}{3} - \frac{x}{9}\sqrt{9-x^2}) + C$$



Lecture 44

Reduction Formulae

Example 1. If $I_n = \int \sin^n x\ dx$ (n is an integer). Express I_n in terms of I_{n-2} and hence evaluate $\int_0^{\pi/2} \sin^5 x\ dx$ (application of Integration by Parts).

Solution.

$$\begin{aligned}
I_n &= \int \sin^n x\ dx \\
&= \int \sin^{n-1} x \sin x\ dx \text{ (Let } u = \sin^{n-1} x \text{ and } v' = \sin x \\
&\qquad \therefore u' = (n-1)\sin^{n-2} x.\cos x \ \&\ v = -\cos x) \\
&= -\cos x \sin^{n-1} x + \int (n-1)\sin^{n-2} \cos x \cos x\ dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} \cos^2 x\ dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2}(1-\sin^2 x)\ dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\ dx - (n-1)\int \sin^n x\ dx \\
&= -\cos x \sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n \\
(n-1)I_n + I_n &= -\cos x \sin^{n-1} x (n-1)I_{n-2} \\
I_n(n-1+1) &= -\cos x \sin^{n-1} x + (n-1)I_{n-2} \\
\therefore I_n &= -\tfrac{1}{n}\cos x \sin^{n-1} x + \tfrac{n-1}{n} I_{n-2} \quad \square \\
\therefore I_5 &= \int_0^{\pi/2} \sin^5 x\ dx \\
&= [-\tfrac{1}{5}\cos x \sin^4 x + \tfrac{4}{5} I_3]_0^{\pi/2} \\
&= [-\tfrac{1}{5}\cos x \sin^4 x + \tfrac{4}{5}(-\tfrac{1}{3}\cos x \sin^2 x + \tfrac{2}{3} I_1)]_0^{\pi/2} \\
&= [-\tfrac{1}{5}\cos x \sin^4 x + \tfrac{4}{5}(-\tfrac{1}{3}\cos x \sin^2 x + \tfrac{2}{3}\int \sin x\ dx)]_0^{\pi/2} \\
&= [-\tfrac{1}{5}\cos x \sin^4 x - \tfrac{4}{5}\cos x \sin^2 x + \tfrac{8}{15}\int \sin x\ dx]_0^{\pi/2} \\
&= [-\tfrac{1}{5}\cos x \sin^4 x - \tfrac{4}{15}\cos x \sin^2 x - \tfrac{8}{15}\cos x]_0^{\pi/2} \\
&= -(-\tfrac{8}{15}\cos 0) \\
&= \tfrac{8}{15}
\end{aligned}$$

Example 2. If $I_n = \int_1^2 (\log_e x)^n\ dx$ show that $I_n = 2(\log_e 2)^n - nI_{n-1}$ hence evaluate $\int_1^2 (\log_e x)^4\ dx$.

$I_n = \int_1^2 (\log_e x)^n \ dx$ (Let $u = (\log_e x)^n$ & $v' = 1$

$\therefore u' = \dfrac{n}{x}(\log_e x)^{n-1}, v = x)$

$= x(\log_e x)^n - n\int_1^2 \frac{1}{x}(\log_e x)^{n-1} x \ dx$

$= [x(\log_e x)^n]_1^2 - n\int_1^2 (\log_e x)^{n-1} \ dx$

$= 2(\ln 2)^n - nI_{n-1}$

$\therefore I_4 = \int_1^2 (\log_e x)^n$

$= 2(\ln 2)^4 - 4I_3$

$= 2(\ln 2)^4 - 4(2(\ln 2)^3 - 3I_2)$

$= 2(\ln 2)^4 - 8(\ln 2)^3 + 12I_2$

$= 2(\ln 2)^4 - 8(\ln 2)^3 + 12(2(\ln 2)^2 - 2I_1)$

$= 2(\ln 2)^4 - 8(\ln 2)^3 + 24(\ln 2)^2 - 24I_1$ .... $(*)$

$= 2(\ln 2)^4 - 8(\ln 2)^3 + 24(\ln 2)^2 - 24\int_1^2 \ln x \ dx$ (Let $u = \log x, v' = 1$

$\therefore u' = \frac{1}{x}, v = x)$

$= 2(\ln 2)^4 - 8(\ln 2)^3 + 24(\ln 2)^2 - 24[[x \ln x]_1^2 - \int_1^2 dx]$

$= 2(\ln 2)^4 - 8(\ln 2)^3 + 24(\ln 2)^2 - 24[x \ln x - x]_1^2$

$= 2(\ln 2)^4 - 8(\ln 2)^3 + 24(\ln 2)^2 - 24(2\ln 2 - 2 + 1)$

$= 2(\ln 2)^4 - 8(\ln 2)^3 + 24(\ln 2)^2 - 48\ln 2 + 24 \quad \square$

$(*)$ Also $I_1 = 2(\ln 2)^1 - 1I_0$

$= 2(\ln 2) - \int_1^2 dx$

$= 2(\ln 2) - [x]_1^2$

$= 2\ln 2 - 1.$

Example 3. Let $I_n = \int \sec^n x \ dx$. Show that $I_n = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

Note: $\frac{d}{dx}\sec x = \frac{d}{dx}(\cos x)^{-1} = -(\cos)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \tan x \sec x$.

$I_n = \int \sec^{n-2} x \sec^2 x \ dx$ [Let $u = \sec^{n-2} x$ $v' = \sec^2 x$

$\therefore u' = (n-2)(\sec x)^{n-3} \tan x \sec x = (n-2)(\sec x)^{n-2} \tan x,$

$v = \tan x]$

$= \tan x \sec^{n-2} x - (n-2)\int \tan x (\sec x)^{n-2} \ dx$

$= \tan x \sec^{n-2} x - (n-2)\int (\sec)^{n-2} \tan^2 x \ dx$

$= \tan x \sec^{n-2} x - (n-2)\int \tan^2 x (\sec x)^{n-2} \ dx$

$= \tan x \sec^{n-2} x - (n-2)\int (\sec^2 x - 1)\sec^{n-2} x \ dx$

$= \tan x \sec^{n-2} x - (n-2)\int \sec^n x \ dx + (n-2)\int \sec^{n-2} x \ dx$

So $\int \sec^n x \ dx + (n-2)\int \sec^n x \ dx = (n-1)\int \sec^n x \ dx$

$= \tan x \sec^{n-2} x + (n-2)\int \sec^{n-2} x \ dx = \tan x \sec^{n-2} x + (n-2)I_{n-2}$

So $\int \sec^n x \ dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2} \quad \square$

Example 4. Let $I_n = \int \tan^n x \ dx$. Then:

$$\begin{aligned}
I_n &= \int \tan^{n-2} x \tan^2 x \ dx \\
&= \int \tan^{n-2} x (\sec^2 x - 1) \ dx \\
&= \int \tan^{n-2} x \sec^2 x \ dx - \int \tan^{n-2} x \ dx \\
&= \int \tan^{n-2} x \sec^2 x \ dx - I_{n-2} \ [\text{Let } u = \tan x \therefore du = \sec^2 x \ dx] \\
&= \int u^{n-2} - I_{n-2} \\
&= \tfrac{u^{n-1}}{n-1} - I_{n-2} \\
&= \tfrac{\tan^{n-1} x}{n-1} - I_{n-2}
\end{aligned}$$