Lecture 60

Harder 3 unit

This can be studied by looking at past papers available at

www.boardofstudies.nsw.edu.au

and

fourunitmaths.cjb.net

In 2002 Bill Pender did an inservice on it available on the net at

members.optusnet.com.au/limkw/SGS_harder3U_2002.zip

However, he said that pi^2/6 is too hard for 4 unit which as you will see on the next page is wrong.

Pi Squared On Six

2002 Ext. 2 hsc Q8a $\Rightarrow \sum_{j=1}^{m} \cot^2 \frac{\pi j}{2m+1} = \frac{m(2m-1)}{3}$

$\therefore \sum_{j=1}^{m} (\operatorname{cosec}^2 \frac{\pi j}{2m+1} - 1) = (\sum_{j=1}^{m} \operatorname{cosec}^2 \frac{\pi j}{2m+1}) - m = \frac{m(2m-1)}{3}$

$\Rightarrow \sum_{j=1}^{m} \operatorname{cosec}^2 \frac{\pi j}{2m+1} = \frac{m(2m-1)}{3} + m = \frac{2m(m+1)}{3}$.

For $0 < \theta < \frac{\pi}{2}$, $\sin\theta < \theta < \tan\theta$. $\therefore \cot\theta < \frac{1}{\theta} < \operatorname{cosec}\theta$ $\therefore \cot^2\theta < \frac{1}{\theta^2} < \operatorname{cosec}^2\theta$.

$\therefore \frac{m(2m-1)}{3} = \sum_{j=1}^{m} \cot^2 \frac{\pi j}{2m+1} < \sum_{j=1}^{m} \frac{(2m+1)^2}{(\pi j)^2} = \frac{(2m+1)^2}{\pi^2} \sum_{j=1}^{m} \frac{1}{j^2} < \sum_{j=1}^{m} \operatorname{cosec}^2 \frac{\pi j}{2m+1} = \frac{2m(m+1)}{3}$

$\therefore 1 - \frac{4m(m+1)}{(2m+1)^2} = \frac{1}{(2m+1)^2} < 1 - \frac{6}{\pi^2} \sum_{j=1}^{m} \frac{1}{j^2} < 1 - \frac{2m(2m-1)}{(2m+1)^2} = \frac{6m+1}{(2m+1)^2} < \frac{6m+3}{(2m+1)^2} = \frac{3}{2m+1}$

$\therefore 0 = \lim_{m\to\infty} \frac{1}{(2m+1)^2} \le \lim_{m\to\infty} (1 - \frac{6}{\pi^2} \sum_{j=1}^{m} \frac{1}{j^2}) = 1 - \frac{6}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{j^2} \le \lim_{m\to\infty} \frac{3}{2m+1} = 0$

Hence $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$.

- Derek Buchanan,

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