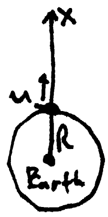
Lecture 52

Dynamics - Variable Acceleration

Example. The acceleration due to gravity at a point outside the earth is inversely proportional to the square of the distance x from the centre, i.e., $\ddot{x} = -\frac{k}{x^2}$. Neglecting air resistance, show that if a particle is projected vertically upwards with speed u from a point on the Earth's surface, its speed v in any position x is given by $v^2 = u^2 - 2gR^2(\frac{1}{R} - \frac{1}{x})$ where R is the radius of the Earth and g the acceleration due to gravity at the Earth's surface. Show that the greatest height H above the Earth's surface reached by the particle is $H = \frac{u^2R}{2gR-u^2}$ and find the speed needed to escape the Earth's influence. ($R = 6400\,\mathrm{km}$, $g = 10\,\mathrm{m/s^2} = 0.01\,\mathrm{km/s^2}$.)

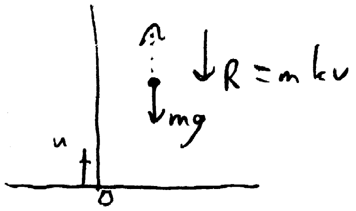
Solution.



Initially, $v = u$ and $x = R \therefore \ddot{x} = -g$.
$\ddot{x} = -\frac{k}{x^2} = -kx^{-2} \therefore \frac{d}{dx}(\frac{1}{2}v^2) = -kx^{-2} \therefore \frac{1}{2}v^2 = kx^{-1} + c = \frac{k}{x} + c$
When $v = u$, $x = R$
$\therefore \frac{1}{2}u^2 = \frac{k}{R} + c \therefore c = \frac{1}{2}u^2 - \frac{k}{R} \therefore \frac{1}{2}v^2 = \frac{k}{x} + \frac{1}{2}u^2 - \frac{k}{R} \therefore v^2 = u^2 + \frac{2k}{x} - \frac{2k}{R} \therefore u^2 - 2k(\frac{1}{R} - \frac{1}{x})$.
But when $x = R$, $\ddot{x} = -g$ $-g = -\frac{k}{R^2}$ $k = gR^2$.
$\therefore v^2 = u^2 - 2gR^2(\frac{1}{R} - \frac{1}{x})$ greatast height is reached when $v = 0$.
$\therefore u^2 - 2gR^2(\frac{1}{R} - \frac{1}{x}) = 0 \therefore 2gR^2(\frac{1}{R} - \frac{1}{x}) = u^2 \therefore \frac{1}{R} - \frac{1}{x} = \frac{u^2}{2gR^2} \therefore \frac{1}{x} = \frac{1}{R} - \frac{u^2}{2gR^2} = \frac{2gR-u^2}{2gR^2}$
$\therefore x = \frac{2gR^2}{2gR-u^2}$
$\therefore$ greatest height above Earth's surface is
$\frac{2gR^2}{2gR-u^2} - R = \frac{2gR^2-R(2gR-u^2)}{2gR-u^2} = \frac{2gR^2-2gR^2+Ru^2}{2gR-u^2} = \frac{u^2R}{2gR-u^2}$.
If it escapes the Earth's influence, $H \to \infty$ (since it keeps going).
So $2gR - u^2 = 0$ and $\therefore u^2 = 2gR$
$\therefore u = \sqrt{2gR}$ (i.e., the escape velocity) $\doteqdot \sqrt{2(0.01)(6400)} = 8\sqrt{2}$ km/s $\doteqdot 11.3$ km/s.

Resisted Motion - from Coroneos, 1982a

Case 1. - projected upwards.



A particle of mass m is projected vertically upwards with velocity u, in a medium whose resistance to the motion varies as the velocity of the particle. Prove that the time to reach the highest point of the path is $\frac{1}{k}\ln(1+\frac{ku}{g})$, where k is a constant, and find this greatest height. $\{g$ is the acceleration due to gravity.$\}$

Solution.

Let the velocity v and the displacement x be measured upward ($\uparrow$) from the point of projection O. Since the resistance R to the motion varies as the velocity v, then $R = Kv$ where K is a positive constant. (Direction of R will be downward since resistance is always in the opposite direction since resistance is always in the opposite direction to the velocity.) For convenience later, let $K = mk$ and $\therefore R = mkv$. Hence, the force acting on the particle in the downward ($\downarrow$) direction, is $mg + R$. Now, by Newton's Second Law, the force on the particle in the upward ($\uparrow$) direction is $m\ddot{x}$. Thus, $m\ddot{x} = -mg - mkv$, i.e., $\ddot{x} = -g - kv$ is the equation of motion of the particle.

Firstly, $\ddot{x} = \frac{dv}{dt}$, $\therefore \frac{dv}{dt} = -g - kv$ & $\therefore \frac{dt}{dv} = \frac{-1}{g+kv}$.
$\therefore t = -\int \frac{dv}{g+kv} = -\frac{1}{k}\ln(g+kv) + c$, where C is a constant ... (1)
By data, when $t = 0$, $v = u$, $\therefore 0 = -\frac{1}{k}\ln(g+ku) + c$, whence $c = \frac{1}{k}\ln(g+ku)$
$\therefore$ (1) becomes $t = \frac{1}{k}\ln(g+ku) - \frac{1}{k}\ln(g+kv) = \frac{1}{k}\ln\frac{g+ku}{g+kv}$ ... (2)
At the highest point on the path $v = 0 \therefore t = \frac{1}{k}\ln\frac{g+ku}{g}$.
That is, the time to reach the highest point is $\frac{1}{k}\ln(1+\frac{ku}{g})$.

Secondly, taking $\ddot{x} = v\frac{dv}{dx}$, then the equation of motion becomes $v\frac{dv}{dx} = -g - kv$,
i.e., $\frac{dv}{dx} = \frac{-(g+kv)}{v}$
and $\therefore \frac{dx}{dv} = \frac{-v}{g+kv}$ ... (3)
Hence, $x = -\int \frac{v\ dv}{g+kv} = -\int \frac{\frac{1}{k}(g+kv) - \frac{g}{v}}{g+kv}\ dv = -\frac{1}{k}\int(1 - \frac{g}{g+kv})\ dv$
$\therefore x = -\frac{1}{k}(v - \frac{g}{k}\ln(g+kv)) + c_1$ ... (4)
Now, when $x = 0$, $v = u \therefore 0 = -\frac{1}{k}[u - \frac{g}{k}\ln(g+ku)] + c_1$
Thus, from (4), we have $x = \frac{1}{k}[u - \frac{g}{k}\ln(g+ku)] - \frac{1}{k}[v - \frac{g}{k}\ln(g+kv)]$
The greatest height H is given, when $v = 0$, by
$H = \frac{1}{k}[u - \frac{g}{k}\ln(g+ku)] - \frac{1}{k}[-\frac{g}{k}\ln g] = \frac{u}{k} - \frac{g}{k^2}\ln(\frac{g+ku}{g}) = \frac{1}{k^2}[uk - g\ln(1+\frac{ku}{g})]$.

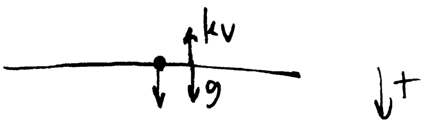
{Alternatively, the distance travelled from $v = u$ to $v = 0$, i.e., the greatest height H, is given by $H = -\int_u^0 \frac{v \ dv}{g+kv}$ ..., see (3) above.}

Lecture 53

Case 2 - dropped down

A particle of mass m falls vertically from rest, in a medium whose resistance is proportional to the velocity. Find the terminal velocity of the particle and derive expressions in terms of **(i)** t **(ii)** v.

Solution.



$\ddot{x} = g - kv$
$\frac{dv}{dt} = g - kv$
$\frac{dv}{g-kv} = dt$
$\therefore -\frac{1}{k}\ln(g - kv) = t + c$
when $t = 0,\ v = 0$
$-\frac{1}{k}\ln g = 0 + c,\ \therefore c = -\frac{1}{k}\ln g.$
$\therefore t = \frac{1}{k}\ln g - \frac{1}{k}\ln(g - kv) = \frac{1}{k}\ln\frac{g}{g-kv}$
$\therefore \ln\frac{g}{g-kv} = kt$
$\therefore \frac{g}{g-kv} = e^{kt}$
$\therefore \frac{g-kv}{g} = e^{-kt}$
$g - kv = ge^{-kt}$
$\therefore kv = g - ge^{-kt}$
$\therefore v = \frac{g}{k}(1 - e^{-kt})$
& $\lim\limits_{t\to\infty} v = \frac{g}{k}$ $\therefore$ terminal velocity $= \frac{g}{k}$ (since as $t \to \infty, e^{-kt} \to 0$)
$\frac{dx}{dt} = \frac{g}{k}(1 - e^{-kt})$
$\therefore x = \frac{g}{k}(1 - e^{-kt})$
$\therefore x = \frac{g}{k}\int(1 - e^{-kt})\ dt = \frac{g}{k}(t + \frac{1}{k}e^{-kt}) + c$
& when $x = 0,\ t = 0 \Rightarrow c = -\frac{g}{k^2}$
$x = \frac{g}{k}(t + \frac{1}{k}e^{-kt}) - \frac{g}{k^2} = \frac{g}{k^2}(kt + e^{-kt} - 1)$
$\ddot{x} = g - kv$
$\ddot{v}\frac{dv}{dx} = g - kv$
$\therefore \frac{v\ dv}{g-kv} = dx$
$\therefore x = \int\frac{v\ dv}{g-kv} = \int\frac{-\frac{1}{k}(g-kv)+\frac{g}{k}}{g-kv}\ dv = \int(-\frac{1}{k} + \frac{g}{k}\frac{1}{g-kv}\ dv) = -\frac{v}{k} - \frac{g}{k^2}\ln(g - kv) + c$

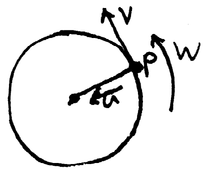
When $x = 0,\ v = 0 \Rightarrow c = \frac{g}{k^2}\ln g$
$x = -\frac{v}{k} - \frac{g}{k^2}\ln(g - kv) + \frac{g}{k^2}\ln g = -\frac{v}{k} + \frac{g}{k^2}\ln\frac{g}{g-kv}.$

Lecture 54

Circular Motion

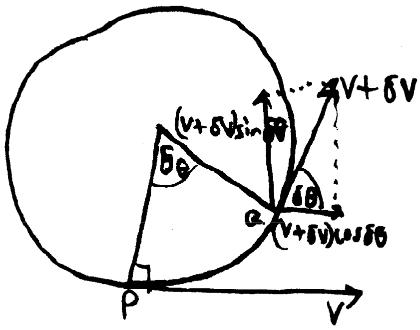
Summary (Physics)

Constant angular velocity



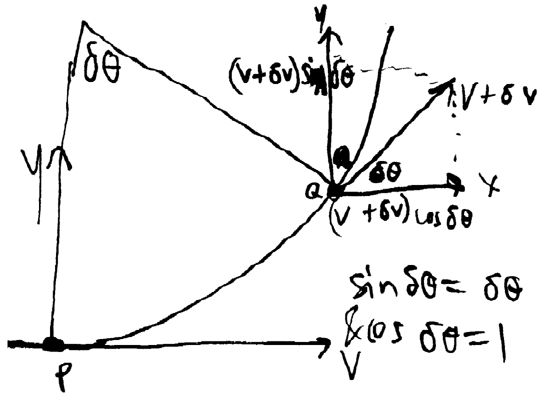
$\omega = \frac{d\theta}{dt}$
1. $v = r\omega$
2. $T = \frac{2\pi}{\omega}$
3. $\theta = \omega t$ (angular displacement)
4. $r\omega^2 = \frac{v^2}{r}$

Tangential and Normal components of Acceleration.



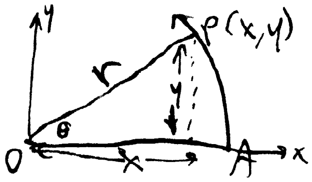
Change in velocity along tangent at
$P = (v + \delta v)\cos\delta\theta - v$ but $\cos\delta\theta \doteqdot 1$
$\doteqdot v + \delta v - v$
$= \delta v.$
Along normal at
$P = (v + \delta v)\sin\delta\theta$ but $\sin\delta\theta \doteqdot \delta\theta$
$= (v + \delta v)\delta\theta$
$= v\delta\theta + \delta v\delta\theta$ but $\delta\theta\delta v$ we ignore
$= v\delta\theta.$
$\therefore$ tangential acceleration from P to $Q = \frac{\delta v}{\delta t}$
$\therefore$ tangential acceleration at $P = \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt} = \frac{d(r\omega)}{dt} = r\frac{d}{dt}(\frac{d\theta}{dt}) = r\frac{d^2\theta}{dt^2} = r\ddot{\theta}$ - this equals zero for constant angular velocity.

Method 1 - as above



Acceleration along tangent at $P = \lim\limits_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt} = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} = r\frac{d}{dt}(\frac{d\theta}{dt}) = r\frac{d^2\theta}{dt^2} = r\ddot{\theta}$
Acceleration along normal at $P = \lim_{\delta t \to 0} \frac{v\delta\theta}{\delta t} = v\frac{d\theta}{dt} = r\omega.\omega = r\omega^2$.

Method 2.
Acceleration in normal/tangential directions.



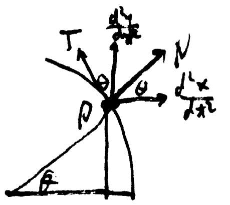
Consider OA to be the x axis.
$x = r\cos\theta$, $y = r\sin\theta$
$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = -r\sin\theta.\dot{\theta}$
$\frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} = r\cos\theta.\dot{\theta}$.
Acceleration:
$\frac{d^2x}{dt^2} = \frac{d}{dt}(-r\sin\theta.\dot{\theta}) = -r\sin\theta.\frac{d\dot{\theta}}{dt} + \dot{\theta}\frac{d}{dt}(-r\sin\theta) = -r\sin\theta.\ddot{\theta} + \dot{\theta}.(-r\cos\theta).\frac{d\theta}{dt}$
$= -r\sin\theta.\ddot{\theta} - r\cos\theta.(\dot{\theta})^2$

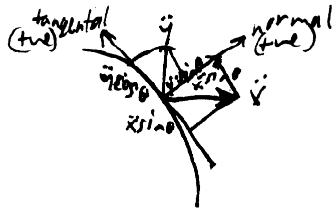
$\frac{d^2y}{dt^2} = r\cos\theta.\frac{d\dot{\theta}}{dt} + \dot{\theta}.\frac{d}{d\theta}(r\cos\theta).\frac{d\theta}{dt} = r\cos\theta.\ddot{\theta} - r\sin\theta.(\dot{\theta})^2$

Lecture 55

Acceleration Components

$\frac{d^2x}{dt^2} = -r\sin\theta.\ddot{\theta} - r\cos\theta.\dot{\theta}^2$
$\frac{d^2y}{dt^2} = r\cos\theta.\ddot{\theta} - r\sin\theta.\dot{\theta}^2$





acceleration in direction PN
$= \ddot{y}\sin\theta + \ddot{x}\cos\theta$
$= r\cos\theta\sin\theta.\ddot{\theta} - r\sin^2\theta.\dot{\theta}^2 - r\sin\theta\cos\theta.\ddot{\theta} - r\cos^2\theta.\dot{\theta}^2$
$= -r(\dot{\theta})^2(\sin^2\theta + \cos^2\theta)$
$= -r\dot{\theta}^2$
$= -r\omega^2 \Rightarrow$ direction is towards centre.

acceleration in direction PT
$= \ddot{y}\cos\theta - \ddot{x}\sin\theta$
$= r\cos^2\theta.\ddot{\theta} - r\sin\theta\cos\theta.\dot{\theta}^2 + r\sin^2\theta.\ddot{\theta} + r\sin\theta\cos\theta.\dot{\theta}^2$
$= r\ddot{\theta}(\cos^2\theta + \sin^2\theta)$
$= r\ddot{\theta}$ (or $r\dot{\omega}$ or $r\frac{d^2\theta}{dt^2}$)

Example

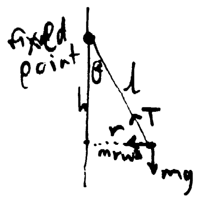
A string 50cm long will break if a mass exceeding 40kg is hung from it. A mass of 2kg is attached to one end of a string and it is revolved in a circle in a horizontal plane. Find the greatest angular velocity without the string breaking (gravity = g).

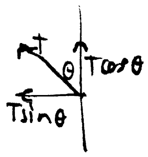
Force $\leq 40g$N (i.e., m/sec^2)

Horizontal forces
$T = ma = mr\omega^2 = 2\text{kg}.0.5\text{m}.\omega^2$
But $T \leq 40g \therefore \omega^2 \leq 40g$ (strng is elastic and not massive)
$\therefore \omega \leq \sqrt{40g}$ rad/sec.

Conical Pendulum

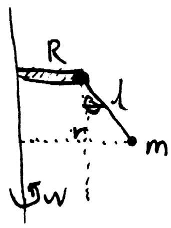
- need to analyse forces vertically and horizontally.



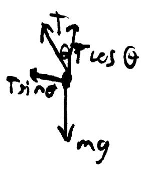


$T\cos\theta = mg$
Horizontal forces: $T\sin\theta = mr\omega^2$
$\frac{T\sin\theta}{T\cos\theta} = \tan\theta = \frac{mr\omega^2}{mg} = \frac{r\omega^2}{g} \leftarrow$ independant of mass
From diagram, $\tan\theta = \frac{r}{h} \therefore \frac{r}{h} = \frac{r\omega^2}{g} \therefore h = \frac{g}{\omega^2} \therefore \omega^2 = \frac{g}{h} \therefore \omega = \sqrt{\frac{g}{h}}$
But period $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{h}{g}}$
$T\sin\theta = mr\omega^2$ but $\sin\theta = \frac{r}{l} \therefore r = l\sin\theta \therefore T\sin\theta = ml\sin\theta\omega^2 \therefore T = ml\omega^2$.

Example



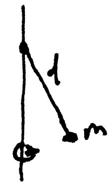
a rod of length R is attached horizontally to a rotating vertical support. A string of lenght l with a mass m attached to the end, is attached to the rod as shown. The support is rotated at speed of ω rad/s. Show that $\omega = \sqrt{\frac{g\tan\theta}{R+l\sin\theta}}$.



Vertically, $T\cos\theta = mg$, horizontally, $T\sin\theta = mr\omega^2$, but $r = R + l\sin\theta$
$\therefore \frac{T\sin\theta}{T\cos\theta} = \tan\theta = \frac{mr\omega^2}{mg} = \frac{r\omega^2}{g} = \frac{(R+l\sin\theta)\omega^2}{g}$
$\therefore \omega^2 = \frac{g\tan\theta}{R+l\sin\theta} \therefore \omega = \sqrt{\frac{g\tan\theta}{R+l\sin\theta}}$

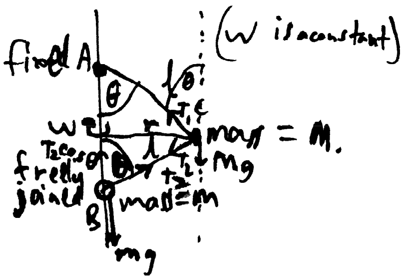
Lecture 56

Tension in String.



$T\sin\theta = mr\omega^2$, $T\cos\theta = mg$, $T = ml\omega^2$
$\therefore T^2\sin^2\theta = m^2r^2\omega^4$, $T^2\cos^2\theta = m^2g^2$
$\therefore T^2\sin^2\theta + T^2\cos^2\theta = T^2(\sin^2\theta+\cos^2\theta) = T^2 = m^2r^2\omega^4 + m^2g^2 = m^2(r^2\omega^4+g^2)$
$\therefore T = m\sqrt{r^2\omega^4+g^2}$

Example.



Show $\theta = \cos^{-1}\left[\frac{(M+2m)g}{Ml\omega^2}\right]$
Resolving vertical forces at B: $T_2\cos\theta = mg \therefore T_2 = \frac{mg}{\cos\theta}$
Resolving forces at C:
Horizontal: $T_1\sin\theta + T_2\sin\theta = Mr\omega^2$ (but $r = l\sin\theta$)
Vertical: $Mg + T_2\cos\theta = T_1\cos\theta \therefore Mg + mg = T_1\cos\theta \therefore T_1 = \frac{Mg+mg}{\cos\theta} = g\left(\frac{M+m}{\cos\theta}\right)$
$T_1\sin\theta + T_2\sin\theta = Ml\sin\theta\omega^2$
$\therefore T_1 + T_2 = Ml\omega^2$
$\therefore \frac{g(M+m)}{\cos\theta} + \frac{mg}{\cos\theta} = Ml\omega^2$
$\therefore \frac{(M+2m)g}{\cos\theta} = Ml\omega^2$
$\therefore \cos\theta = \frac{(M+2m)g}{Ml\omega^2}$

$$\therefore \theta = \cos^{-1}\left(\frac{(M+2m)g}{Ml\omega^2}\right)$$

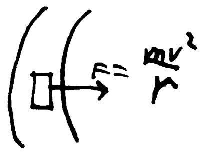
Lecture 57

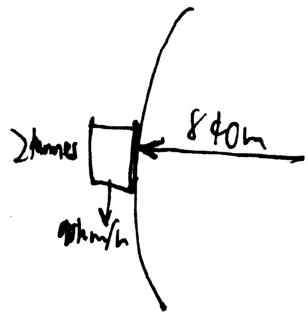
Motion Around a Curved Track

Example - from Coroneos Supplement Set 4E Q1ii

A motor car of mass 2t is rounding a curve of radius 840m on a level track at 90kmh^{-1}. What is the force of friction between the wheels and the ground?

Solution

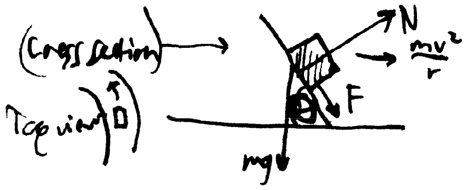




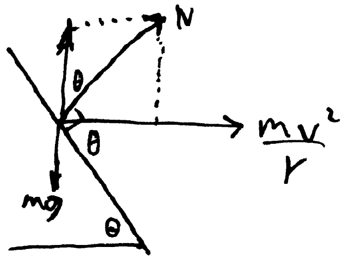
Note frictional force supplies centripetal force.
90km/h=$\frac{90,\ 000}{3600} = 25$m/s.
Frictional forces $= \frac{mv^2}{r} = \frac{2000\times25^2}{840}$kgms^{-2} $= 1488$N.

Lecture 58

Motion around a banked track



If it is going slow, it will slide down.
If it is going fast, it will slide up.
There will be no frictional forces if it does not slide up or down. This is the ideal banking of the track.
So ideal banking of a track implies $F = 0$.
We wish to determine the angle of banking to avoid side-slip.



$N\cos\theta = mg$.
horizontal forces: $N\sin\theta = \frac{mv^2}{r}$
Eliminate N:
$\frac{N\sin\theta}{N\cos\theta} = \tan\theta = \frac{(\frac{mv^2}{r})}{mg} = \frac{v^2}{rg}$ i.e., θ is independant of mass and is dependant on r and velocity.

Example. A curved railway track of radius 690m is designed for trains travelling at an average speed of 48 km/h. Find the ideal banking angle and how much should the outer track be raised if the rail gauge=1.44m (distance between tracks).

Solution.

48km/h$= \frac{48(1000)}{3600} = 13\frac{1}{3}$
$\tan\theta = \frac{v^2}{rg} = \frac{(13\frac{1}{3})^2}{690\times 9.8}$
$\therefore \theta = 1°30'$

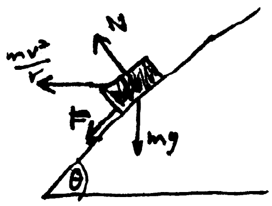
$\sin 1^\circ 30' = \frac{h}{1.44}$
$\therefore h = 1.44 \sin 1^\circ 30' = 0.00377\text{m} = 3.77\text{cm}$

Lecture 59

Circular Motion around a curved track (cont'd)

Example 1. A car of mass m rounds a curve of radius r banked at an angle θ to the horizontal with speed v. if F is the sideways frictional force between the tyres and the road and N is the normal reaction of the road on the tyre, show that $F = mg\cos\theta(\frac{v^2}{gr} - \tan\theta)$, $N = mg\cos\theta(\frac{v^2}{gr}\tan\theta + 1)$.

Solution:



horizontally: $\frac{mv^2}{r} = N\sin\theta + F\cos\theta$
vertically: $N\cos\theta = mg + F\sin\theta$
eliminate N: $N\sin\theta = \frac{mv^2}{r} - F\cos\theta$

$$N\cos\theta = mg + F\sin\theta$$
$$\frac{N\sin\theta}{N\cos\theta} = \frac{\frac{mv^2}{r} - F\cos\theta}{mg + F\sin\theta}$$
$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{\frac{mv^2}{r} - F\cos\theta}{mg + F\sin\theta}$$
$$mg\sin\theta + F\sin^2\theta = \frac{mv^2}{r}\cos\theta - F\cos^2\theta$$
$$F\sin^2\theta + F\cos^2\theta = \frac{mv^2}{r}\cos\theta - mg\sin\theta$$
$$F = mg\cos\theta(\frac{v^2}{rg} - \tan\theta).$$

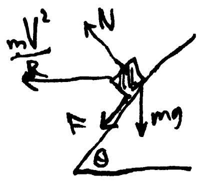
Eliminate F: $(\frac{mv^2}{r} = N\sin\theta + F\cos\theta) \times \sin\theta$

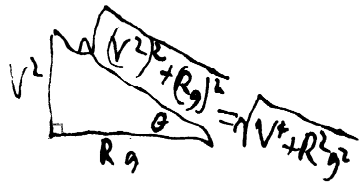
$$N\cos\theta = mg + F\sin\theta) \times \cos\theta$$
$$\frac{mv^2}{r}\sin\theta = N\sin^2\theta + F\cos\theta\sin\theta. \text{ Subtract.}$$
$$N\cos^2\theta = mg\cos\theta + F\sin\theta\cos\theta$$
$$\frac{mv^2}{r}\sin\theta - N\cos^2\theta = N\sin^2\theta - mg\cos\theta$$
$$N\sin^2\theta + N\cos^\theta = \frac{mv^2}{r}\sin\theta + mg\cos\theta$$
$$N = mg\cos\theta(\frac{v^2}{rg}\tan\theta + 1)$$

Example 2. A cart travels at vm/s along a curved track of radius Rm. Find the inclination of the track to the horizontal if there is to be no tendancy for the car to slip sideways. If the speed of the car is Vm/s prove that the sideways frictional force on the wheels of the car of mass m is $\frac{mg(V^2-v^2)}{\sqrt{v^4+R^2g^2}}$.

Solution.

horizontally: $\frac{mv^2}{R} = N \sin\theta.$
vertically: $mg = N\cos\theta$
eliminate $\boldsymbol{N}$: $\frac{N\sin\theta}{N\cos\theta} = \frac{\frac{mv^2}{R}}{mg}$. So $\tan\theta = \frac{v^2}{Rg}$ $\therefore \theta = \tan^{-1}\frac{v^2}{gR}$.





horizontally: $\frac{mV^2}{R} = N\sin\theta + F\cos\theta$
vertically: $mg = N\cos\theta + F\sin\theta$
$F = mg\cos\theta(\frac{V^2}{Rg} - \tan\theta) = \frac{mgRg}{\sqrt{v^4+R^2g^2}}(\frac{V^2}{Rg} - \frac{v^2}{Rg}) = \frac{mg(V^2-v^2)}{\sqrt{v^4+R^2g^2}}$