

Lecture 17

Implicit differentiation.

Making y the subject: If $xy = 1$, $y = x^{-1}$ & $\frac{dy}{dx} = -x^{-2}$. But $xy - y^2 = 1$ is harder to be changed to the subject of y .

Note: $\frac{d}{dx}(f(y)) = f'(y) \cdot \frac{dy}{dx}$

Example 1. Find $\frac{dy}{dx}$ given $xy - y^2 = 1$.

$$\begin{aligned}1 \cdot y + x \cdot 1 \cdot \frac{dy}{dx} - 2y \cdot \frac{dy}{dx} &= 0 \\x \frac{dy}{dx} - 2y \frac{dy}{dx} &= -y \\ \frac{dy}{dx} &= \frac{-y}{x-2y} \\ &= \frac{y}{2y-x} \quad \square\end{aligned}$$

Example 2. Find $\frac{dy}{dx}$ if $x^2y^3 = 1$.

$$\begin{aligned}2xy^3 + x^2 \cdot 3y^2 \frac{dy}{dx} &= 0 \\3x^2y^2 \frac{dy}{dx} &= -2xy^3 \\ \frac{dy}{dx} &= \frac{-2xy^3}{3x^2y^2} \\ &= \frac{-2y}{3x} \quad \square\end{aligned}$$



Lecture 18

Implicit functions examples from Set 6M (Coroneos, 1982c)

Question 7. If $x^2y = (x + y)^3$, show that $\frac{dy}{dx} = \frac{y}{x}$.

$$\begin{aligned}\frac{d}{dx}x^2y &= \frac{d}{dx}(x + y)^3. \\ \therefore y\left(\frac{d}{dx}x^2\right) + x^2\frac{d}{dy}y\frac{dy}{dx} &= \frac{d}{dx}(x^3 + 3x^2y + 3xy^2 + y^3). \\ \therefore 2xy + x^2\frac{dy}{dx} &= \frac{d}{dx}x^3 + \frac{d}{dx}3x^2y + \frac{d}{dx}3xy^2 + \frac{d}{dx}y^3 \\ &= 3x^2 + y\frac{d}{dx}3x^2 + 3x^2\frac{d}{dy}y\frac{dy}{dx} + y^2\frac{d}{dx}3x + 3x\frac{d}{dy}y^2\frac{dy}{dx} + \frac{d}{dy}y^3\frac{dy}{dx} \\ &= 3x^2 + 6xy + 3x^2\frac{dy}{dx} + 3y^2 + 6xy\frac{dy}{dx} + 3y^2\frac{dy}{dx}. \\ \therefore \frac{dy}{dx}(2x^2 + 6xy + 3y^2) &= -3x^2 - 4xy - 3y^2. \\ \therefore \frac{dy}{dx} &= \frac{-3x^2 - 4xy - 3y^2}{2x^2 + 6xy + 3y^2} \\ &= \frac{-3(x+y)^2 + 2xy}{3(x+y)^2 - x^2} \cdot \frac{x+y}{x+y} \\ &= \frac{-3(x+y)^3 + 2x^2y + 2xy^2}{3(x+y)^3 - x^3 - x^2y} \\ &= \frac{-3x^2y + 2x^2y + 2xy^2}{3x^2y - x^3 - x^2y} \quad (\text{since } x^2y = (x + y)^3) \\ &= \frac{2xy^2 - x^2y}{2x^2y - x^3} \\ &= \frac{2y^2 - xy}{2xy - x^2} \\ &= \frac{y}{x} \cdot \frac{2y - x}{2y - x} \\ &= \frac{y}{x}. \quad \square\end{aligned}$$

Question 8. If $xy + 5x + 8 = 0$, show that $x^2\frac{dy}{dx} = 8$.

$$\begin{aligned}\frac{d}{dx}(xy + 5x + 8) &= \frac{d}{dx}0. \\ \therefore y\frac{d}{dx}x + x\frac{d}{dy}y\frac{dy}{dx} + \frac{d}{dx}5x + \frac{d}{dx}8 &= 0. \\ \therefore y + x\frac{dy}{dx} + 5 &= 0. \\ \therefore x\frac{dy}{dx} &= -y - 5. \\ \therefore \frac{dy}{dx} &= \frac{-y-5}{x} \\ \therefore x^2\frac{dy}{dx} &= x^2\left(\frac{-y-5}{x}\right) \\ &= -xy - 5x \\ &= 8 \quad (\text{since } xy + 5x + 8 = 0). \quad \square\end{aligned}$$

Question 9. If $ax^2 + by^2 = c$ show that $\frac{d^2y}{dx^2} = \frac{-ac}{b^2y^3}$.

$$\frac{d}{dx}ax^2 + b\frac{d}{dy}y^2\frac{dy}{dx} = \frac{d}{dx}c.$$

$$\therefore 2ax + 2by\frac{dy}{dx} = 0$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{-2ax}{2by} \\ &= \frac{-ax}{by}\end{aligned}$$

$$\begin{aligned}\therefore \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{dx}\frac{-a}{b}xy^{-1} \\ &= \frac{-a}{b}\left(y^{-1} \cdot 1 + x \cdot \frac{dy^{-1}}{dy} \cdot \frac{dy}{dx}\right) \\ &= \frac{-a}{b}\left(\frac{1}{y} - \frac{x}{y^2} \cdot \frac{-ax}{by}\right) \\ &= -a\left(\frac{by^2 + ax^2}{b^2y^3}\right) \\ &= \frac{-ac}{b^2y^3} \quad (\text{since } ax^2 + by^2 = c). \quad \square\end{aligned}$$



Lecture 19

The Conic Sections

The Circle (Cartesian equation)

$$x^2 + y^2 = a^2$$

centre at origin and radius a .

$$(x - h)^2 + (y - k)^2 = a^2$$

centre (h, k) and radius a .

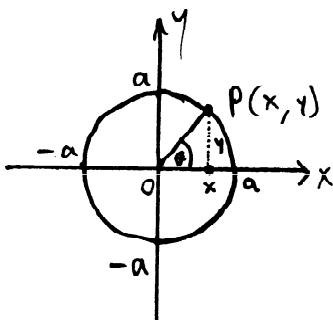
Parametric Equations

Idea of parametric equations is to express x and y in terms of a 3rd variable such as t or θ .

E.g., find the cartesian equation of the following curve:

$$\text{parametric equations } \begin{cases} x = 1 + t \\ y = t^2 \end{cases} \Rightarrow t = x - 1$$

$$\therefore y = (x - 1)^2 \text{ - cartesian equation.}$$



$$x = a \cos \theta$$

$$y = a \sin \theta$$

-from the diagram.

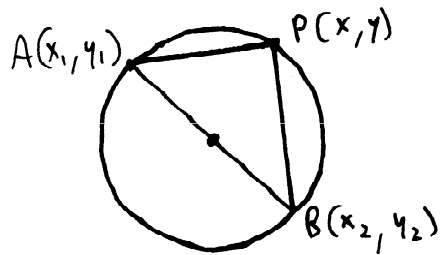
-parametric equations of a circle centre $(0, 0)$, radius a .

With centre (h, k) , parametric equations are:

$$x = h + a \cos \theta$$

$$y = k + a \sin \theta.$$

Circle on diameter $A(x_1, y_1), B(x_2, y_2)$:



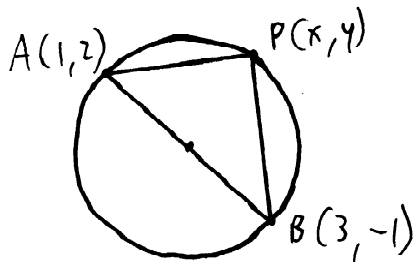
Since $AP \perp BP$, $\text{grad}AP \cdot \text{grad}BP = -1$.

$$\therefore \frac{y-y_1}{x-x_1} \cdot \frac{y-y_2}{x-x_2} = -1$$

$$(y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$$

$$\therefore (y-y_1)(y-y_2) + (x-x_1)(x-x_2) = 0.$$

Fig. 1. Find the equation of the circle with diameter A, B where $A = (1, 2)$ and $B = (3, -1)$. Find the centre and radius.



$$\frac{y-2}{x-1} \cdot \frac{y+1}{x-3} = -1$$

$$(y-2)(y+1) + (x-1)(x-3) = 0$$

$$\therefore y^2 - y^2 + x^2 - 4x + 1 = 0$$

$$\therefore y^2 - y + \frac{1}{2} + x^2 - 4x + 4 + 1 = 4\frac{1}{2}.$$

$$(x-2)^2 + (y-\frac{1}{2})^2 = 3\frac{1}{4} \therefore \text{the centre} = (2, \frac{1}{2}), \text{radius} = \frac{\sqrt{13}}{2}. \quad \square$$

Example 1 from Coroneos, 1982b, p25.

Find the equation of the circle

(i) centre the origin to pass through the point $(-3, 7)$

(ii) centre $(1, -4)$ to touch the line $2x - 3y = 9$

(iii) which touches the coordinates axes and passes through $(8, 1)$.

(i) $x^2 + y^2 = (-3-0)^2 + (7-0)^2$ i.e., $x^2 + y^2 = 58$.

(ii) $(x - 1)^2 + (y + 4)^2 = \left(\frac{|2(1) - 3(-4) - 9|}{\sqrt{2^2 + (-3)^2}} \right)^2$ i.e., $(x - 1)^2 + (y + 4)^2 = \frac{25}{13}$.

(iii) $(x - a)^2 + (y - a)^2 = a^2$ where $(8 - a)^2 + (1 - a)^2 = a^2$.

Hence $64 - 16a + a^2 + 1 - 2a + a^2 = a^2$

$\therefore a^2 - 18a + 65 = 0$

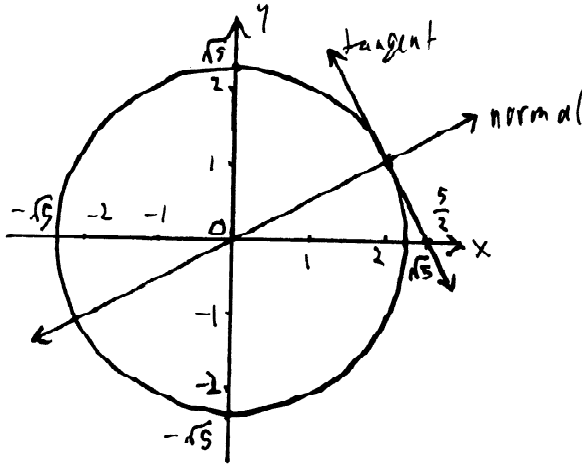
$\therefore (a - 5)(a - 13) = 0 \therefore a = 5, 13$

$\therefore (x - 5)^2 + (y - 5)^2 = 25$ or $(x - 13)^2 + (y - 13)^2 = 169$. \square



Lecture 20

Fig. 1. Find the equation of the tangent and normal to the circle $x^2 + y^2 = 5$ at $P(2, 1)$.



Since it is a circle with centre $(0,0)$, then the normal does as well \therefore the equation of the normal can be found and also the tangent since the normal \perp tangent. However by calculus methods,

$$x^2 + y^2 = 5$$

$$\therefore 2x + 2y \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$\therefore m_T = \frac{-2}{1} = -2 \text{ and } m_N = \frac{1}{2}.$$

Using $y - y_1 = m(x - x_1)$:

Tangent

$$y - 1 = -2(x - 2)$$

$$y - 1 = -2x + 4$$

$$2x + y - 5 = 0$$

Normal

$$y - 1 = \frac{1}{2}(x - 2)$$

$$2y - 2 = x - 2$$

$$x - 2y = 0. \quad \square$$

Fig. 2. Find the equation of the tangent and normal to $x^2 + y^2 = a^2$ at

(i) $P(x, y)$ (ii) $P(a \cos \theta, a \sin \theta)$.

(i) $x^2 + y^2 = a^2$

$$\therefore 2x + 2y \frac{dy}{dx} = 0$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{-x}{y} \\ \therefore m_T &= \frac{-x_1}{y_1} \\ \text{and } m_N &= \frac{y_1}{x_1} \quad \therefore \text{since } y - y_1 = m(x - x_1) \end{aligned}$$

Tangent

$$\begin{aligned} y - y_1 &= \frac{-x_1}{y_1}(x - x_1) \\ y_1y - y_1^2 &= -x_1x + x_1^2 \\ x_1x + y_1y &= x_1^2 + y_1^2 = a^2 \\ \therefore x_1x + y_1y &= a^2 \end{aligned}$$

Normal

$$\begin{aligned} y - y_1 &= \frac{y_1}{x_1}(x - x_1) \\ x_1y - x_1y_1 &= y_1x - x_1y_1 \\ \therefore y_1x - x_1y &= 0 \quad \square \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \frac{dy}{dx} &= \frac{-x}{y} \\ m_T &= \frac{-a \cos \theta}{a \sin \theta} = \frac{-\cos \theta}{\sin \theta} = -\cot \theta \\ \therefore m_N &= \frac{\sin \theta}{\cos \theta} = \tan \theta. \end{aligned}$$

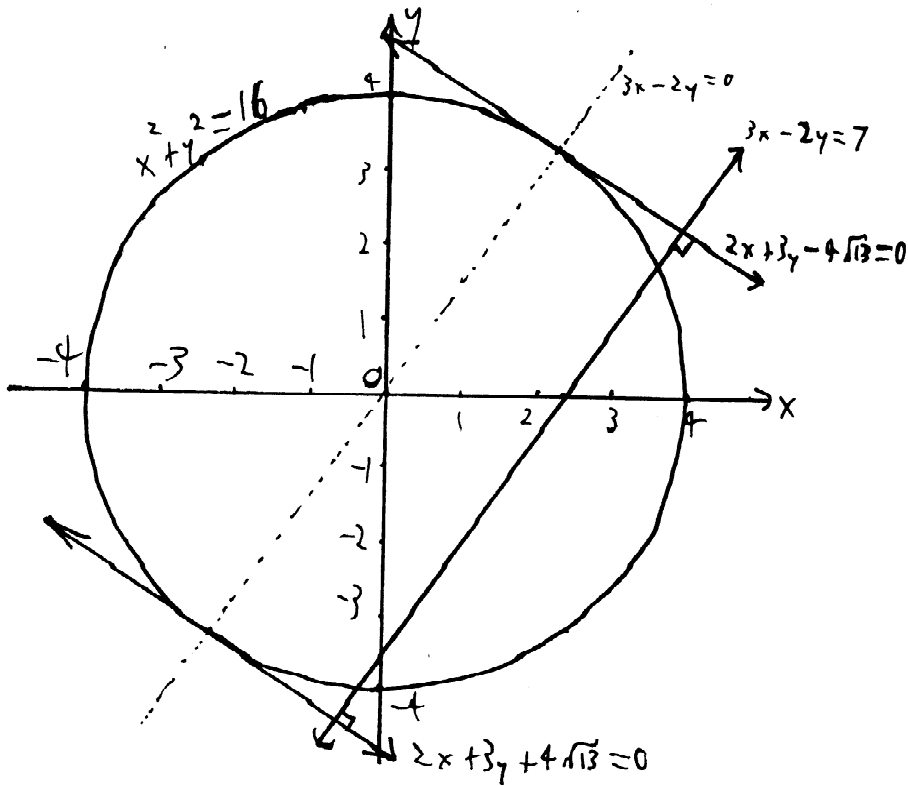
Tangent

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - a \sin \theta &= \frac{-\cos \theta}{\sin \theta}(x - a \cos \theta) \\ y \sin \theta - a \sin^2 \theta &= -x \cos \theta + a \cos \theta \\ x \cos \theta + y \sin \theta &= a \sin^2 \theta + a \cos^2 \theta \\ \therefore x \cos \theta + y \sin \theta &= a \end{aligned}$$

Normal

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - a \sin \theta &= \frac{\sin \theta}{\cos \theta}(x - a \cos \theta) \\ y \cos \theta - a \sin \theta \cos \theta &= x \sin \theta - a \sin \theta \cos \theta \\ \therefore x \sin \theta - y \cos \theta &= 0 \quad \square \end{aligned}$$

Fig. 3. Find the equation of the tangent to the circle $x^2 + y^2 = 16$ and perpendicular to $3x - 2y = 7$.



Line has $m = \frac{3}{2}$.

$$\therefore m_T = -\frac{2}{3}$$

$$\therefore 2x + 3y = k$$

$$\therefore 2x + 3y - k = 0$$

Perpendicular distance of tangent to centre of circle = radius of circle.

$$\therefore d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} = 4$$

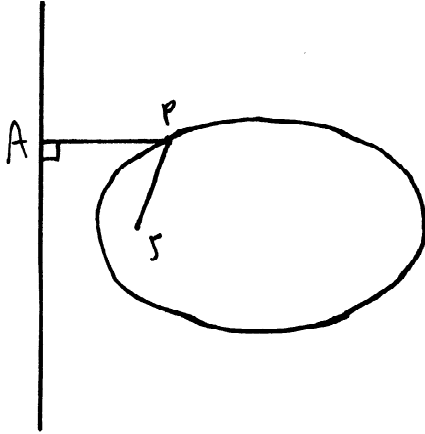
$$\therefore \frac{|k|}{\sqrt{13}} = 4 \text{ (since } c \text{ is constant and } x_1 \text{ and } y_1 = 0.)$$

$$k = \pm 4\sqrt{13} \therefore 2x + 3y \pm 4\sqrt{13}. \quad \square$$



Lecture 21

Ratio of distance from a fixed point (focus) to the distance from a fixed line (directrix) equals a constant (eccentricity, e).



$$\frac{PS}{PM} = e$$

$e > 1 \Rightarrow$ hyperbola

$e = 1 \Rightarrow$ circle

$e < 1 \Rightarrow$ ellipse.

Ellipse.

General equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

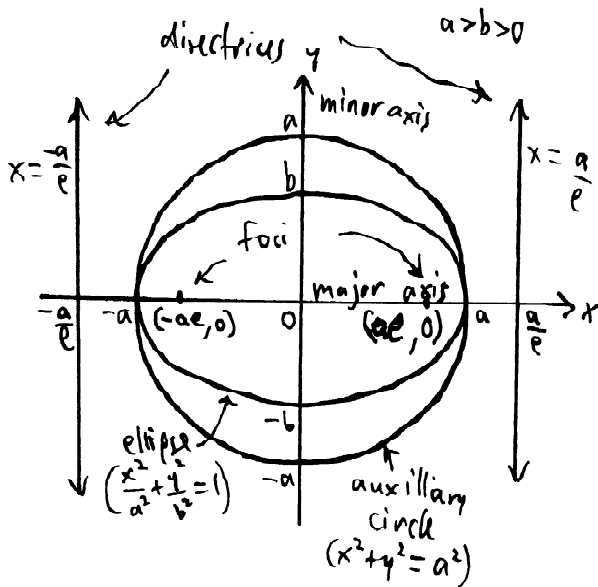
$$a > b.$$

where foci $(\pm ae, 0)$

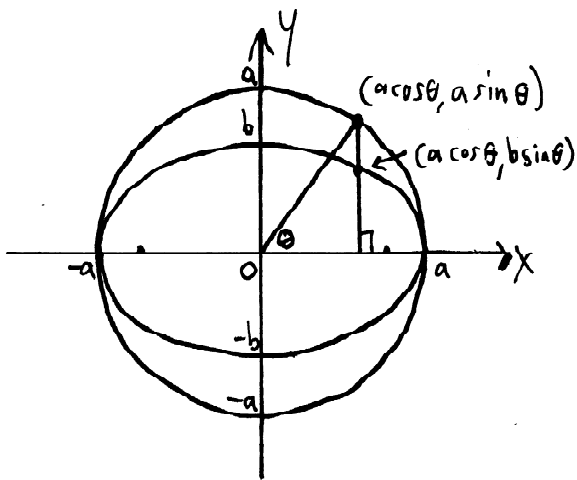
directrix $x = \pm \frac{a}{e}$

eccentricity: $e^2 = 1 - \frac{b^2}{a^2}$

$$\therefore e = \sqrt{1 - \frac{b^2}{a^2}}$$

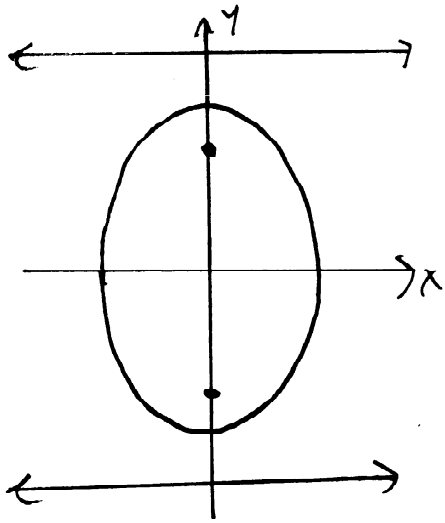


Parametric equation



Parametric equations of ellipse are $x = a \cos \theta, y = b \sin \theta$

Note for $b > a$ we have



Eg. Find the eccentricity, foci, directrices, and sketch the ellipse $4x^2 + 9y^2 = 18$. Also find the parametric equation.

$$\frac{4x^2}{18} + \frac{9y^2}{18} = 1$$

$$\frac{2x^2}{9} + \frac{y^2}{2} = 1$$

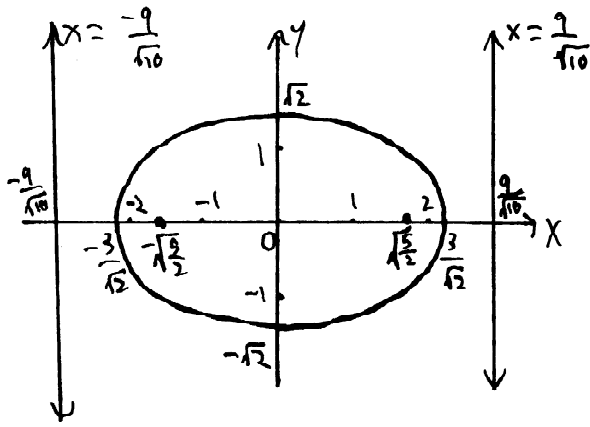
$$\frac{x^2}{9/2} + \frac{y^2}{2} = 1$$

$$\therefore a = \frac{3}{\sqrt{2}} \text{ and } b = \sqrt{2}$$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{2}{9/2}} = \frac{\sqrt{5}}{3}$$

$$\text{foci: } (\pm ae, 0) = \left(\pm \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{5}}{3}, 0 \right) = \left(\pm \sqrt{\frac{5}{2}}, 0 \right)$$

$$\text{directrices } x = \pm \frac{a}{e} = \pm \frac{3/\sqrt{2}}{\sqrt{5}/3} = \pm \frac{3}{\sqrt{2}} \cdot \frac{3}{\sqrt{5}} = \pm \frac{9}{\sqrt{10}}$$



Parametric equations: $x = \frac{3}{\sqrt{2}} \cos \theta$, $y = \sqrt{2} \sin \theta$, $0 \leq \theta < 2\pi$.



Lecture 22

Tangents and Normals to ellipses

$$\begin{aligned}\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\ \frac{2y}{b^2} \frac{dy}{dx} &= \frac{-2x}{a^2} \\ \frac{dy}{dx} &= \frac{-b^2x}{a^2y}.\end{aligned}$$

At $P(a \cos \theta, b \sin \theta)$

Tangent:-

$$\begin{aligned}y - y_1 &= m(x - x_1) \\ m_{\text{tan}} &= \frac{-b^2(\cos \theta)}{a^2(b \sin \theta)} = \frac{-b \cos \theta}{a \sin \theta} \\ \therefore \text{equation is } y - b \sin \theta &= \frac{-b \cos \theta}{a \sin \theta}(x - a \cos \theta). \\ ay \sin \theta - ab \sin^2 \theta &= -bx \cos \theta + ab \cos^2 \theta \\ bx \cos \theta + ay \sin \theta &= ab(\sin^2 \theta + \cos^2 \theta) \\ \frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} &= 1\end{aligned}$$

Normal:-

$$\begin{aligned}m_{\text{norm}} &= \frac{-1}{m_{\text{tan}}} = \frac{a \sin \theta}{b \cos \theta} \\ \therefore \text{equation is } y - b \sin \theta &= \frac{a \sin \theta}{b \cos \theta}(x - a \cos \theta) \\ by \cos \theta - b^2 \sin \theta \cos \theta &= ax \sin \theta - a^2 \sin \theta \cos \theta \\ ax \sin \theta - by \cos \theta - a^2 \sin \theta \cos \theta + b^2 \sin \theta \cos \theta &= 0 \\ \frac{ax}{\cos \theta} - \frac{by}{\sin \theta} &= a^2 - b^2\end{aligned}$$

Tangent:-

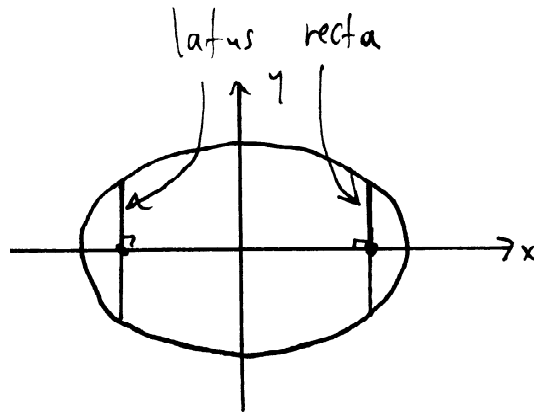
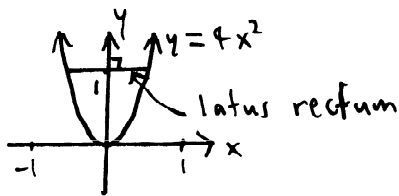
$$\begin{aligned}ax \sec \theta - by \operatorname{cosec} \theta &= a^2 - b^2 \\ \text{At } P(x_1, y_1), \text{ equation is} \\ y - y_1 &= \frac{-b^2x_1}{a^2y_1}(x - x_1) \\ a^2yy_1 - a^2y_1^2 &= -b^2x_1(x - x_1) \\ a^2yy_1 - a^2y_1^2 &= -b^2xx_1 + b^2x_1^2 \\ b^2xx_1 - b^2x_1^2 + a^2yy_1 - a^2y_1^2 &= 0 \\ b^2xx_1 + a^2yy_1 &= b^2x_1^2 + a^2y_1^2 \\ \frac{xx_1}{a^2} + \frac{yy_1}{b^2} &= \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \text{ since } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ \therefore \frac{x_1x}{a^2} + \frac{y_1y}{b^2} &= 1\end{aligned}$$

Normal:-

$$\begin{aligned}\text{On } \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1, P(x_1, y_1) \\ m_{\text{norm}} &= \frac{-1}{m_{\text{tan}}} = \frac{a^2y_1}{b^2x_1}\end{aligned}$$

$$\begin{aligned} \text{and } \therefore (y - y_1) &= \frac{a^2 y_1}{b^2 x_1} (x - x_1) \\ b^2 x_1 y - b^2 x_1 y_1 &= a^2 y_1 x - a^2 y_1 x_1 \\ b^2 x_1 y - a^2 y_1 x &= b^2 x_1 y_1 - a^2 y_1 x_1 = x_1 y_1 (b^2 - a^2) \\ \frac{b^2 x_1 y - a^2 y_1 x}{x_1 y_1} &= b^2 - a^2 \\ \therefore \frac{a^2 y_1 x - b^2 x_1 y}{x_1 y_1} &= a^2 - b^2 \\ \therefore \frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} &= a^2 - b^2 \end{aligned}$$

Note: A latus rectum (plural - latus recta) is the perpendicular chord to the major axis of symmetry of the shape passing through the focus (plural - foci).

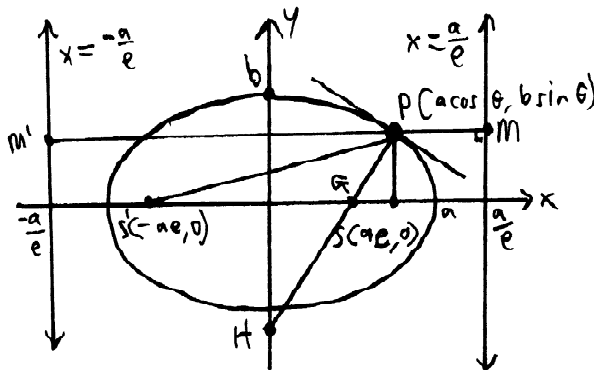


Lecture 23

Example. P is the point $(a \cos \theta, b \sin \theta)$ on $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with eccentricity e , foci S, S' .

i. Show that $SP = a(1 - e \cos \theta)$ and find an expression for $S'P$. Show that $SP + S'P$ is independent of the position of P .

ii. If the normal at P intersects the major axis at G and minor axis at H , show that $\frac{PG}{PH} = 1 - e^2$.



$$\frac{PS}{PM} = e$$

$$\therefore PS = PM \cdot e = \left(\frac{a}{e} - a \cos \theta\right)e = a - ae \cos \theta = a(1 - e \cos \theta)$$

$$\frac{PS'}{PM'} = e$$

$$S'P = ePM' = e\left(\frac{a}{e} + a \cos \theta\right) = a + ae \cos \theta = a(1 + e \cos \theta)$$

$$\therefore SP + S'P = a(1 - e \cos \theta) + a(1 + e \cos \theta) = a(1 - e \cos \theta + 1 + e \cos \theta) = 2a$$

which is independent of the position of P .

$$\text{Normal: } ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$$

$$\text{major axis: } y = 0$$

$$\therefore ax \sec \theta = a^2 - b^2$$

$$x = \frac{a^2 - b^2}{a \sec \theta}$$

$$\therefore G\left(\frac{a^2 - b^2}{a \sec \theta}, 0\right)$$

$$\text{Minor axis: } x = 0 \text{ by } \operatorname{cosec} \theta = \frac{a^2 - b^2}{by}$$

$$y = \frac{-(a^2 - b^2)}{b \operatorname{cosec} \theta} \text{ and therefore } H = \left(0, \frac{-(a^2 - b^2)}{b \operatorname{cosec} \theta}\right)$$

$$\begin{aligned}
\therefore \frac{PG}{PH} &= \sqrt{\frac{(a \cos \theta - \frac{a^2 - b^2}{a \sec \theta})^2 + (b \sin \theta)^2}{(a \cos \theta)^2 + (b \sin \theta - \frac{b^2 - a^2}{\operatorname{cosec} \theta})^2}} \\
&= \sqrt{\frac{(a \cos \theta - a \cos \theta + \frac{b^2}{a} \cos \theta)^2 + (b \sin \theta)^2}{(a \cos \theta)^2 + (b \sin \theta - b \sin \theta + \frac{a^2}{b} \sin \theta)^2}} \\
&= \sqrt{\frac{\frac{b^4}{a^2} \cos^2 \theta + b^2 \sin^2 \theta}{a^2 \cos^2 \theta + \frac{a^4}{b^2} \sin^2 \theta}} \\
&= \sqrt{\frac{b^6 \cos^2 \theta + a^2 b^4 \sin^2 \theta}{a^4 b^2 \cos^2 \theta + a^6 \sin^2 \theta}} \\
&= \frac{b^2}{a^2} \sqrt{\frac{b^2 \cos^2 \theta + a^2 \sin^2 \theta}{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} \\
&= \frac{b^2}{a^2} \\
&= 1 - e^2 \quad \text{since } e^2 = 1 - \frac{b^2}{a^2}.
\end{aligned}$$



Lecture 24

From Coroneos, 1982b, Set 2D Q8.

8. P is the point $(a \cos \theta, b \sin \theta)$ on the ellipse E and Q is the corresponding point on the auxiliary circle C (i.e., P, Q have the same abscissa). State the coordinates of Q .

(i) Find the equation of the tangent at P to E and at Q to C . Prove that these tangents meet on the major axis of the ellipse.

(ii) Show that the perpendicular distance from a focus S of E to the tangent at Q to C is equal to SP .

(iii) Find the equation of OQ and of the normal to E at P . Show that these meet on the circle $x^2 + y^2 = (a + b)^2$.

$$Q = (a \cos \theta, a \sin \theta)$$

(i) When E is the locus of P , the equation of E is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and therefore $\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0$. Hence $\frac{dy}{dx} = \frac{-b^2 x}{a^2 y}$ and thus at P , $\frac{dy}{dx} = \frac{-b^2 \cos \theta}{a^2 b \sin \theta} = \frac{-b \cos \theta}{a \sin \theta}$.

$$\therefore \tan_P : y - b \sin \theta = \frac{-b \sin \theta}{a \cos \theta} (x - a \cos \theta). \text{ Hence } ay \sin \theta - ab \sin^2 \theta = -b \cos \theta + ab \cos^2 \theta.$$

$$\therefore bx \cos \theta + ay \sin \theta = ab(\sin^2 \theta + \cos^2 \theta) = ab.$$

When C is the locus of Q , the equation of C is $x^2 + y^2 = a^2$ and therefore $2x + 2y \frac{dy}{dx} = 0$ and therefore $\frac{dy}{dx} = \frac{-x}{y}$.

$$\therefore \text{at } Q, \frac{dy}{dx} = \frac{-a \cos \theta}{a \sin \theta} = \frac{-\cos \theta}{\sin \theta}$$

$$\therefore \tan_Q : y - a \sin \theta = \frac{-\cos \theta}{\sin \theta} (x - a \cos \theta)$$

$$\therefore y \sin \theta - a \sin^2 \theta = -x \cos \theta + a \cos^2 \theta$$

$$\therefore x \cos \theta + y \sin \theta = a(\sin^2 \theta + \cos^2 \theta) = a.$$

When $y = 0$ for \tan_P , $bx \cos \theta = ab$

$$\therefore x = \frac{a}{\cos \theta}$$

and for \tan_Q , $x \cos \theta = a$ and therefore $x = \frac{a}{\cos \theta}$ which is the same point for $\tan_P((\frac{a}{\cos \theta}, 0))$ and therefore the two tangents meet on the major axis of the ellipse (x -axis ($y = 0$)).

(ii) When a focus of E is S , S is $(ae, 0)$

$$\therefore SP = \sqrt{(a \cos \theta - ae)^2 + (b \sin \theta - 0)^2}$$

$$= \sqrt{a^2(\cos \theta - e)^2 + b^2 \sin^2 \theta}$$

$$= \sqrt{a^2(\cos \theta - e)^2 + a^2(1 - e^2)(1 - \cos^2 \theta)}$$

$$= a\sqrt{\cos^2 \theta - 2e \cos \theta + e^2 + 1 - \cos^2 \theta - e^2 + e^2 \cos^2 \theta}$$

$$= a\sqrt{e^2 \cos^2 \theta - 2e \cos \theta + 1}$$

$$= a\sqrt{(e \cos \theta - 1)^2}$$

$$= a|e \cos \theta - 1|.$$

The perpendicular distance from S to \tan_Q is $\left| \frac{ae \cos \theta + 0 \sin \theta - a}{\sqrt{\sin^2 \theta + \cos^2 \theta}} \right| = a|e \cos \theta - 1| = SP$.

(iii) Equation OQ is $y = \frac{a \sin \theta}{a \cos \theta} x = \frac{\sin \theta}{\cos \theta} x \therefore x \sin \theta - y \cos \theta = 0$ (since O is the origin).

Norm $_P$: $y - b \sin \theta = \frac{a \sin \theta}{b \cos \theta} (x - a \cos \theta)$

and therefore $ax \sin \theta - b^2 \cos \theta \sin \theta = ax \sin \theta - a^2 \sin \theta \cos \theta$

$\therefore ax \sin \theta - by \cos \theta = (a^2 - b^2) \cos \theta \sin \theta$

$\therefore \frac{ax \sin \theta}{\cos \theta \sin \theta} - \frac{by \cos \theta}{\cos \theta \sin \theta} = \frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$

\therefore where OQ and Norm $_P$ meet, $x \sin \theta - y \cos \theta = 0$

$\therefore x \sin \theta = y \cos \theta$ and $ax \sin \theta - by \cos \theta = (a^2 - b^2) \cos \theta \sin \theta$

$\therefore ax \sin \theta - bx \sin \theta = (a^2 - b^2) \cos \theta \sin \theta$

$\therefore x \sin \theta (a - b) = (a - b)(a + b) \cos \theta \sin \theta$

$\therefore x \sin \theta = (a + b) \sin \theta \cos \theta$

$\therefore x = (a + b) \cos \theta$ and $y \cos \theta = (a + b) \cos \theta \sin \theta$

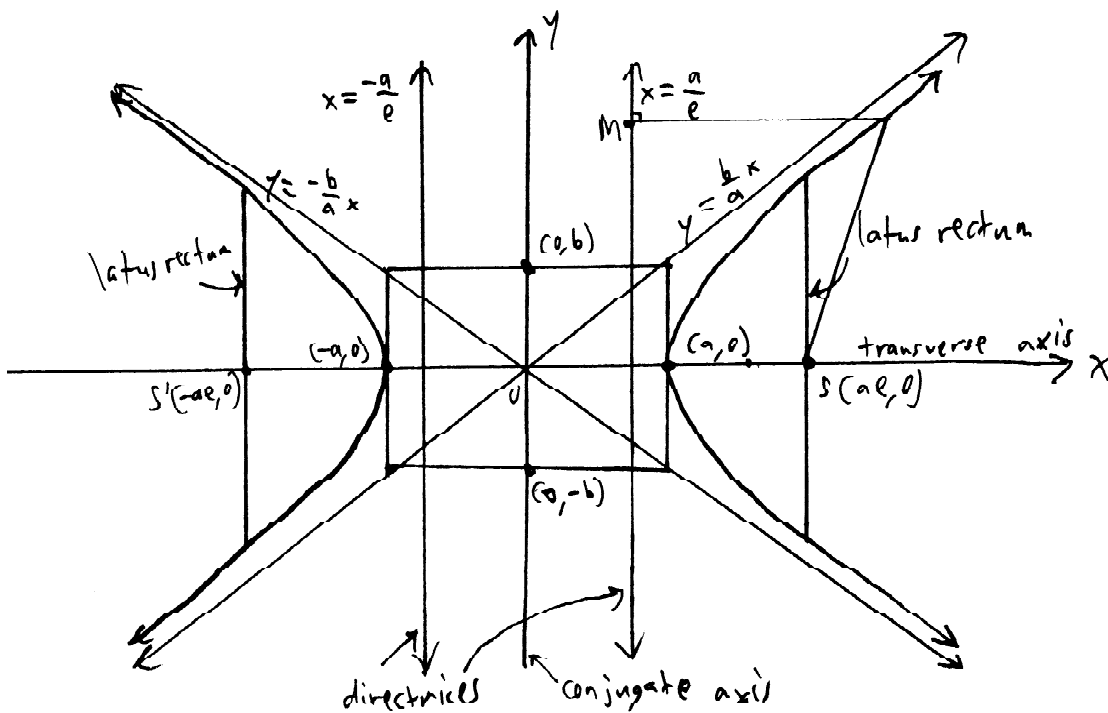
$\therefore y = (a + b) \sin \theta$ which are the parametric equations of the circle, centre $(0, 0)$ and radius $(a + b)$ units which has cartesian equation $x^2 + y^2 = (a + b)^2$ and therefore OQ and Norm $_P$ meet on the circle $x^2 + y^2 = (a + b)^2$.

The Hyperbola

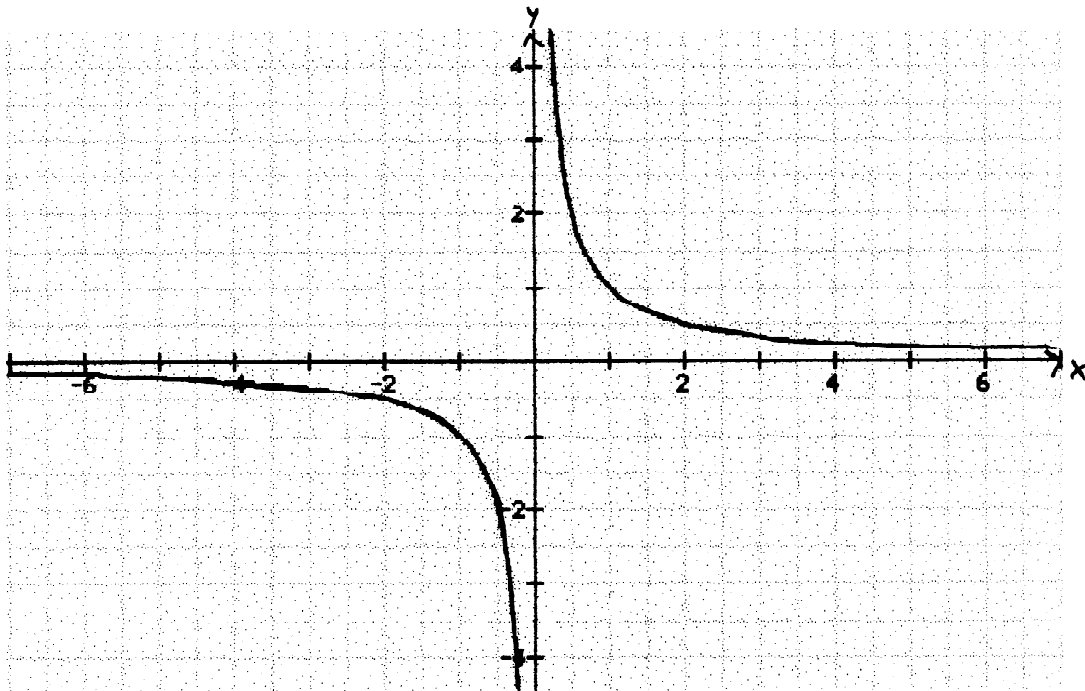
General Equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ eccentricity $e^2 = 1 + \frac{b^2}{a^2} \therefore e = \sqrt{1 + \frac{b^2}{a^2}} > 1$, foci $(\pm ae, 0)$, directrices $x = \pm \frac{a}{e}$, asymptotes $y = \pm \frac{b}{a}x$

Parametric equations

$x = a \sec \theta, y = b \tan \theta$ (Note: $e = \frac{|PS|}{|PM|}$ therefore $PS > PM$).



Note $xy = 1$ is a rectangular hyperbola, i.e.,



Example. For the hyperbola $3x^2 - y^2 = 27$ find the vertex, foci, directrices, asymptotes, eccentricity and sketch the hyperbola. Find the acute angle between the asymptotes.

$$3x^2 - y^2 = 27.$$

$$\frac{x^2}{9} - \frac{y^2}{27} = 1$$

$$a = 3$$

$$b = \sqrt{27} = 3\sqrt{3}$$

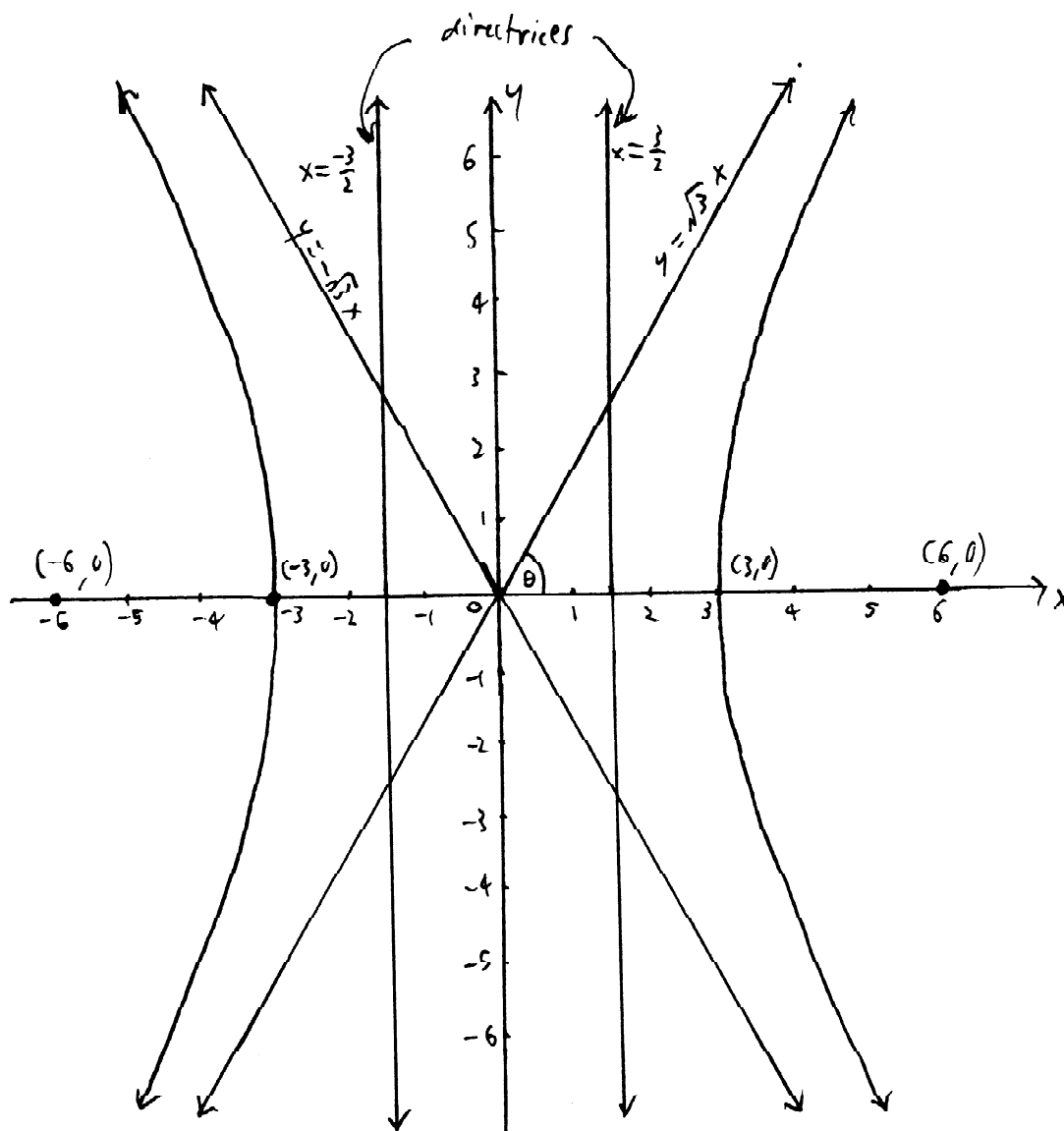
Vertices $(3, 0), (-3, 0)$.

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{27}{9}} = 2$$

Foci: $(\pm, 0)$

Directrices: $x = \pm \frac{3}{2}$

Asymptotes: $y = \pm \sqrt{3}x$



$$\tan \theta = \sqrt{3}$$

$$\therefore \theta = 60^\circ$$

$\therefore \angle AOB$ is the acute angle between the asymptotes and therefore it is 60° .



Lecture 25

Supp. Set 2F, Q1,9 (Coroneos, 1982b):

1. For the hyperbola $H : x^2/16 - y^2/9 = 1$, find the coordinates of the foci S, S' . P is any point $(4 \sec \theta, 3 \tan \theta)$ on H .

(i) Show that $PS = 5 \sec \theta - 4$ and $PS' = 5 \sec \theta + 4$; and hence prove the difference of the focal distances to P is independent of the position of P on H .

(ii) Prove the tangent at P has equation $x \sec \theta/4 - y \tan \theta/3 = 1$. If this tangent meets the transverse axis in T , show that $S'P : PS = S'T : TS$. Deduce that the tangent at P bisects the angle $S'PS$.

(iii) Show the normal at P has equation $4x/\sec \theta + 3y/\tan \theta = 25$. If this normal meets the x -axis in G , prove that $S'P : PS = S'G : SG$.

Solution.

For hyperbola $H : x^2/16 - y^2/9 = 1$, foci S and S' are $(\pm\sqrt{16}\sqrt{1+\frac{9}{16}}, 0) = (\pm 5, 0)$ and P on H is $(4 \sec \theta, 3 \tan \theta)$.

$$(i) PS = \sqrt{1 + \frac{9}{16}}(4 \sec \theta - \frac{\sqrt{16}}{\sqrt{1+\frac{9}{16}}}) = \frac{5}{4}(4 \sec \theta - \frac{16}{5}) = 5 \sec \theta - 4$$

$$PS' = \sqrt{1 + \frac{9}{16}}(4 \sec \theta + \frac{\sqrt{16}}{\sqrt{1+\frac{9}{16}}}) = \frac{5}{4}(4 \sec \theta + \frac{16}{5}) = 5 \sec \theta + 4.$$

\therefore the difference of the focal distances to P is $(5 \sec \theta + 4) - (5 \sec \theta - 4) = 8$ which is independent of θ and \therefore independent of the position of P on H .

(ii) For H , $\frac{x}{8} - \frac{2y}{9} \frac{dy}{dx} = 0$ and therefore $\frac{dy}{dx} = \frac{9x}{16y} = \frac{36 \sec \theta}{48 \tan \theta} = \frac{3 \sec \theta}{4 \tan \theta}$ at P and therefore \tan_P is $y - 3 \tan \theta = \frac{3 \sec \theta}{4 \tan \theta}(x - 4 \sec \theta)$ i.e., $4y \tan \theta - 12 \tan^2 \theta = 3x \sec \theta - 12 \sec^2 \theta$ i.e., $3x \sec \theta - 4y \tan \theta = 12 \sec^2 \theta - 12 \tan^2 \theta = 12(\sec^2 \theta - \tan^2 \theta) = 12$ i.e., $\frac{x \sec \theta}{4} - \frac{y \tan \theta}{3} = 1$. If this meets transverse axis in T , then for T , $y = 0$ and therefore $\frac{x \sec \theta}{4} = 1$ and therefore $x = \frac{4}{\sec \theta}$ and therefore T is $(\frac{4}{\sec \theta}, 0)$ and therefore $S'T : TS = (\frac{4}{\sec \theta} + 5) : (5 - \frac{4}{\sec \theta}) = (5 \sec \theta + 4) : (5 \sec \theta - 4)$. $S'P : PS = (5 \sec \theta + 4) : (5 \sec \theta - 4) = S'T : TS$ (from (i)) and therefore if $S'P$ is produced to I such that $SI \parallel TP$, then $\frac{S'P}{PI} = \frac{S'T}{TS}$ (by intercept theory) $= \frac{S'P}{PS}$ (above) and therefore $\frac{PI}{S'P} = \frac{PS}{S'P}$ and therefore $PI = PS$. Therefore $\triangle IPS$ is isosceles with $\angle PIS = \angle ISP$ (base \angle 's of isosceles $\triangle IPS$) $= \angle SPT$ (alternate). But $\angle PIS = \angle S'PT$ (corresponding) and therefore $\angle S'PT = \angle TPS$ and therefore the tangent at P (i.e., TP) bisects $\angle S'PS$.

(iii) Norm $_P$ is $y - 3 \tan \theta = \frac{-4 \tan \theta}{3 \sec \theta}(x - 4 \sec \theta)$ and therefore $3y \sec \theta - 9 \tan \theta \sec \theta = -4x \tan \theta + 16 \sec \theta \tan \theta$ and therefore $4x \tan \theta + 3y \sec \theta = 16 \sec \theta \tan \theta + 9 \tan \theta \sec \theta = 25 \sec \theta \tan \theta$ and therefore $\frac{4x}{\sec \theta} + \frac{3y}{\tan \theta} = 25$. Where G is the x -intercept, $y = 0$ and therefore $\frac{4x}{\sec \theta} = 25$ and therefore $x = \frac{25 \sec \theta}{4}$ and therefore G is $(\frac{25 \sec \theta}{4}, 0)$ and therefore

$S'G : SG = (\frac{25 \sec \theta}{4} + 5) : (\frac{25 \sec \theta}{4} - 5) = (\frac{25 \sec \theta + 20}{25 \sec \theta - 20}) = (5 \sec \theta + 4) : (5 \sec \theta - 4) = S'P : SP$
and therefore $S'P : SP = S'G : SG$. \square

9. Write down the equations of the asymptotes of the hyperbola $H : x^2/9 - y^2/16 = 1$.

(i) Prove that the part of the tangent at $P(3 \sec \theta, 4 \tan \theta)$ on H which is intercepted between the asymptotes is bisected at P .

(ii) Show that this tangent forms with the asymptotes a triangle of constant area.

(iii) Prove that the tangent intercepted between the asymptotes subtends a constant angle at a focus.

Solution. For the hyperbola $H : x^2/9 - y^2/16 = 1$, asymptotes are $y = \pm \frac{\sqrt{16}}{\sqrt{9}}x = \pm \frac{4}{3}x$.

(i) For $x^2/9 - y^2/16 = 1$, $2x/9 - (2y/16) \frac{dy}{dx} = 0$ and therefore $\frac{dy}{dx} = \frac{16x}{9y} = \frac{16(3 \sec \theta)}{9(4 \tan \theta)} = \frac{4 \sec \theta}{3 \tan \theta}$
at $P(3 \sec \theta, 4 \tan \theta)$ on H and therefore Tan_P is $y - 4 \tan \theta = \frac{4 \sec \theta}{3 \tan \theta}(x - 3 \sec \theta)$ and therefore
 $3y \tan \theta - 12 \tan^2 \theta = 4x \sec \theta - 12 \sec^2 \theta$ and therefore $4x \sec \theta - 3y \tan \theta = 12 \sec^2 \theta + 2 \tan^2 \theta = 12(\sec^2 \theta - \tan^2 \theta) = 12$ and if this tangent crosses the asymptote $y = \frac{4}{3}x$ at A ,
then for A , $y = \frac{4}{3}x$ and $4x \sec \theta - 3y \tan \theta = 12$ and therefore $4x \sec \theta - 3(\frac{4}{3}x) \tan \theta = 12$
and therefore $4x \sec \theta - 4x \tan \theta = 12$ and therefore $x(4 \sec \theta - 4 \tan \theta) = 12$ and therefore
 $x = \frac{12}{4 \sec \theta - 4 \tan \theta} = \frac{12}{4(\sec \theta - \tan \theta)} = \frac{3}{\sec \theta - \tan \theta}$ and therefore $y = \frac{4}{3}(\frac{3}{\sec \theta - \tan \theta}) = \frac{4}{\sec \theta - \tan \theta}$
and therefore A is $(\frac{3}{\sec \theta - \tan \theta}, \frac{4}{\sec \theta - \tan \theta})$. If Tan_P crosses the asymptote $y = -\frac{4}{3}x$ at B ,
then for B , $y = -\frac{4}{3}x$ and $4x \sec \theta - 3y \tan \theta = 12$ and therefore $4x \sec \theta - 3(-\frac{4}{3}x) \tan \theta = 12$
and therefore $4x \sec \theta + 4x \tan \theta = 12$. $\therefore x(4 \sec \theta + 4 \tan \theta) = 12$. $\therefore x = \frac{12}{4 \sec \theta + 4 \tan \theta} =$
 $\frac{12}{4(\sec \theta + \tan \theta)} = \frac{3}{\sec \theta + \tan \theta}$ and therefore $y = -\frac{4}{3}(\frac{3}{\sec \theta + \tan \theta}) = \frac{-4}{\sec \theta + \tan \theta}$ and therefore B is
 $(\frac{3}{\sec \theta + \tan \theta}, \frac{-4}{\sec \theta + \tan \theta})$ and therefore the midpoint of AB is
 $(\frac{\frac{3}{\sec \theta - \tan \theta} + \frac{3}{\sec \theta + \tan \theta}}{2}, \frac{\frac{4}{\sec \theta - \tan \theta} - \frac{4}{\sec \theta + \tan \theta}}{2})$
 $= (3(\frac{\frac{1}{\sec \theta - \tan \theta} + \frac{1}{\sec \theta + \tan \theta}}{2}), 4(\frac{\frac{1}{\sec \theta - \tan \theta} - \frac{1}{\sec \theta + \tan \theta}}{2}))$
 $= (3(\frac{\sec \theta + \tan \theta + \sec \theta - \tan \theta}{2(\sec^2 \theta - \tan^2 \theta)}), 4(\frac{\sec \theta + \tan \theta - (\sec \theta - \tan \theta)}{2(\sec^2 \theta - \tan^2 \theta)}))$
 $= (3(\frac{2 \sec \theta}{2(1)}), 4(\frac{2 \tan \theta}{2(1)}))$
 $= (3 \sec \theta, 4 \tan \theta)$
 $= P$

and therefore the part of the tangent at $P(3 \sec \theta, 4 \tan \theta)$ on H which is intercepted between the asymptotes is bisected at P .

(ii) Where O is the origin $(0, 0)$,

$$\begin{aligned} OA &= \sqrt{(\frac{3}{\sec \theta - \tan \theta})^2 + (\frac{4}{\sec \theta - \tan \theta})^2} \\ &= \sqrt{\frac{9}{(\sec \theta - \tan \theta)^2} + \frac{16}{(\sec \theta - \tan \theta)^2}} \\ &= \sqrt{\frac{25}{(\sec \theta - \tan \theta)^2}} \\ &= \frac{5}{\sec \theta - \tan \theta}. \end{aligned}$$

$$\begin{aligned}
OB &= \sqrt{\left(\frac{3}{\sec \theta + \tan \theta}\right)^2 + \left(\frac{4}{\sec \theta + \tan \theta}\right)^2} \\
&= \sqrt{\frac{9}{(\sec \theta + \tan \theta)^2} + \frac{16}{(\sec \theta + \tan \theta)^2}} \\
&= \sqrt{\frac{25}{(\sec \theta + \tan \theta)^2}} \\
&= \frac{5}{\sec \theta + \tan \theta}.
\end{aligned}$$

$$\angle AOB = 2\angle AOX$$

$$\begin{aligned}
&= 2 \tan^{-1}\left(\frac{4}{\sec \theta - \tan \theta} / \frac{3}{\sec \theta - \tan \theta}\right) \\
&= 2 \tan^{-1} \frac{4}{3}
\end{aligned}$$

$$\therefore \text{Area } \triangle AOB = \frac{1}{2} \cdot OB \cdot OB \cdot \sin \angle AOB$$

$$= \frac{1}{2} \cdot \frac{5}{\sec \theta - \tan \theta} \cdot \frac{5}{\sec \theta + \tan \theta} \cdot \sin\left(2 \tan^{-1} \frac{4}{3}\right)$$

$$= \frac{25}{\sec^2 \theta - \tan^2 \theta} \left(\sin \tan^{-1} \frac{4}{3} \cos \tan^{-1} \frac{4}{3}\right)$$

$$= 25 \left(\frac{4}{\sqrt{4^2+3^2}} \cdot \frac{3}{\sqrt{4^2+3^2}}\right) \text{unit}^2$$

$$= 25 \left(\frac{12}{4^2+3^2}\right) \text{unit}^2$$

$$= 25 \left(\frac{12}{25}\right) \text{unit}^2$$

= 12unit² which is a constant and therefore the tangent forms with the asymptotes of a triangle of constant area.

(iii) The focus S is $(\sqrt{9}, \sqrt{1 + \frac{16}{9}}, 0) = (3(\frac{5}{3}), 0) = (5, 0)$ and therefore

$$\begin{aligned}
\angle ASB &= 180^\circ - \tan^{-1} \frac{\left(\frac{4}{\sec \theta - \tan \theta} - 0\right) / \left(\frac{3}{\sec \theta - \tan \theta} - 5\right) - \left(\frac{-4}{\sec \theta + \tan \theta} - 0\right) / \left(\frac{3}{\sec \theta + \tan \theta} - 5\right)}{1 + \left(\frac{4}{\sec \theta - \tan \theta} - 0\right) / \left(\frac{3}{\sec \theta - \tan \theta} - 5\right) \left(\frac{-4}{\sec \theta + \tan \theta} - 0\right) / \left(\frac{3}{\sec \theta + \tan \theta} - 5\right)} \\
&= 180^\circ - \tan^{-1} \frac{\left(\frac{4}{\sec \theta - \tan \theta}\right) / \left(\frac{3-5(\sec \theta - \tan \theta)}{\sec \theta - \tan \theta}\right) - \left(\frac{-4}{\sec \theta + \tan \theta}\right) / \left(\frac{3-5(\sec \theta + \tan \theta)}{\sec \theta + \tan \theta}\right)}{1 + \left(\frac{4}{\sec \theta - \tan \theta}\right) / \left(\frac{3-5(\sec \theta - \tan \theta)}{\sec \theta - \tan \theta}\right) \left(\frac{-4}{\sec \theta + \tan \theta}\right) / \left(\frac{3-5(\sec \theta + \tan \theta)}{\sec \theta + \tan \theta}\right)} \\
&= 180^\circ - \tan^{-1} \frac{(4/(3-5(\sec \theta - \tan \theta))) + (4/(3-5(\sec \theta + \tan \theta)))}{1 - (4/(3-5(\sec \theta - \tan \theta)))(4/(3-5(\sec \theta + \tan \theta)))} \\
&= 180^\circ - \tan^{-1} \frac{(4(3-5(\sec \theta + \tan \theta)) + 4(3-5(\sec \theta - \tan \theta))) / ((3-5(\sec \theta - \tan \theta))(3-5(\sec \theta + \tan \theta)))}{((3-5(\sec \theta - \tan \theta))(3-5(\sec \theta + \tan \theta)) - 4(4)) / ((3-5(\sec \theta - \tan \theta))(3-5(\sec \theta + \tan \theta)))} \\
&= 180^\circ - \tan^{-1} \frac{(4(3-5(\sec \theta + \tan \theta)) + 4(3-5(\sec \theta - \tan \theta)))}{((3-5(\sec \theta - \tan \theta))(3-5(\sec \theta + \tan \theta)) - 4(4))} \\
&= 180^\circ - \tan^{-1} \frac{12 - 20 \sec \theta - 20 \tan \theta + 12 - 20 \sec \theta + 20 \tan \theta}{9 - 15(\sec \theta + \tan \theta) - 15(\sec \theta - \tan \theta) + 25(\sec^2 \theta - \tan^2 \theta) - 16} \\
&= 180^\circ - \tan^{-1} \frac{24 - 20 \sec \theta}{25(1) - 7 - 15 \sec \theta - 15 \tan \theta - 15 \sec \theta + 15 \tan \theta} \\
&= 180^\circ - \tan^{-1} \frac{4(6 - 10 \sec \theta)}{18 - 30 \sec \theta} \\
&= 180^\circ - \tan^{-1} \frac{4(6 - 10 \sec \theta)}{3(6 - 10 \sec \theta)} \\
&= 180^\circ - \tan^{-1} \frac{4}{3} \\
&= 180^\circ - 53^\circ 8'
\end{aligned}$$

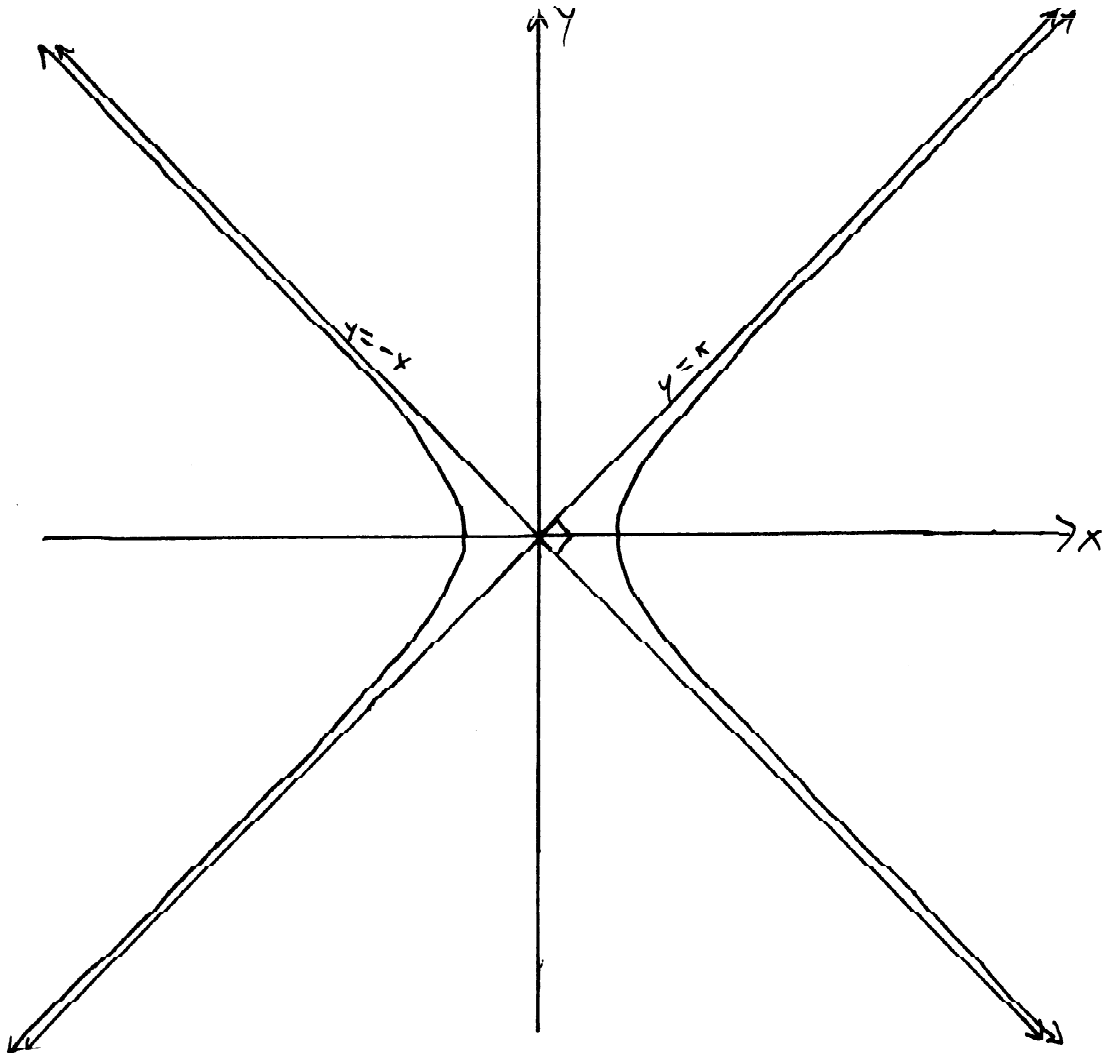
= 126°52' which is a constant and therefore the tangent intercepted between the asymptotes subtends a constant angle at a focus (53°8' at one focus and 126°52' at the other).



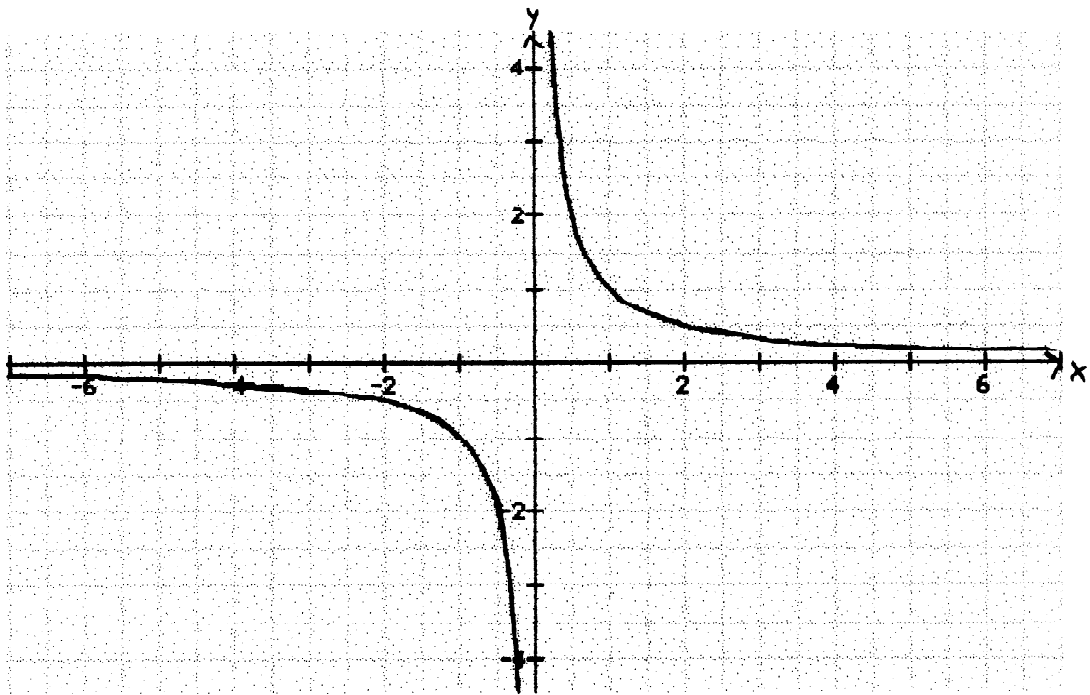
Lecture 26

Rectangular Hyperbola - asymptotes are at 90° to each other.

$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ or $x^2 - y^2 = a^2$ i.e., asymptotes: $y = \pm x$, eccentricity: $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{2}$.
For



swiveled around the origin, we get $xy = c^2$ of parametric equations $x = cp, y = \frac{c}{p}$



Tangents and normals at $P(cp, \frac{c}{p})$:

$$xy = c^2$$

$$y = c^2 x^{-1}$$

$$\therefore \frac{dy}{dx} = -c^2 x^{-2} = -\frac{c^2}{x^2} = -\frac{c^2}{(cp)^2} (a + p) = -\frac{1}{p^2}.$$

$$\text{Tan}_P: y - \frac{c}{p} = -\frac{1}{p}(x - cp)$$

$$x + p^2 y = cp + cp$$

$$\therefore x + p^2 y = 2cp$$

$$\text{Norm}_P: y - \frac{c}{p} = p^2(x - cp)$$

$$p^3 x - py = p^4 c - c = c(p^4 - 1).$$

