

The Optimal Debt Structure using Stochastic Interest Rates

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Abstract

I extend the Leland and Toft paper from constant interest rate assumption to stochastic interest rate assumption by embedding the Longstaff and Schwartz model. In this paper, the following topics have been discussed: 1) optimal debt structure in stochastic and constant interest rate assumptions, 2) bond price behaviors in both interest rate assumptions, 3) default probability in both interest rate assumptions, 4) sensitivity analysis of stochastic interest rate parameters on optimal debt structures, and 5) default asset values in different $D(t)$ functions.

Introduction

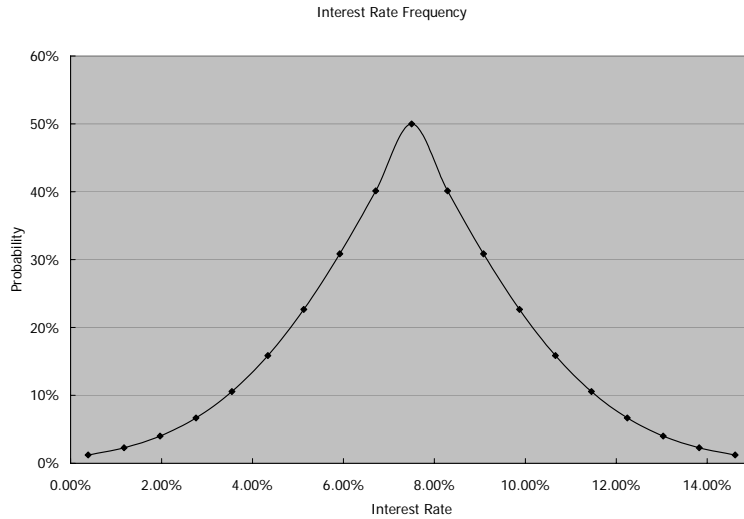
Since the Merton's framework (1974), structural models have become one of the most popular ways to examine the credit risk of a company. Opposed to reduced form models, structural models evaluate the credit risk from the economic viewpoint of a company. Normally in a structural model, the asset value is assumed under a stochastic process with a pre-set default threshold. The default probability can therefore be calculated.

After Merton, Geske (1977), Ingersoll (1976), Leland (1994), Longstaff and Schwartz (1995), and Leland and Toft (1996) improved the structural models on credit risk modeling. The objective of this paper is to extend the Leland and Toft model. Leland and Toft assume the interest rate to be constant, calculate the value of the finite life debt, and then analyze the behaviors of a firm. The interest rate is the drift term in the asset value stochastic process and also the discount factor of cash flows. To extend this model, I use the Longstaff and Schwartz model to calculate the finite life debt with stochastic interest rate assumption, and apply it to Leland and Toft's framework.

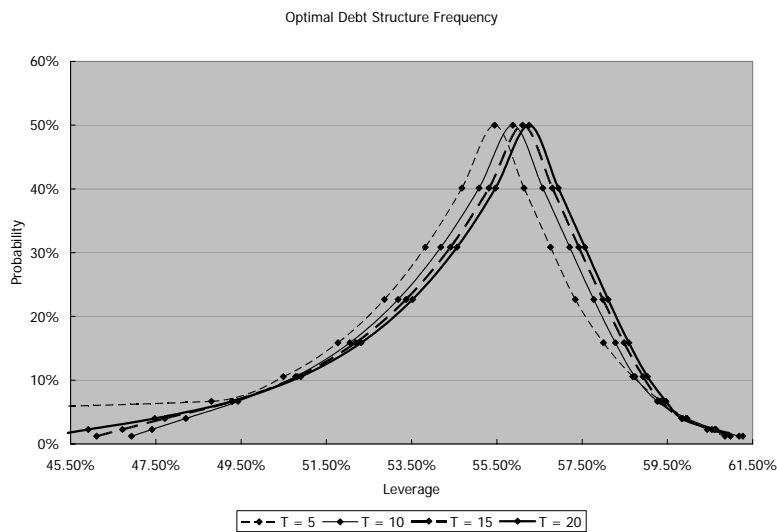
The Objective of This Paper

The objective of this paper is to tell the impact of stochastic interest rate to the corporate behaviors. But how sensitive would the interest rate affect the corporate

behaviors; i.e., does the interest rate really matter? Here is the way I interpret this issue. Leland and Toft assume that the expected interest rate is constant in their model. In other words, the interest rate follows a geometric Brownian motion with no drift. By using the parameters setup in Leland and Toft's paper and moreover assuming the volatility of interest rate process is 3.16%, the probability distribution function of interest rate is as follows:



It is anticipatable that the distribution of the interest rate would be symmetric. However, the frequencies of corporate behaviors would not behave in a symmetric distribution way because they are on conditional of surviving at any given of time. The probability distributions of optimal debt structure from the Leland and Toft model on $T = 5$, $T = 10$, $T = 15$ and $T = 20$ are plotted as follows:



Interestingly, the optimal debt structure tends to skew left when T grows. And also, expectedly, the default asset value would skew right in line with the diagram above. Therefore, the interest rate affects not only the expected corporate behaviors, but also the probability distribution function of the corporate behaviors. In summary, the interest rate would affect the corporate behaviors on higher moment conditions. In this paper, however, I am going to discuss the impact of the stochastic interest rate to the optimal debt structure on the first order condition only.

Overview of the Leland and Toft Model

The research methodology in Leland and Toft's paper is shown in the following steps:

Step 1. Get the close-form solution of single debt value from Harrison (1990), Rubinstein and Reiner (1991):

$$d(V; V_B, t) = \frac{c(t)}{r} + e^{-rt} \left[p(t) - \frac{c(t)}{r} \right] [1 - F(t)] + \left[\rho(t)V_B - \frac{c(t)}{r} \right] G(t)$$

$$F(t) = N[h_1(t)] + \left(\frac{V}{V_B} \right)^{-2a} N[h_2(t)] \quad G(t) = \left(\frac{V}{V_B} \right)^{-a+z} N[q_1(t)] + \left(\frac{V}{V_B} \right)^{-a-z} N[q_2(t)]$$

$$h_1(t) = \frac{(-b - a\sigma^2 t)}{\sigma\sqrt{t}}; h_2(t) = \frac{(-b + a\sigma^2 t)}{\sigma\sqrt{t}} \quad q_1(t) = \frac{(-b - z\sigma^2 t)}{\sigma\sqrt{t}}; q_2(t) = \frac{(-b + z\sigma^2 t)}{\sigma\sqrt{t}}$$

$$a = \frac{\left(r - \delta - \frac{\sigma^2}{2} \right)}{\sigma^2}; b = \ln\left(\frac{V}{V_B} \right) \quad z = \frac{\sqrt{(a\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}$$

Step 2. Extend Leland (1994)'s result of firm value:

$$v(V; V_B) = V + \frac{\tau C}{r} \left[1 - \left(\frac{V}{V_B} \right)^{-x} \right] - \alpha V_B \left(\frac{V}{V_B} \right)^{-x}$$

$$x = a + z$$

Step 3. Assume a rolling over mechanism on the corporate debt. Therefore the default asset value can be independent of time. The close-form solution of total value of debt can be found by taking the integral of decomposed single debts:

$$D(V; V_B, T) = \int_{t=0}^T d(V; V_B, t) dt$$

$$= \frac{C}{r} + \left(P - \frac{C}{r} \right) \left(\frac{1 - e^{-rT}}{rT} - I(T) \right) + \left((1 - \alpha)V_B - \frac{C}{r} \right) J(T)$$

$$\begin{aligned}
I(T) &= \frac{1}{T} \int_0^T e^{-rt} F(t) dt \\
&= \frac{1}{rT} (G(T) - e^{-rT} F(T)) \\
J(T) &= \frac{1}{T} \int_0^T G(t) dt \\
&= \frac{1}{z\sigma\sqrt{T}} \left(-\left(\frac{V}{V_B}\right)^{-a+z} N[q_1(T)]q_1(T) + \left(\frac{V}{V_B}\right)^{-a-z} N[q_2(T)]q_2(T) \right)
\end{aligned}$$

Step 4. The value of equity can therefore be calculated by subtracting total value of debt from firm value.

Step 5. Leland and Toft use smooth-pasting process constraint to solve the default asset value, and endogenously get a close-form solution.

Smooth-pasting process:

$$\frac{\partial v}{\partial V} \Big|_{V=V_B} = \frac{\partial D}{\partial V} \Big|_{V=V_B} \quad \frac{\partial E}{\partial V} \Big|_{V=V_B} = 0$$

Close-form solution of endogenous default point:

$$V_B = \frac{\frac{C}{r} \left(\frac{A}{rT} - B \right) - \frac{AP}{rT} - \tau \frac{Cx}{r}}{1 + \alpha x - (1 - \alpha)B}$$

$$x = a + z$$

$$A = 2ae^{-rT} N(a\sigma\sqrt{T}) - 2zN(z\sigma\sqrt{T}) - \frac{2}{\sigma\sqrt{T}} n(z\sigma\sqrt{T}) + \frac{2e^{-rT}}{\sigma\sqrt{T}} n(a\sigma\sqrt{T}) + (z - a)$$

$$B = -\left(2z + \frac{2}{z\sigma^2 T} \right) N(z\sigma\sqrt{T}) - \frac{2}{\sigma\sqrt{T}} n(z\sigma\sqrt{T}) + (z - a) + \frac{1}{z\sigma^2 T}$$

Step 6. To generalize the case, assume that the single debt is issued at par. Analyze the behavior of the company given any change in its leverage ratio, credit spread and so forth.

Apply Stochastic Interest Rate to the Leland and Toft Model

The interest rate is assumed to be constant in Leland and Toft's model. To fulfill the two purposes of interest rate in the structural model, I make the following assumptions to calibrate the impact of stochastic interest rate:

1. The dynamic of short-term riskless interest rate follows the Vasicek model (1977) assumptions: $dr = (\alpha - \beta r)dt + \eta dW$. Although Vasicek model is not empirically accurate due to the negative interest rate, its close-form solution of D(t) function makes it easy to be implemented:

$$D(t) = A(t)\exp(-r_0 B(t))$$

$$A(t) = \exp\left[\left(\frac{\eta^2}{2\beta^2} - \frac{\alpha}{\beta}\right)t + \left(\frac{\alpha}{\beta^2} - \frac{\eta^2}{\beta^3}\right)(1 - e^{-\beta t}) + \frac{\eta^2}{4\beta^3}(1 - e^{-2\beta t})\right]$$

$$B(t) = \frac{1}{\beta}(1 - e^{-\beta t})$$

Therefore, I replace the close-form solution of firm value from Leland's paper (1994) with the following to get the firm value:

$$v(V; V_B) = V + \tau C \left[\sum_{t=0}^{\infty} D(t) \left[1 - \left(\frac{V}{V_B} \right)^{-x} \right] - \alpha V_B \left(\frac{V}{V_B} \right)^{-x} \right]; D(t) \text{ is given by Vasicek model.}$$

$$x = a + z$$

2. Again I install the Vasicek model into the value of single debt. Longstaff and Schwartz estimate the value of fixed coupon debts under Vasicek's stochastic interest rate assumption. Instead of using the close-form solution of single debt from the Leland and Toft model, thus, I use the Longstaff and Schwartz model to compute the value of single debt and fit it into Leland and Toft's framework. The value of a zero coupon debt in the Longstaff and Schwartz model is shown as follows:

$$P(X, r, T) = D(r, t) - \omega D(r, T) Q(X, r, T)$$

$$Q(X, r, T, n) = \sum_{i=1}^n q_i,$$

$$q_1 = N(a_1), q_i = N(a_i) - \sum_{j=1}^{i-1} q_j N(b_{ij}), i = 2, 3, 4, \dots, n;$$

$$a_i = \frac{-\ln X - M\left(\frac{iT}{n}, T\right)}{\sqrt{S\left(\frac{iT}{n}\right)}}, b_{ij} = \frac{M\left(\frac{jT}{n}, T\right) - M\left(\frac{iT}{n}, T\right)}{\sqrt{S\left(\frac{iT}{n}\right) - S\left(\frac{jT}{n}\right)}};$$

$$M(t, T) = \left(\frac{\alpha - \rho\sigma\eta}{\beta} - \frac{\eta^2}{\beta^2} - \frac{\sigma^2}{2} \right) t + \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{\eta^2}{2\beta^3} \right) \exp(-\beta T) (\exp(\beta t) - 1) +$$

$$\left(\frac{r}{\beta} - \frac{\alpha}{\beta^2} + \frac{\eta^2}{\beta^3} \right) (1 - \exp(-\beta t)) - \left(\frac{\eta^2}{2\beta^3} \right) \exp(-\beta t) (1 - \exp(-\beta t));$$

$$S(t) = \left(\frac{\rho\sigma\eta}{\beta} + \frac{\eta^2}{\beta^2} + \sigma^2 \right) t - \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{2\eta^2}{\beta^3} \right) (1 - \exp(-\beta t)) + \left(\frac{\eta^2}{2\beta^3} \right) (1 - \exp(-2\beta t))$$

, where X is asset to default asset value ratio, r is short rate, and T means time to maturity

The $P(X, r, T)$ function above denotes the value of a zero coupon bond given the default asset value ratio, short rate, and time to maturity. Therefore the value of single debt is

$$d = \int_{t=0}^T c(t)P(t)dt + P(t) \times \text{Par value of debt}, \text{ where } c(t) \text{ is the coupon at time } t.$$

Unfortunately, Longstaff and Schwartz do not incorporate dividends into their model. This issue limits the application of this research to firms with no dividend payout.

In addition, the endogenous default asset value in the stochastic interest rate assumption cannot be found in this framework. Therefore, the close-form solution of total debt value,

$$D(V; V_B, T) = \int_{t=0}^T d_t(V; V_B, T) dt^\dagger, \text{ cannot be solved, and I cannot get the endogenous default asset value via the implication of smooth-pasting process in this framework.}$$

Framework Setup

Notations and Base Case Assumptions

I summarize the notations and the base case assumptions in the following table. The notations are basically following those from the Leland and Toft, and the Longstaff and Schwartz models.

| Notations | Descriptions | Base Case Assumptions |
|-----------|--|--|
| V | Asset value | $V_0 = 100$, or $V = 100$ |
| σ | Asset value volatility | $\sigma = 20\%$ |
| r | Risk-free rate, short rate | $r = 7.5\%$ |
| C | Debt coupon | |
| P | Debt principal | |
| D | Market value of debt | |
| E | Market value of equity | |
| v | Levered firm value | |
| τ | Corporate tax rate | $\tau = 35\%$ |
| δ | Dividend payout ratio <i>This parameter is not implemented in Longstaff and Schwartz model.</i> | $\delta = 0$ |
| T | Bond maturity | |
| V_B | Default asset value | Use the V_B generated by equation 13 from Leland and |

[†] Here the d_t is the value of a fraction of single debt. Please refer to section II in Leland and Toft's paper.

| Notations | Descriptions | Base Case Assumptions |
|-----------------|--|-------------------------------------|
| | | Toft's paper as the base assumption |
| ω | Default cost fraction <i>In Longstaff and Schwartz model, the default cost fraction was denoted ω while in the Leland and Toft model, it is denoted as α. To avoid the confusion from the drift term of stochastic interest rate, I use ω in this research.</i> | $\omega = 30\%$ |
| α, β | Elements of drift term in stochastic interest rate: $dr = (\alpha - \beta r)dt + \eta dW$ | $\alpha = 0.06, \beta = 1$ |
| η | Volatility of the stochastic interest rate: $dr = (\alpha - \beta r)dt + \eta dW$ | $\eta^2 = 0.001$ |
| ρ | Instantaneous correlation between the interest rate and asset value processes. | $\rho = -0.25$ |
| X | The ratio of V to V_B | |

Because the default asset value under the stochastic interest rate assumption is still unsolved in this state, I use the default asset value generated from the Leland and Toft model as the base case. Moreover, because this framework is an extension of the Leland and Toft model by using the Longstaff and Schwartz model, the assumptions of both models must be complied in this framework.

The Extent of Discrete Form

Since I am not able to get the analytical solution of the integral of $c(t) \times P(X, r, t)$ with respect to t, I solve it numerically in a discrete form. To illustrate the impact of discrete form on the single and total debt value, I make the following comparison between the result of integrals in the continuous form and that in the discrete one.

| $V = 100, \sigma = 20\%, r = 7.5\%, \tau = 35\%, \delta = 0, C = 4, P = 51.85, T = 10, V_B = 45.64$ | | |
|---|--------------------|------------|
| $\alpha = 0, \beta = 0, \eta = 0, \rho = 0$ | | |
| | Single Debt | Total Debt |
| Single debt value from the Leland and Toft's model | 51.85 [‡] | 51.93 |
| Single and total debt value from integrating the zero coupon | | |

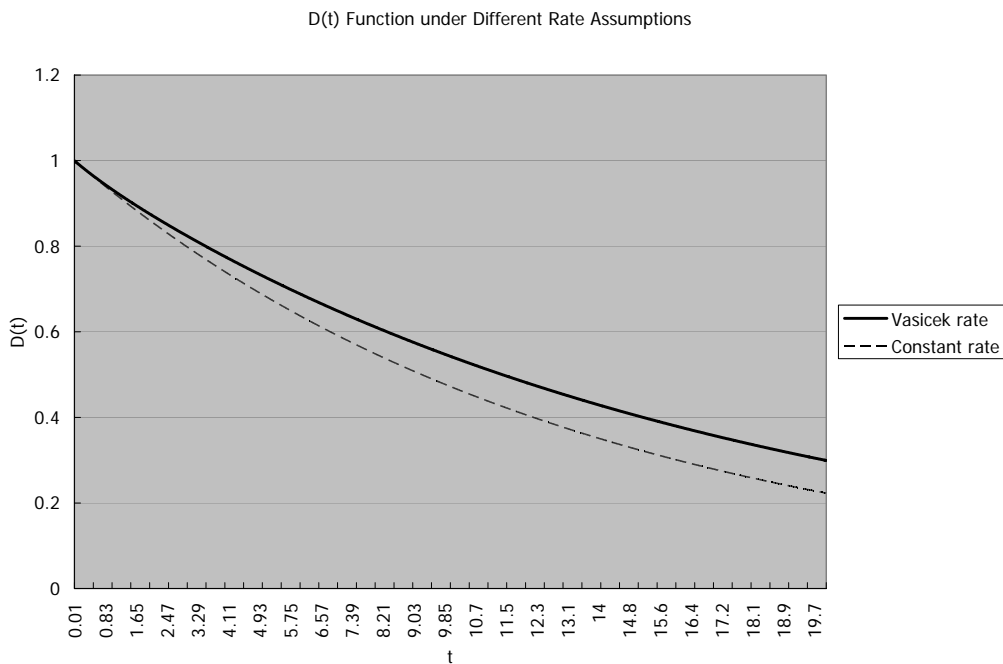
[‡] Apparently, the single debt is assumed to be a par bond.

| | | |
|---|-------------|------------|
| $V = 100, \sigma = 20\%, r = 7.5\%, \tau = 35\%, \delta = 0, C = 4, P = 51.85, T = 10, V_B = 45.64$ | | |
| $\alpha = 0, \beta = 0, \eta = 0, \rho = 0$ | Single Debt | Total Debt |
| bond value from Longstaff and Schwartz model multiplied the future cash flows\ | | |
| Weekly dt | 51.95 | 51.97 |
| Monthly dt | 52.02 | 52.00 |

The table above inferred that it does not make much difference on the single and total debt values when I choose a continuous integral (Leland and Toft's model), one week or one month discrete form. In the research thereafter, I am going to use the monthly discrete form to solve either the single or total debt value.

Base Case Analysis

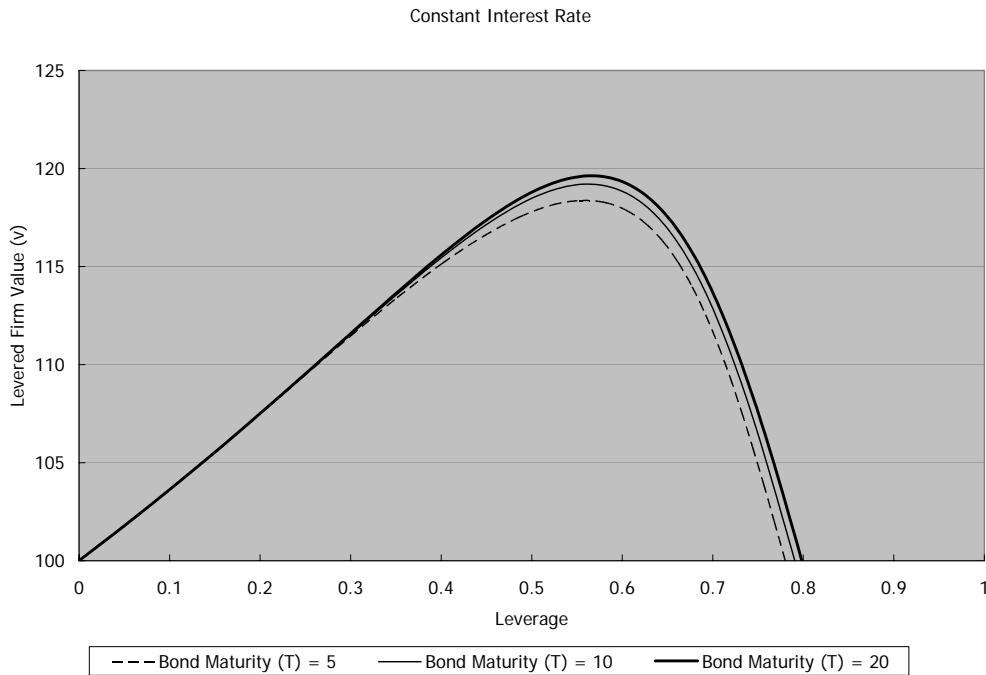
Under the base case assumptions, the $D(t)$ function under the Vasicek model is shown in the following chart:



The dynamic of the Vasicek model is $dr = (\alpha - \beta r)dt + \eta dW$. When $\alpha = 1$, $\beta = 0.06$ and $\eta^2 = 0.001$, $D(t)$ function generated by Vasicek model will be greater than that from a constant interest rate assumption when t is getting large. In the following sections, I am going to analyze the impact of the stochastic interest rate assumption to optimal debt structures, bond price behaviors, and default probabilities.

Optimal Debt Structure

By using Leland and Toft's paper, the levered firm value to the leverage ratio chart of $T = 5$, $T = 10$, $T = 20$ is plotted as below.



$$V = 100, \sigma = 20\%, r = 7.5\%, \tau = 35\%, \omega = 30\%, \delta = 0, V_B \text{ is determined endogenously}$$

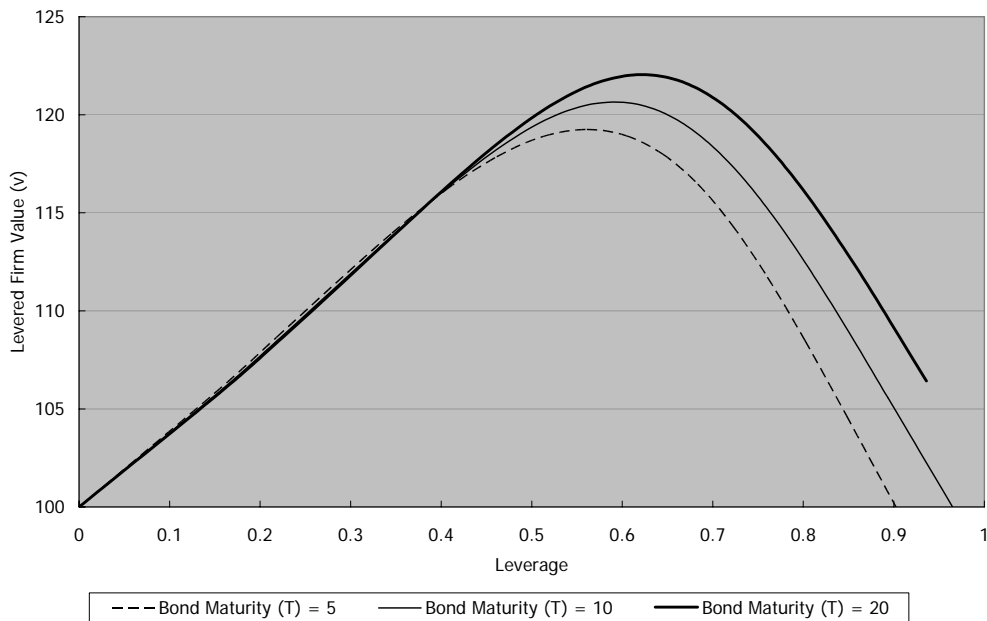
Inferred by Leland and Toft's paper, the optimal debt structure appears at highest levered firm value level. The optimal debt structures of the three bond maturities in Leland and Toft's paper are summarized below:

| Maturity | Default asset value | Levered firm value | Leverage ratio |
|----------|---------------------|--------------------|----------------|
| 5 years | 61.85 | 118.37 | 55.28% |
| 10 years | 60.07 | 119.20 | 55.89% |
| 20 years | 59.09 | 119.63 | 56.57% |

With the increase of bond maturity, the default asset value will decrease, optimal levered firm value would increase and the optimal debt structure would increase.

Considering the stochastic interest rate, however, the optimal debt structure behaves in a different way. By using the based case assumption, the levered firm value to the leverage ratio chart of $T = 5$, $T = 10$, $T = 20$ is plotted as below.

Stochastic Interest Rate



Base case assumption: $V = 100, \sigma = 20\%, r = 7.5\%, \tau = 35\%, \omega = 30\%, \delta = 0, \alpha = 0.06, \beta = 1, \eta^2 = 0.001, \rho = -0.25$, and V_B is given by the Leland and Toft model.

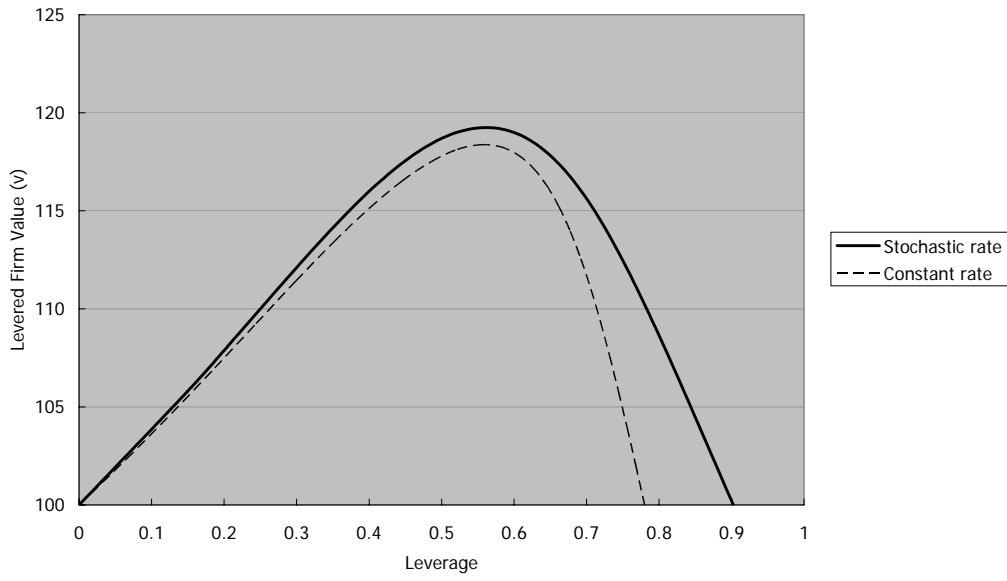
Again, the optimal debt structure is defined as the leverage ratio where the maximum levered firm value appears. The optimal debt structures in the three bond maturities under the stochastic interest rate assumption are summarized below:

| Maturity | Default asset value | Levered firm value | Leverage ratio |
|----------|---------------------|--------------------|----------------|
| 5 years | 60.73 | 119.24 | 55.50% |
| 10 years | 57.64 | 120.55 | 58.75% |
| 20 years | 60.03 | 121.98 | 62.25% |

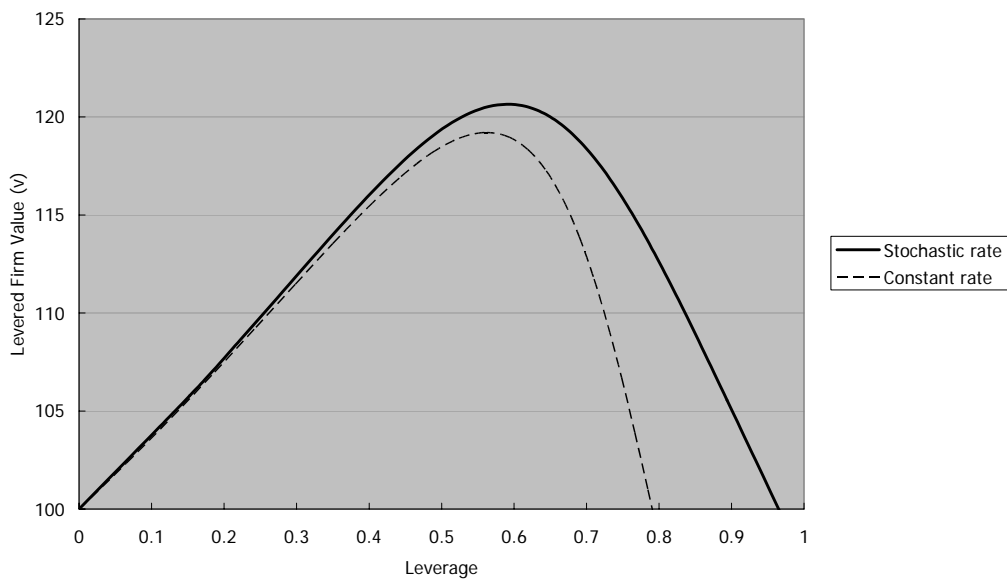
The shape of the curve in the stochastic interest rate does not dramatically change; when the maturity of bonds increases, the levered firm value increases, and the optimal leverage ratio increases as well.

Cross-sectionally, on the other hand, the stochastic interest rate under the base case assumptions intensifies the optimal debt structure. And the greater the bond maturity, the more growth it generates to the optimal debt structure.

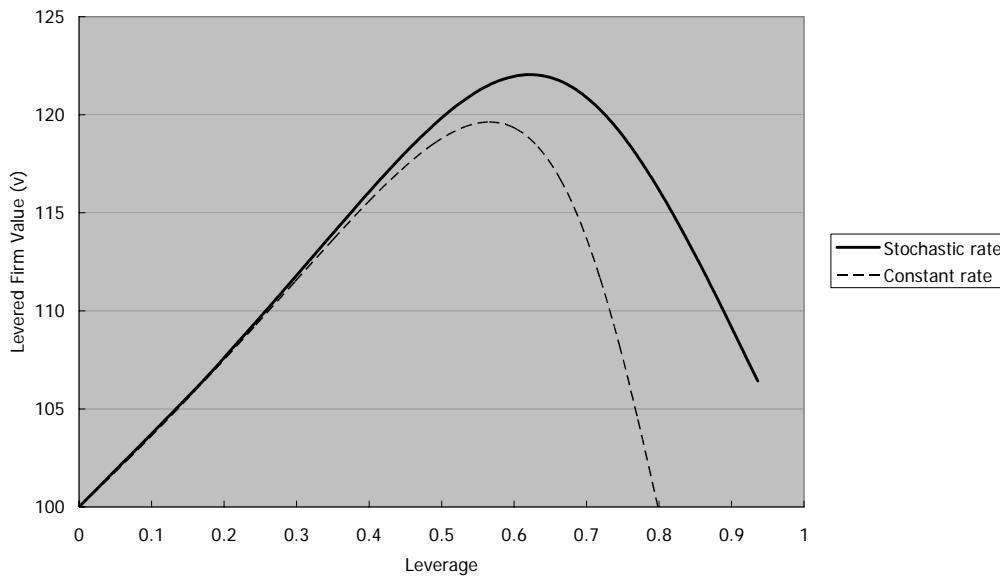
Cross-sectional Analysis, T=5



Cross-sectional Analysis, T=10



Cross-sectional Analysis, T=20



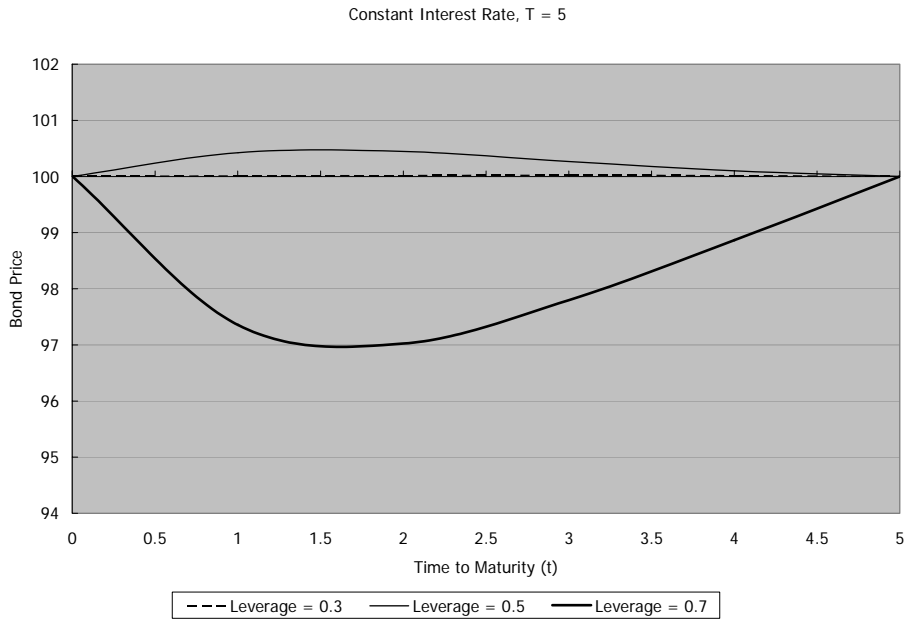
In the previous three cases, the firm value to leverage ratio curves shift up and right when adapting the stochastic interest rate into Leland and Toft’s framework. And interestingly, with the increase in the time to maturity, the difference on the two curves turns out to be more significant.

| Maturity | Increase in optimal levered firm value | Increase in optimal leverage ratio |
|----------|--|------------------------------------|
| 5 years | 0.87 | 0.22% |
| 10 years | 1.35 | 2.86% |
| 20 years | 2.35 | 5.68% |

In the 5 years maturity scenario, the optimal debt structure increases by 0.22% while the optimal levered firm value increases by 0.87. However, in the 10 years and 20 years maturity scenario, the increase in optimal debt structure increase by 2.86% and 5.68% respectively; while the optimal levered firm value increased by 1.35 and 2.35 respectively.

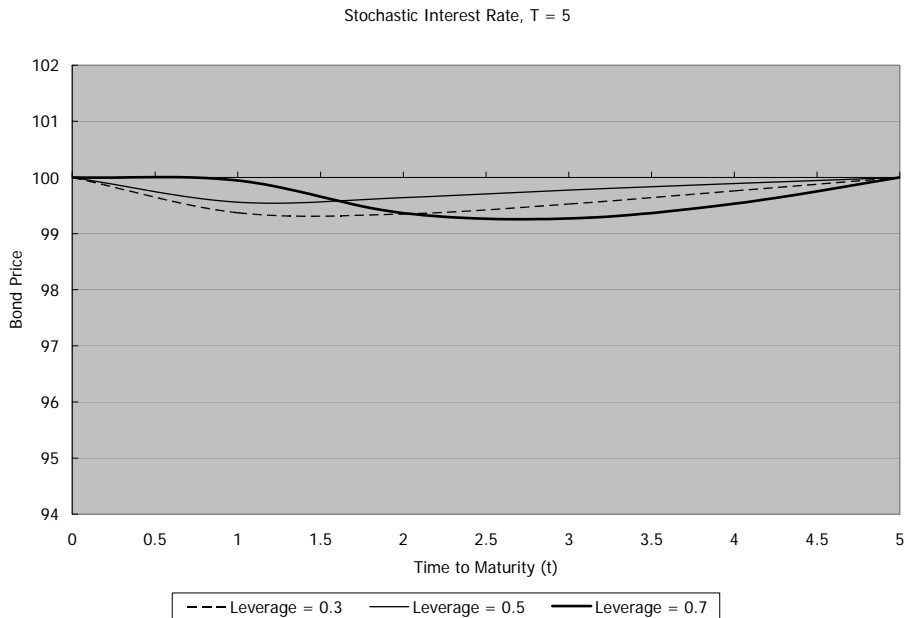
Bond Price Movement and Durations

The bond price movements over its life behave more complex. In the Leland and Toft model, the price of a 5-year par bond changes over its life. It is the result of the change in default probability over time with respect to the decrease in life.



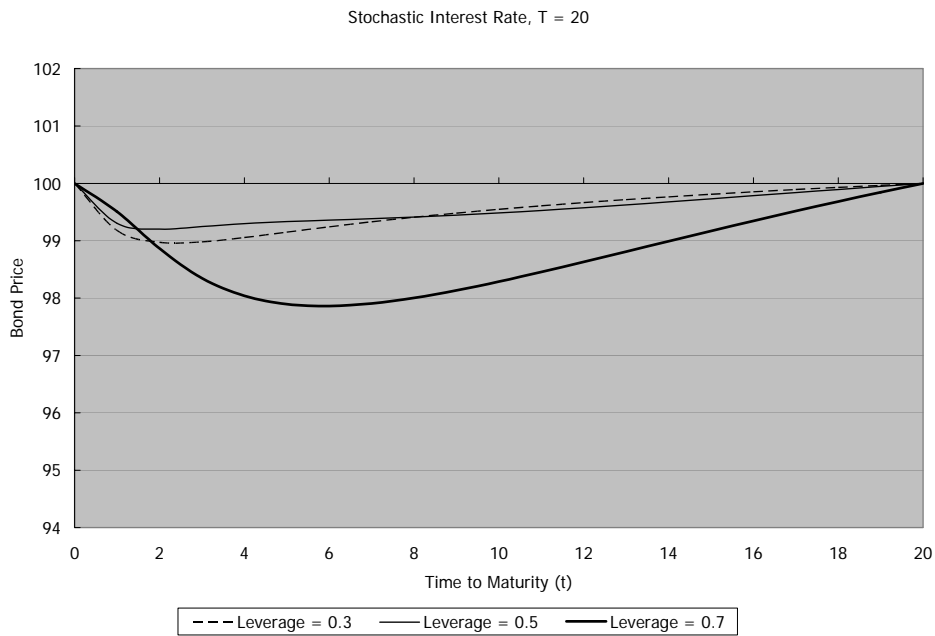
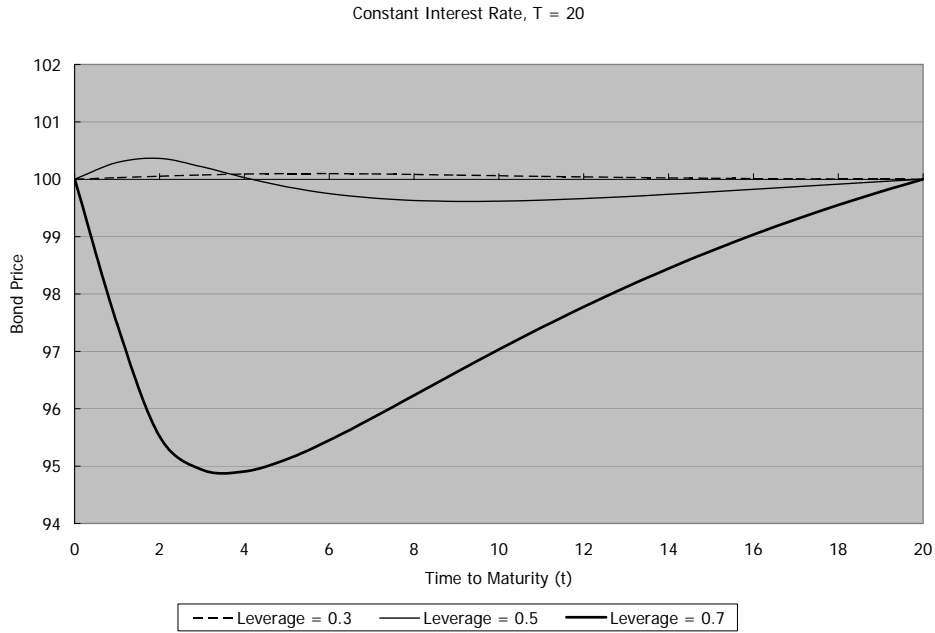
$V = 100, \sigma = 20\%, r = 7.5\%, \tau = 35\%, \omega = 30\%, \delta = 0, V_B$ is determined endogenously, and Par = 100.

On account of the change in default probability, it is intuitive that the par bond price will not stay constant even in the stochastic interest rate environment. However, in the base assumption, the bond price movement of a 5-year par bond over its life behaves differently than the constant interest rate assumption:



Base case assumption: $V = 100, \sigma = 20\%, r = 7.5\%, \tau = 35\%, \omega = 30\%, \delta = 0, \alpha = 0.06, \beta = 1, \eta^2 = 0.001, \rho = -0.25$, and V_B is given by the Leland and Toft model, and Par = 100.

Similar things happen to the 20-year par bond:



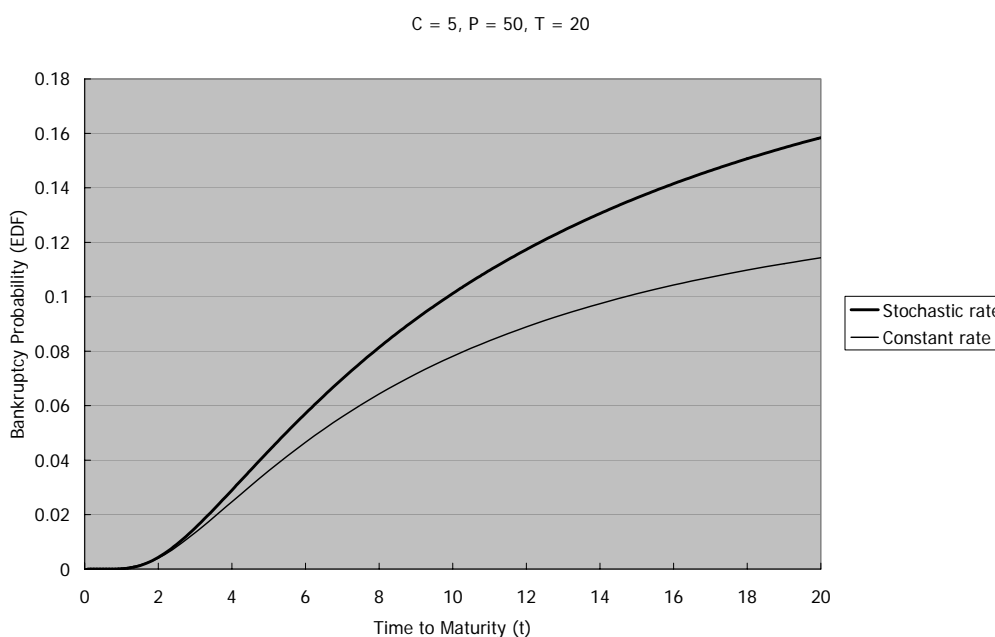
By comparing the bond price movements under constant and stochastic interest rate assumptions, the volatility of the bond price movement over the life of the bond in higher

leverage level in the stochastic interest rate assumption is smaller than that in the constant interest rate assumption. And in the stochastic interest rate assumption, the bond price tends not to gain a premium over the life of the bond.

With a view to the fact that the par bond price would change even if the interest rate level remains the same over its life, it can be anticipated that the effective duration will not equal to Macaulay duration[§]. This issue usually fails the delta-neutral strategies since the bond investors may under-hedge their position by considering Macaulay duration only.

Default Probability

In the Leland and Toft model, the default probability of a corporate bond can be calculated over time by calculating the *F function* in their paper. And the default probability in the stochastic interest rate is equivalent to the default probability from Longstaff and Schwartz model, which is defined as *Q function* in their paper. Under risk neutral measure^{**}, the comparison of default probability of a synthetic bond on constant and stochastic interest rate is illustrated below:



Base case assumption: $V = 100, \sigma = 20\%, r = 7.5\%, \tau = 35\%, \omega = 30\%, \delta = 0, \alpha = 0.06, \beta = 1,$

$\eta^2 = 0.001, \rho = -0.25,$ and V_B is given by the Leland and Toft model.

[§] Refer to Leland and Toft's paper section III.

^{**} Here on the base case assumption, the short rate under risk neutral probability is 7.5%.

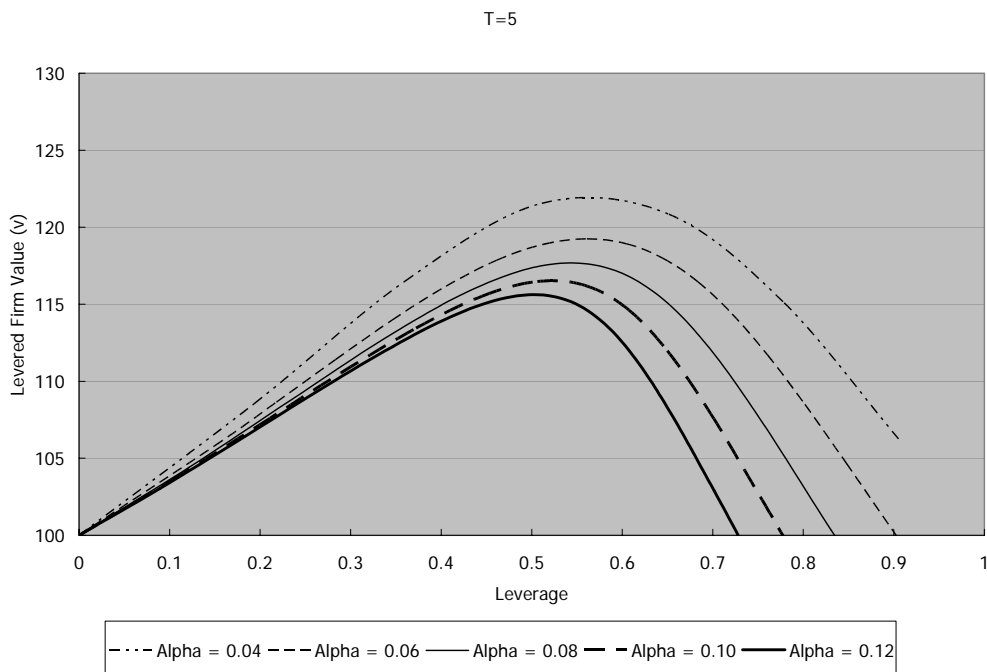
The default probability under the stochastic interest rate assumption is higher than the default probability under the constant interest rate assumption, and the greater the time to maturity, the bigger difference they have. Nonetheless, the deficit of the default probability may result from the assumption of default asset value. I will discuss this issue in the later paragraph.

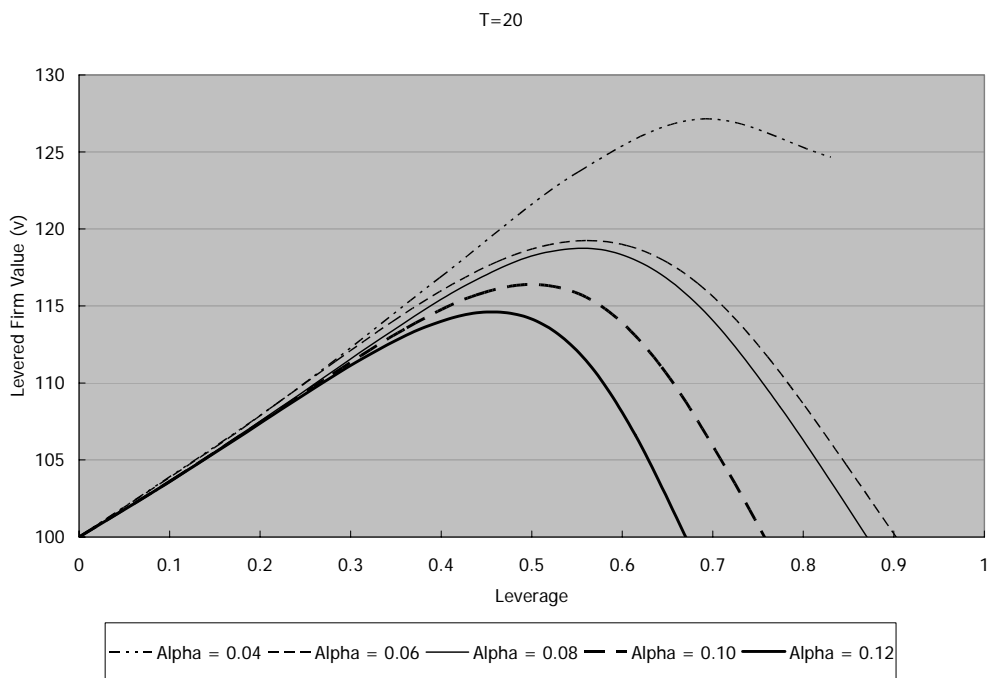
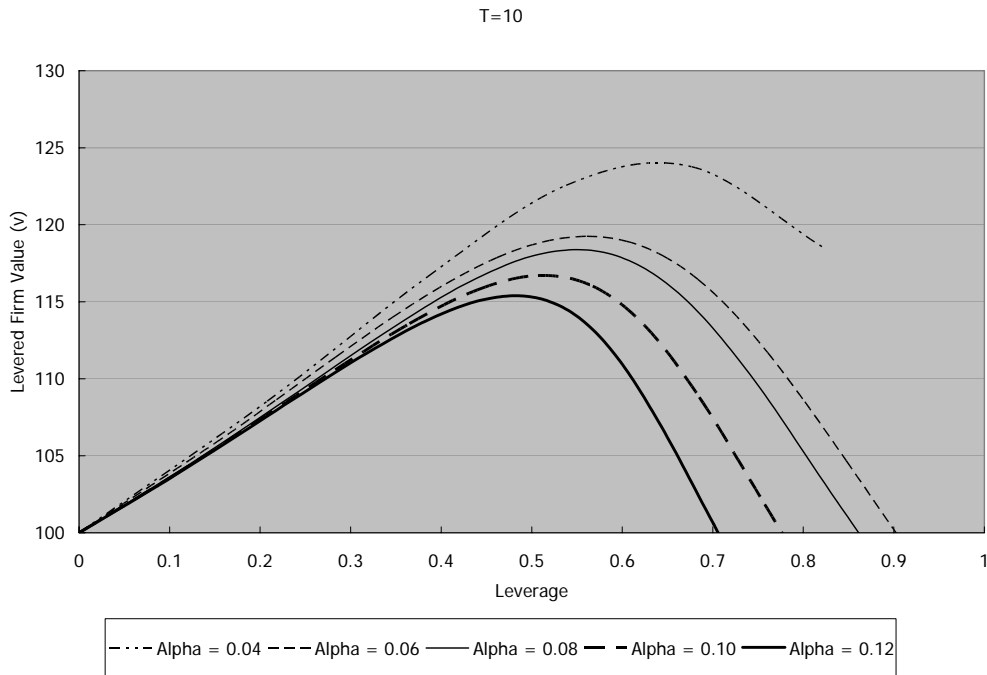
Sensitivity of Stochastic Interest Rate Parameters

In this section, I am going to test the sensitivity of the stochastic interest rate parameters on the optimal debt structure. In the Longstaff and Schwartz model, the stochastic interest rate parameters include α (alpha), β (beta), η (eta) and ρ (rho). Alpha, beta and eta are the elements of the interest rate process and rho is the correlation coefficient of asset and interest rate process.

Alpha

I simulate the alpha parameter in the stochastic interest rate process from alpha = 0.04 to alpha = 0.12, assuming that other parameters remain the same. The optimal debt structures under bond maturity of 5 years, 10 years and 20 years are plotted as follows:



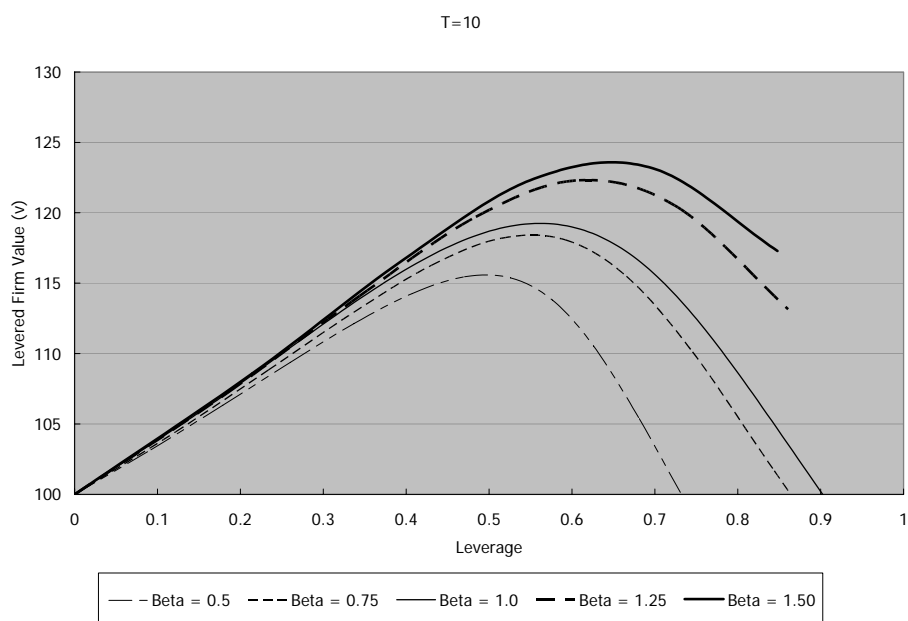
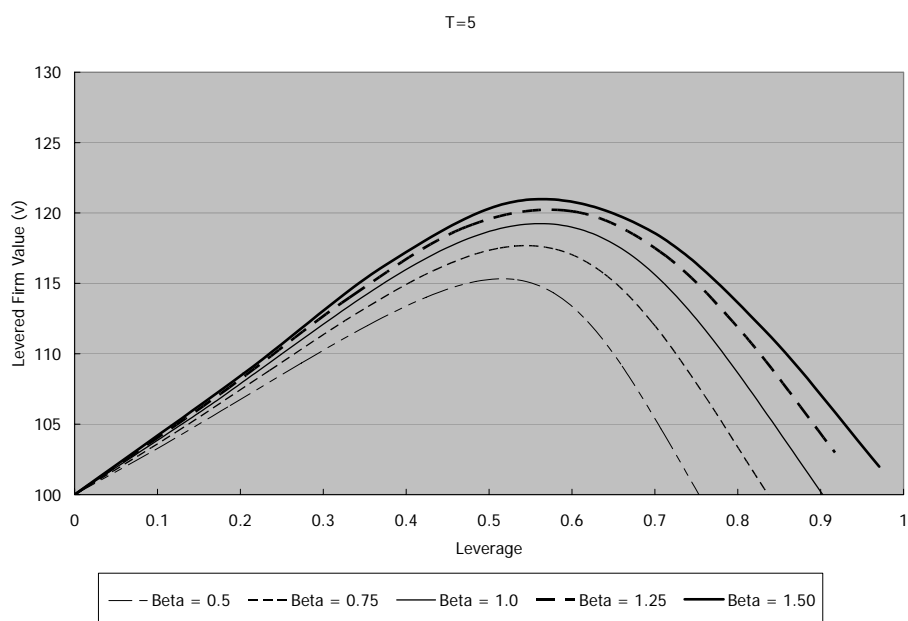


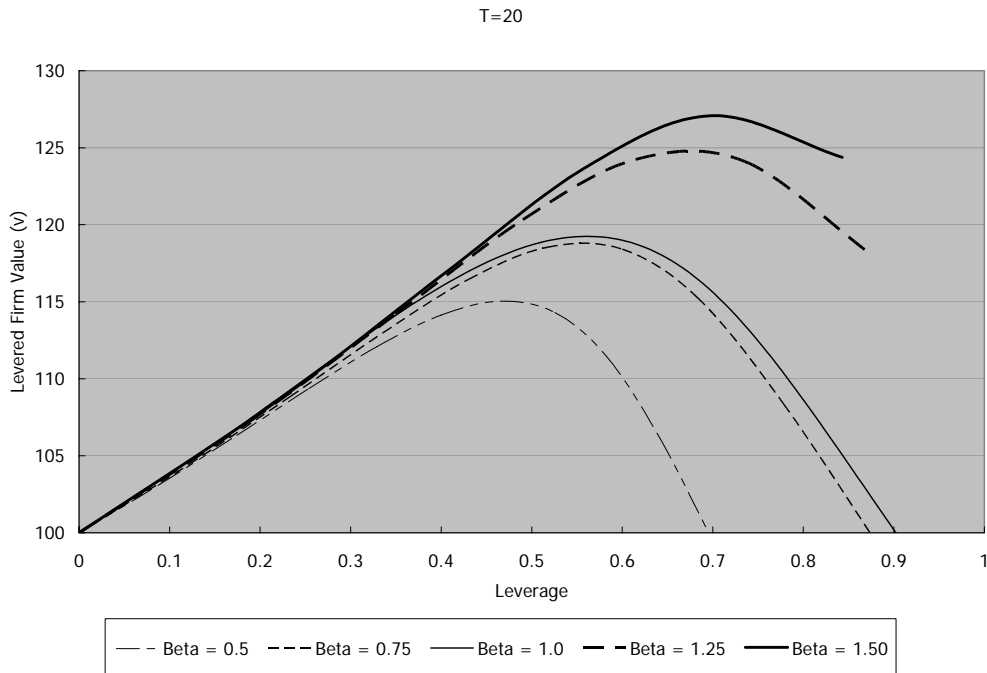
Apparently, the alpha represents a negative correlation to the optimal firm value and debt structure. And similar to the result in the cross-sectional comparison before, the greater bond maturity is, the more significant this effect is. For $T = 5$, the optimal debt structure

decreases from 56.5% to 48.0% when alpha increase from 0.04 to 0.12. Similarly, for $T = 10$, the figures decrease from 64.0% to 46.5% and for $T = 20$, the figures decrease from 71.5% to 44.3%.

Beta

Assuming the based case assumption except beta, I simulate the optimal debt structure with assuming beta from beta = 0.5 to 1.5. The optimal debt structure charts under different beta assumptions and bond maturities are illustrated as below:



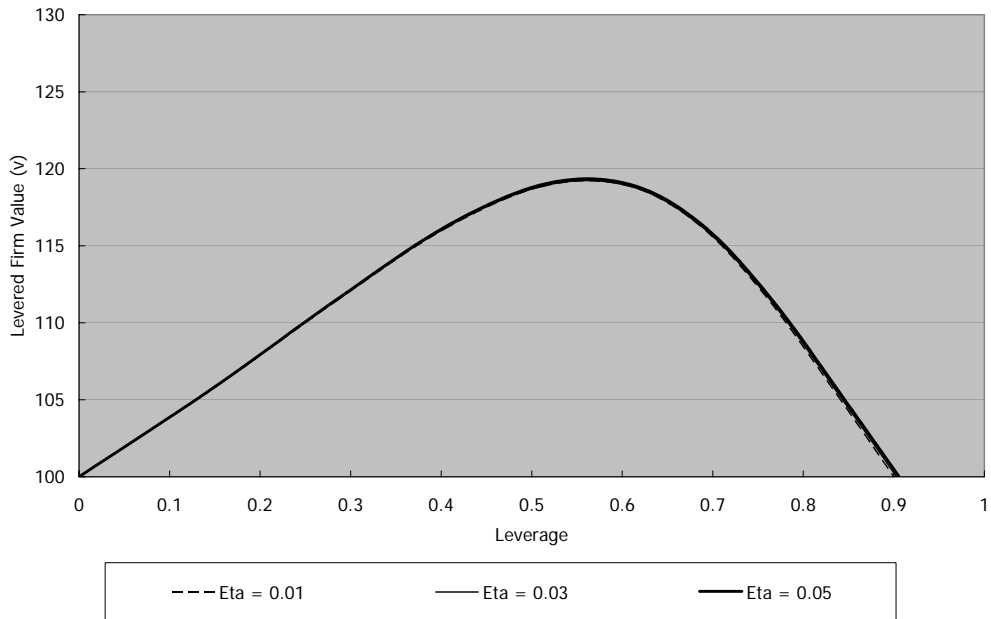


From the above charts, beta has a positive correlation with the optimal leverage ratio and levered firm value. And similarly, when the bond maturity is larger, the impact of beta is getting more significant. When beta for $T = 5$ increases from 0.5 to 1.50, the optimal leverage ratios increases from 49.5% to 57.0%. For $T = 10$, the optimal leverage ratio increases from 48.8% to 64.5%. And for $T = 20$, the figure increases from 45.3% to 72.3%.

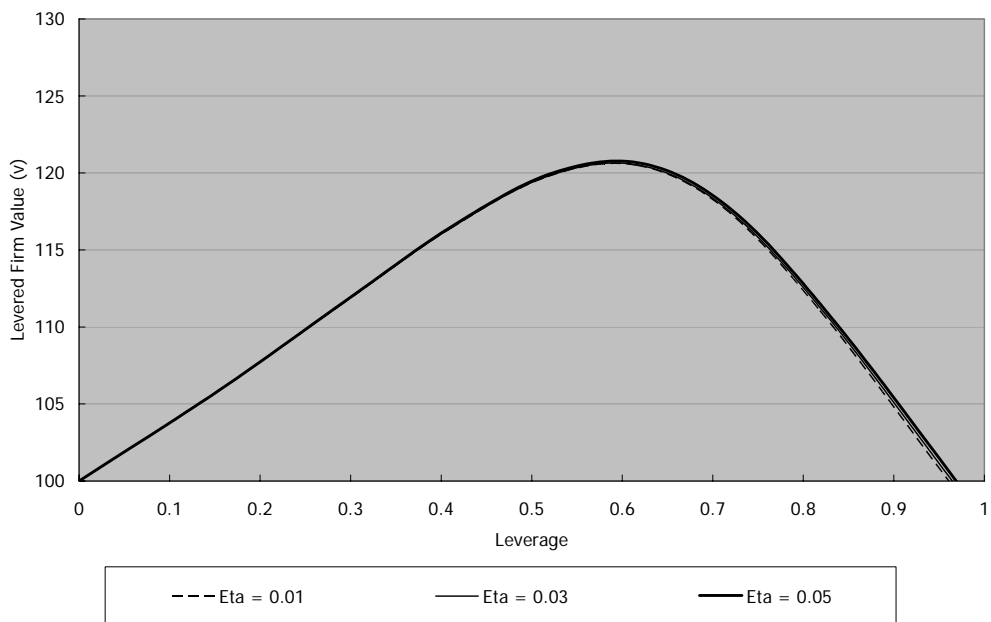
Eta

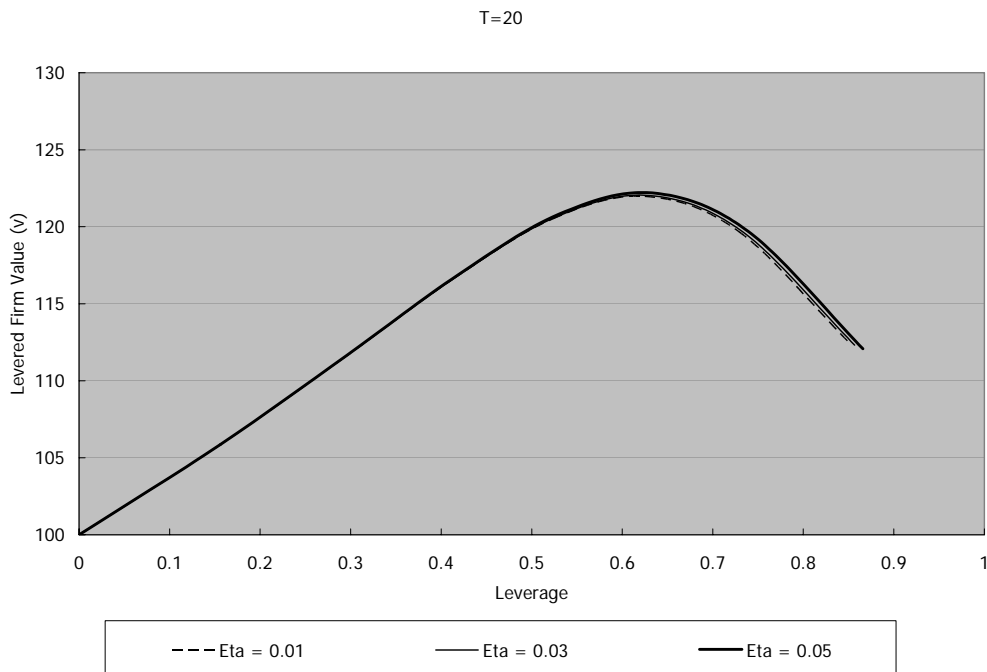
Eta denotes the volatility of the interest rate process and it is reasonable that the eta figure should be comparably smaller than equity volatility. I simulate the optimal debt structure figures from $\eta = 0.01$ to 0.05, assuming that all other parameters follow the base case assumptions. The firm value to debt structures charts are plotted as follows:

T=5



T=10



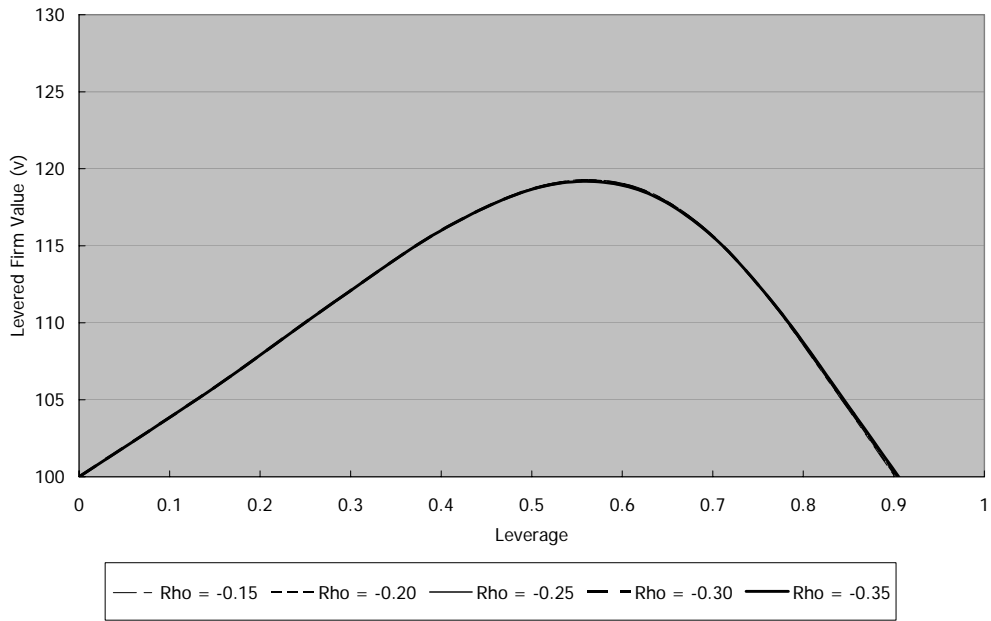


Interestingly, the lines in the three charts almost combine to each other, despite the changes in eta's in the charts. For eta equaling 0.01, 0.03, and 0.05, the optimal debt structure is 55.8%, 55.7%, 55.7% respectively when bond maturity equals 5 years. When bond maturity is 10 years, the figures are 58.9%, 59.0%, and 59.1% respectively. And when bond maturity is 20 years, the figures are 62.0%, 62.2% and 62.4% respectively. The optimal debt structure and eta represents a positive correlation, and the impact of the changes in eta becomes more significant when bond maturity gets larger.

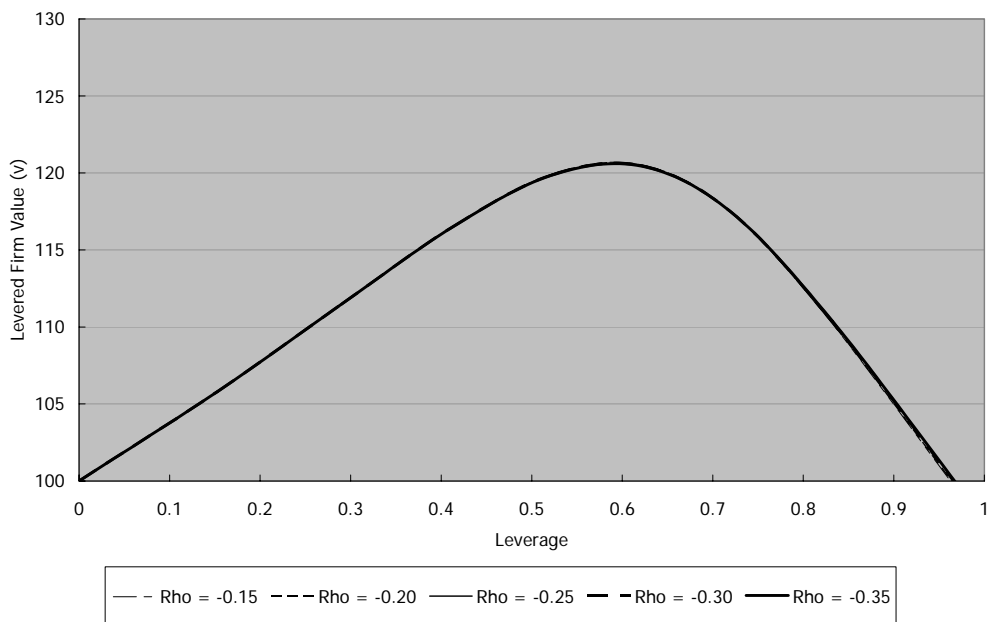
Rho

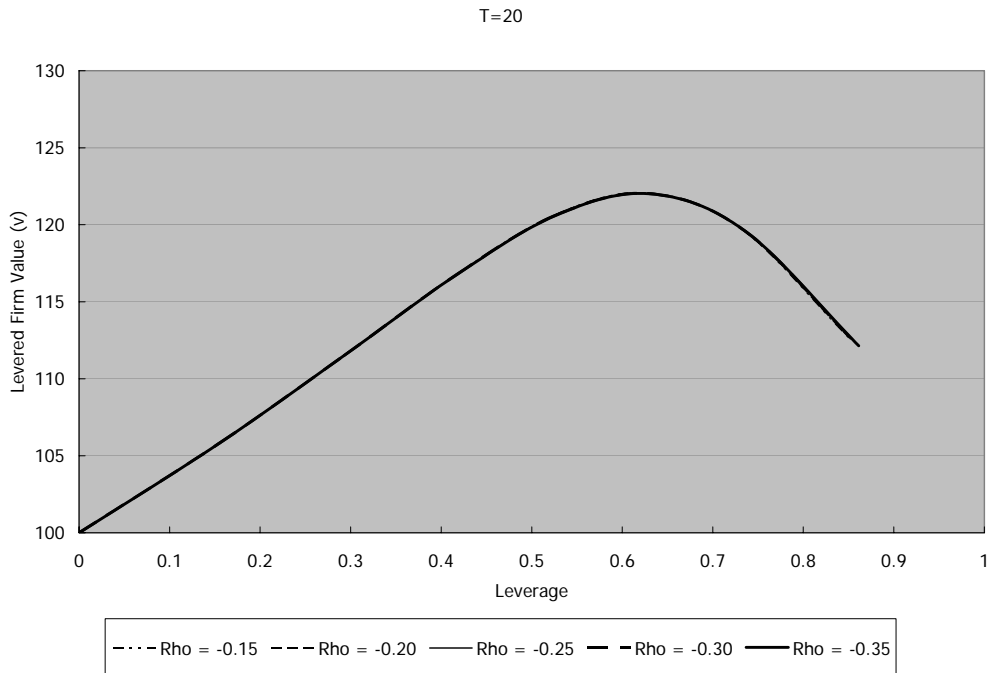
I simulate the optimal debt structure charts by assuming rho changes from -0.15 to -0.35 with bond maturity equals 5 years, 10 years and 20 years; other variables stay constant. The results of the simulations are illustrated below:

T=5



T=10





Again, the lines combine to each other in the three charts given rho changes from -0.15 to -0.35. For rho equaling -0.15, -0.20, -0.25, -0.30, and -0.35, the optimal debt structure is 55.8%, 55.8%, 55.7%, 55.6%, and 55.6% respectively when bond maturity equals 5 years. When bond maturity is 10 years, the figures are 59.00%, 58.98%, 58.98%, 58.98% and 58.95% respectively. And when bond maturity is 20 years, the figures are 62.13%, 62.15%, 62.17%, 62.23% and 62.25% respectively. The correlation between rho and the optimal structure is not clear.

Sensitivity Analysis

Based on the simulations above, the sensitivity of alpha, beta, eta and rho is categorized as the table below:

| | Maturity | Max | Min | Optimal Leverage | | $\frac{d(\text{optimal leverage})}{d(\text{parameter})}$ |
|-------|----------|------|------|------------------|-------|--|
| | | | | Max | Min | |
| Alpha | 5 | 0.12 | 0.04 | 0.565 | 0.480 | 1.062 |
| | 10 | 0.12 | 0.04 | 0.640 | 0.465 | 2.187 |
| | 20 | 0.12 | 0.04 | 0.715 | 0.443 | 3.406 |
| Beta | 5 | 1.5 | 0.5 | 0.570 | 0.495 | 0.075 |
| | 10 | 1.5 | 0.5 | 0.645 | 0.488 | 0.157 |
| | 20 | 1.5 | 0.5 | 0.722 | 0.453 | 0.270 |

| | Maturity | Max | Min | Optimal Leverage | | $d(\text{optimal leverage})$ |
|-----|----------|-------|-------|------------------|-------|------------------------------|
| | | | | Max | Min | $d(\text{parameter})$ |
| Eta | 5 | 0.05 | 0.01 | 0.558 | 0.557 | 0.025 |
| | 10 | 0.05 | 0.01 | 0.591 | 0.589 | 0.050 |
| | 20 | 0.05 | 0.01 | 0.624 | 0.620 | 0.100 |
| Rho | 5 | -0.15 | -0.35 | 0.558 | 0.556 | 0.010 |
| | 10 | -0.15 | -0.35 | 0.590 | 0.589 | 0.002 |
| | 20 | -0.15 | -0.35 | 0.622 | 0.621 | 0.006 |

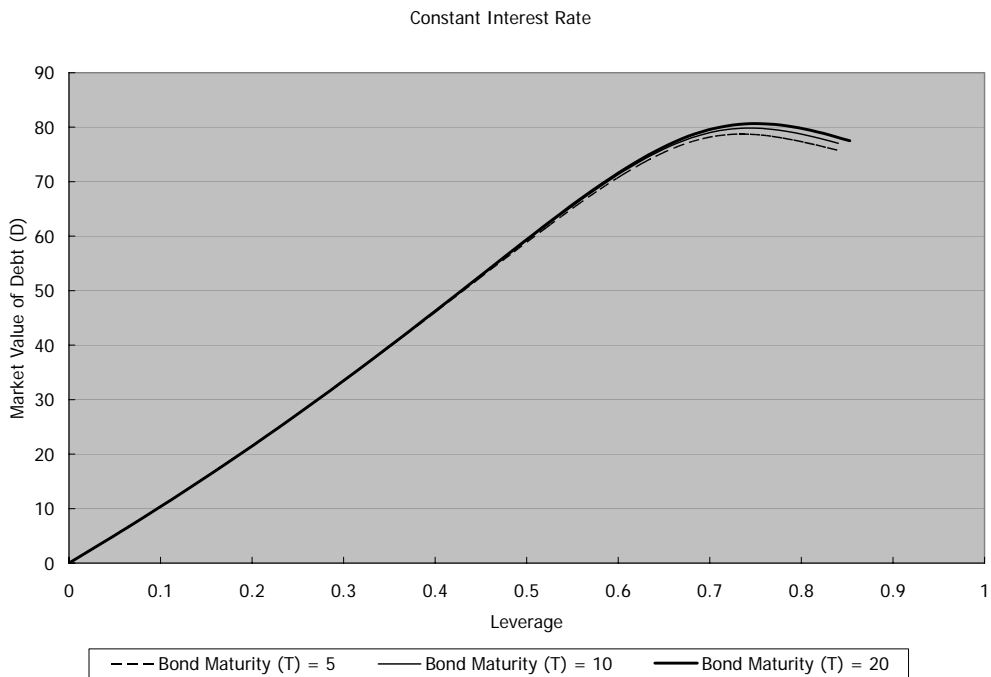
For alpha, beta and eta, the bigger time to maturity, the greater sensitivity to the optimal debt structure. However, for rho, the relationship between the bond maturity and the sensitivity to optimal debt structure is unclear. In addition, alpha is the most sensitive factor among the four factors; the beta is the second high, then eta, and then rho.

In the previous section, I preliminarily conclude that in the base case, the impact of the stochastic interest rate assumption to optimal debt structure gets more significant when the bond maturity is larger. From the sensitivity analysis here, it is obvious that the substantial impact of the increase in bond maturity is due to the alpha, beta and eta factors; while rho seems to have negative impact to the optimal debt structure with less impact. Furthermore, depending on the shape of the $D(t)$ function, the optimal debt structure may be greater or smaller; while the stochastic and constant interest rate assumptions affect the shape of $D(t)$ function.

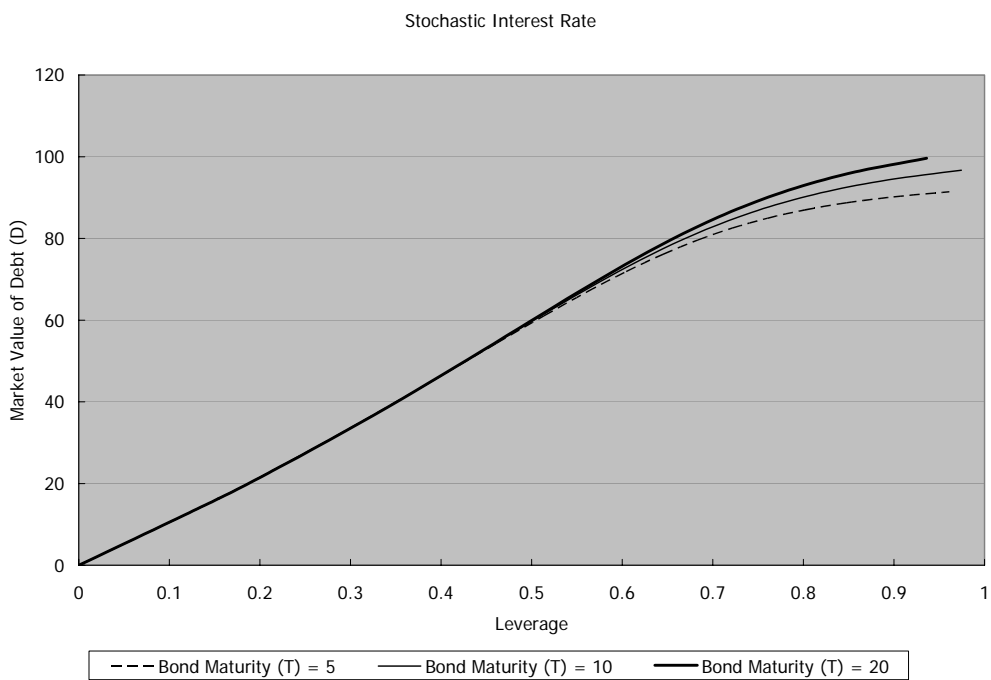
Default Asset Values in Different $D(t)$ Functions

Under the base case assumption, ironically, when bond maturity gets larger, the optimal debt structure will fall over 100%. First of all, a debt structure cannot be over 100%. Furthermore, let's consider a state when there is no optimal debt structure under 100%; i.e. the firm value to leverage is always upward sloping. Economically, a firm can keep leveraging without being down-graded. Obviously, it is not the case.

The chart below plots the debt value to leverage in different bond maturities from the Leland and Toft model. The debt values reverse eventually before the 100% leverage ratio, which economically makes sense.

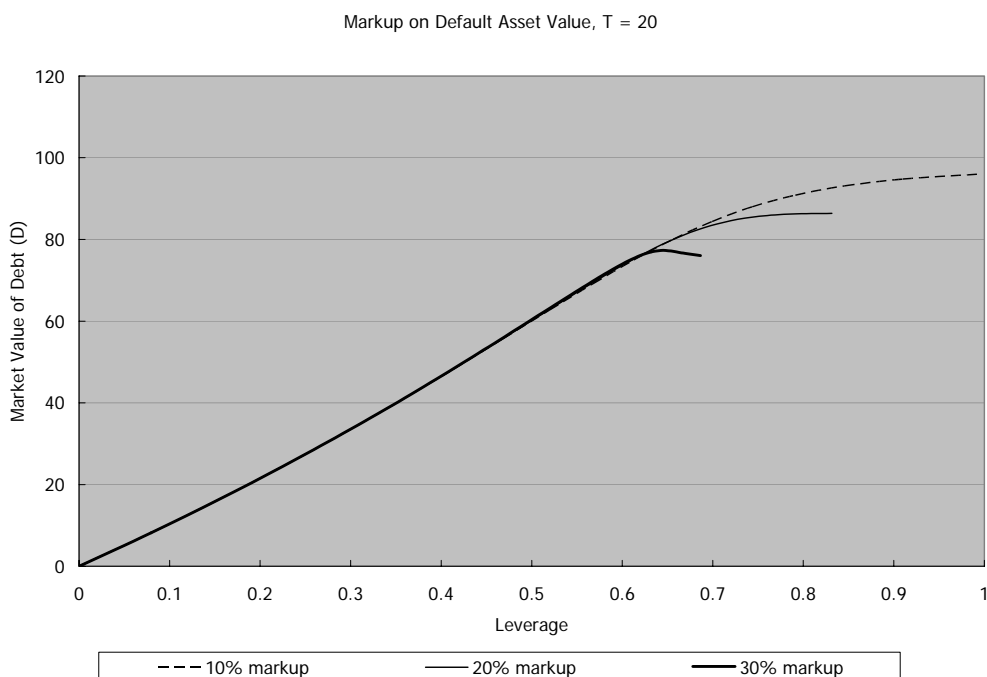


However, in the base assumption as shown in the chart below, the debt value does not reverse somewhere before the 100% leverage ratio.



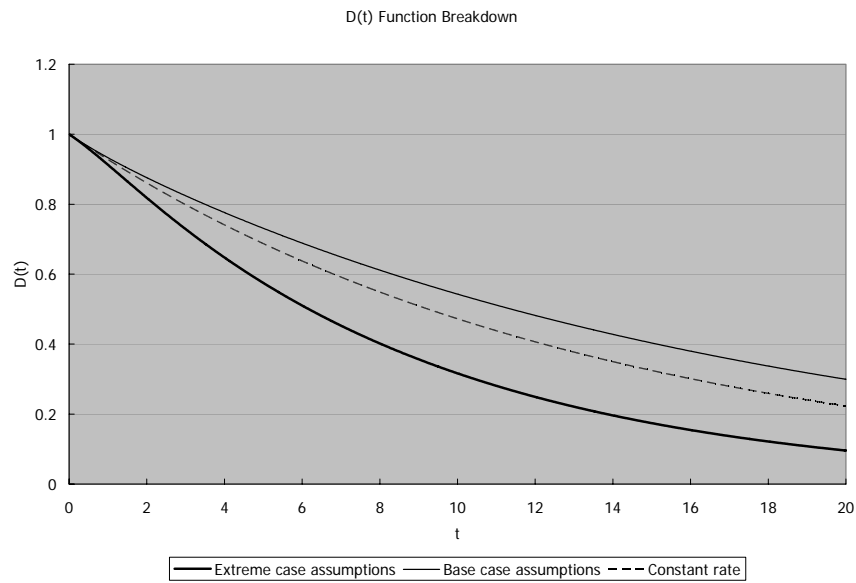
I determine that the default asset value in the base case assumption is too low and thus the market value of debt is over-valued. Leland and Toft use smooth-pasting process to get the close-form solution of default asset value. The implication of smooth-pasting process in Leland and Toft's paper is to maximize the equity value when the company defaults. Unfortunately, the same methodology can not be copied to the research framework in this report due to the lack of close-form solutions of debt value.

From the debt to leverage chart above, the higher the bond maturity, the bigger deficit on the default asset value in this framework. Hence, I test the sensitivity of default asset value to the 20-year bond in the base case assumption. Given other variables staying the same in the base case assumption, I scale up the default asset value from the Leland and Toft model by 10%, 20% and 30% and observe the change in debt value:

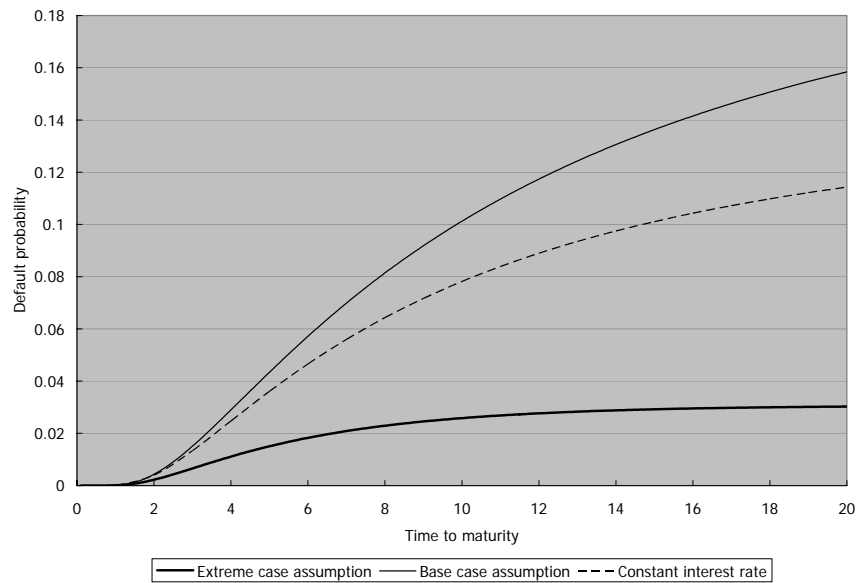


In the case of a 20-year bond in the base case assumptions, the value of debt will not reverse without a more than 20% markup on the default asset value. With a 10% markup on the default asset value in the 20-year bond, the debt value does not reverse before the leverage ratio equal to 1. However, with a 30% markup on the default asset value, the debt value reverses when leverage ratio is 65.5%. The breakeven markup percentage on the default asset value in this case is approximately 20%; with a 20% markup on the default asset value, the debt value reverse when the leverage ratio is exactly 100%.

Please be aware that the $D(t)$ function in the base case assumption is larger than that in the constant interest rate assumption. Therefore, I test the default probability of an extreme case where the $D(t)$ function is smaller than that in constant interest rate assumption by choosing a rather high alpha. Here I set alpha equal to 0.12 as an extreme case assumption. The $D(t)$ function comparison of extreme case, base case, and constant interest rate assumptions is shown as follows:



Thus, the default probabilities associated with the three cases are:



Apparently, the default probability in the extreme case is lower than that in constant interest rate and in base case assumption. Therefore, it is expectable that I should markdown the default asset value from the Leland and Toft model in order to get the default asset value in the extreme case.

From the analysis in this section, the default asset value is actually affected by the shape of $D(t)$ function. Vertically the higher the $D(t)$ function over time is, the higher the default asset value would be, and vice versa. Inferred by this result, the $D(t)$ function would affect the optimal debt structure as well: the higher the $D(t)$ function over time is, the lower the optimal debt structure would be.

Conclusion

This framework provides an extension of the Leland and Toft model to the stochastic interest rate assumption. Despite the drawbacks of no dividend payout and unavailability to implement the default asset value, this paper has the following findings:

- Taking the stochastic interest rate assumption to the Leland and Toft model, the optimal debt structure still exists; the levered firm value to leverage line still behaves as a concave curve.
- Also, the shape of default probability does not change as well. It increases over time to maturity in a concave form as well.
- Depending on the shape of the $D(t)$ function, the default asset value under the stochastic interest rate assumption is either higher or lower than the default asset value under the constant interest rate assumption. The higher comparable $D(t)$ curve over time, the higher default probability curve over time.
- Depending on the shape of the $D(t)$ function, the optimal leverage ratio will either be greater or smaller than that in the Leland and Toft model. So is the optimal levered firm value. The higher comparable $D(t)$ curve over time, the higher optimal debt structure and levered firm value.
- Same as the conclusion in the constant interest rate assumption, price of a par bond in the stochastic interest rate assumption changes over time even if the interest rate remains unchanged. But the pattern of the bond price movement is different: 1) for a more risky bond, the volatility of the price movement over its life is lower, and 2) the bond price tends not to gain a premium over its life.
- Among all Vasicek model parameters, alpha has the biggest sensitivity to the optimal debt structure; the sensitivity of eta and rho is very low.

For the future research on the behaviors of a firm under a stochastic interest rate assumption, there are two issues worth stepping forward. First of all, the probability distribution functions of corporate behaviors under a stochastic interest rate assumption. Secondary, a naïve Monte Carlo simulation may be necessary in order to determinate of default asset value. To do so, nevertheless, a “simulation on simulation” may be necessary to find the maximized equity value on default and the distribution probability functions that the company behaves. Apparently, it will require a lot of computer powers.

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