Calculus Chapter 4

Chapter 4-1

Antiderivatives

Antiderivatives:

The process/operation of working backward from the derivative to the original function is called Integration.

The Antiderivative (original function) is the result of this operation.

Not so hard……. ☺

Suppose there is the derivative

$$f'(x) = 2x + 4$$

Find the antiderivative.

$$f(x) = x^2 + 4x + C$$

Wait! What is that C ??????

"C" is the *constant of integration* which **MUST** always be present in an antiderivative unless you are given some initial conditions in which you will be able to determine its value.

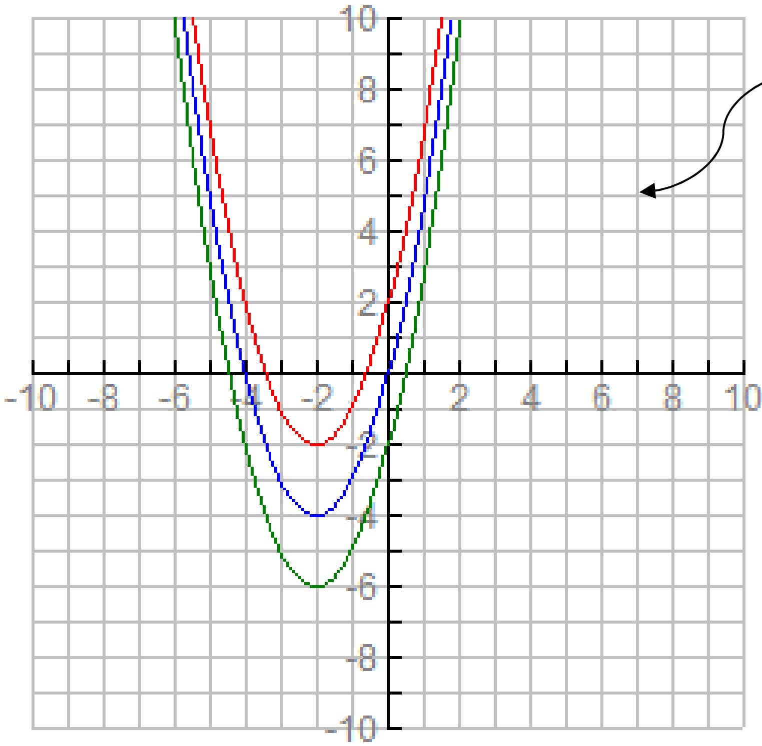
"+" , because a constant can be positive or negative

Thus:

$$f(x) = x^2 + 4x + C$$

is the General Solution for the differential equation

$$f'(x) = 2x + 4$$



Partial Graph of the General Solution

Differential Equation→in x and y / $f(x)$ is an equation that involves, x, y / $f(x)$, and derivatives of y / $f(x)$.

Notation for Antiderivatives:

Integrand

Constant of integration

$$\int f(x)\, dx = F(x) + C$$

Variable of integration

$$\frac{dy}{dx} = f(x)$$

$$(dx)\frac{dy}{dx} = f(x)[dx]$$

$$\int 1dy = \int f(x)[dx]$$

$$y = \int f(x)[dx] = F(x) + C$$

Operation for the above is called ***Indefinite Integration.***

Basic Integration Rules→pg 243

Note:
$$\frac{dy}{dx}\left[\int f(x)\, dx\right] = f(x)$$

Also be sure to read over the examples in the book noting that *rewriting* before integrating helps in some cases.

Initial Conditions & Particular Solutions

Example: Find the general solution, then the particular solution that satisfies the initial condition $F(1)=0$ of

$$F'(x) = \frac{1}{x^2}, \quad x > 0 .$$

Rewrite
Integrate BOTH sides→general solution
Use initial condition→particular solution

You have already worked problems using the Position Function involving gravity in which you have worked backward. Remember???

❀

Chapter 4.2

Area of a Plane Region—Upper & Lower Sums

(or leading up to the Definite Integral)

Summation Formulas *important to know*

$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

To Evaluate a sum→substitute the appropriate values of n.

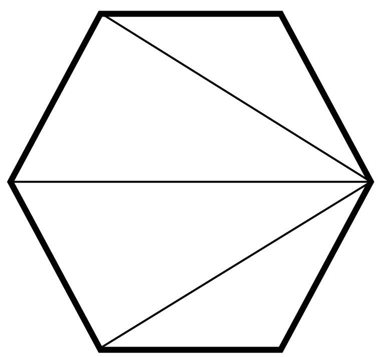
Ex: Evaluate/find the sum. Verify the results on your calculator

$$\sum_{k=2}^{5} (k+1)(k-3) = ?$$

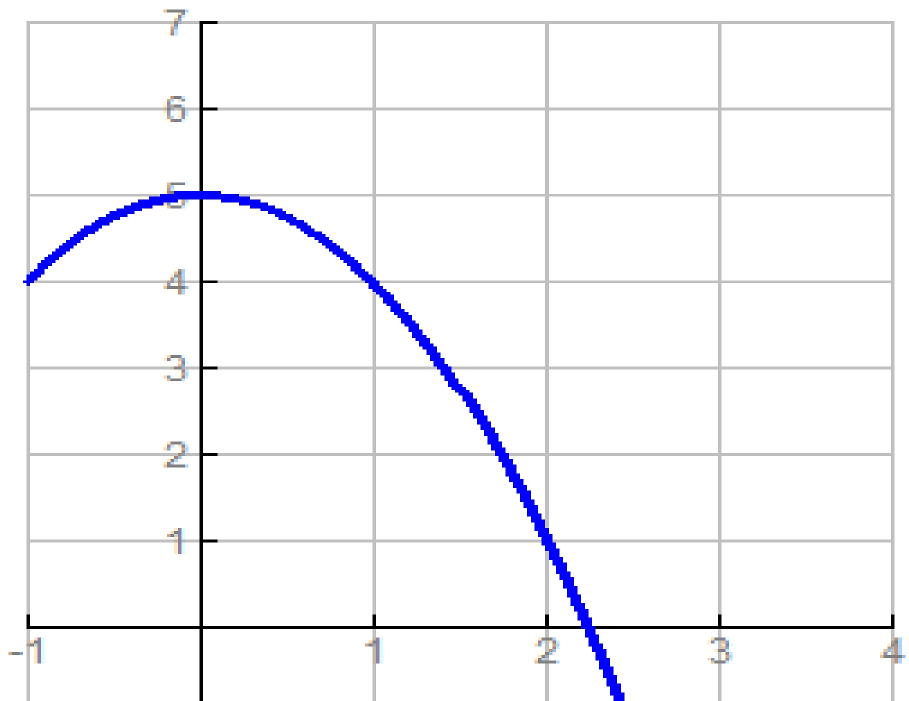
AREA

Area is an important concept in calculus.

Just as the sum of the parts make a whole—the sum of the areas make a whole area…….

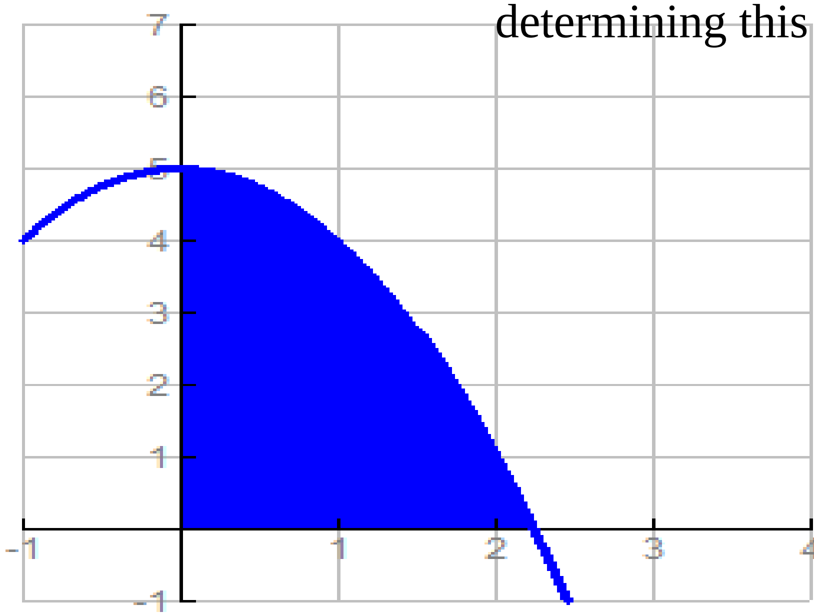


Wait!!!   Suppose the area is curvy???

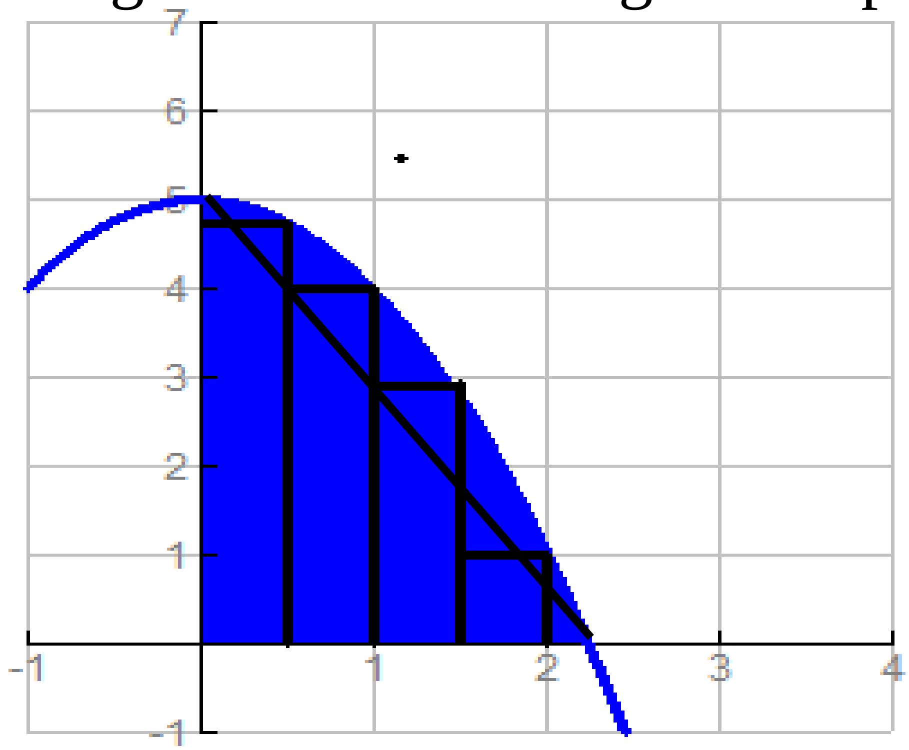


Find the area under the curve bounded by the y-axis and x-axis.

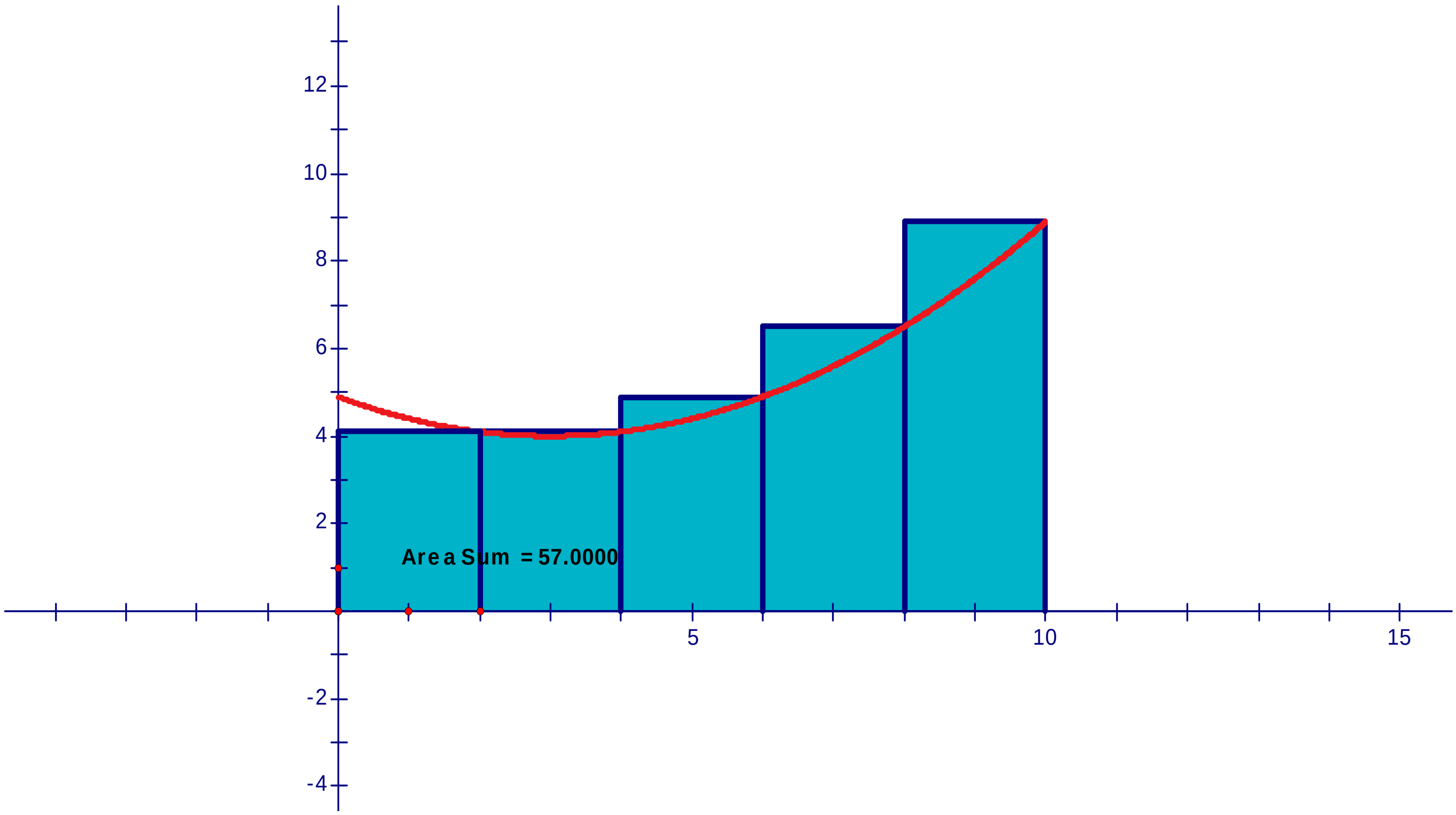
How would you go about determining this area?



You can approximate the area by finding the area of a triangle or by dividing the area into rectangular regions and summing them up.



Gee!! Does that give you a good estimate? Well, the method of rectangles seems the best.



There are three methods/formulas for calculating the sum: Right (*upper*) Sum, Left (*lower*) Sum and Midpoint Sum.

Note:

$$\Delta x = \frac{b-a}{n}$$ for all sums on the interval $[\,a, b\,]$

a & b are the endpoints and n is the number or intervals or regions

Right Sum→the right point of the rectangle lies on the function.

$$\sum_{i=1}^{n} f\left(x_i\right)\Delta x, \quad \text{where } x_i = a + \Delta x\, i$$

thus: $$\sum_{i=1}^{n} f\left(a + \Delta x\, i\right)\Delta x$$

Left Sum→the left point of the rectangle lies on the function.

$$\sum_{i=1}^{n} f\left(x_i\right)\Delta x, \quad \text{where } x_i = a + \Delta x\left(i - 1\right)$$

thus: $$\sum_{i=1}^{n} f\left(a + \Delta x\left(i - 1\right)\right)\Delta x$$

OR

$$\sum_{i=0}^{n-1} f\left(a + \Delta x\, i\right)\Delta x$$

Midpoint Sum→the midpoint of the rectangle lies on the function.

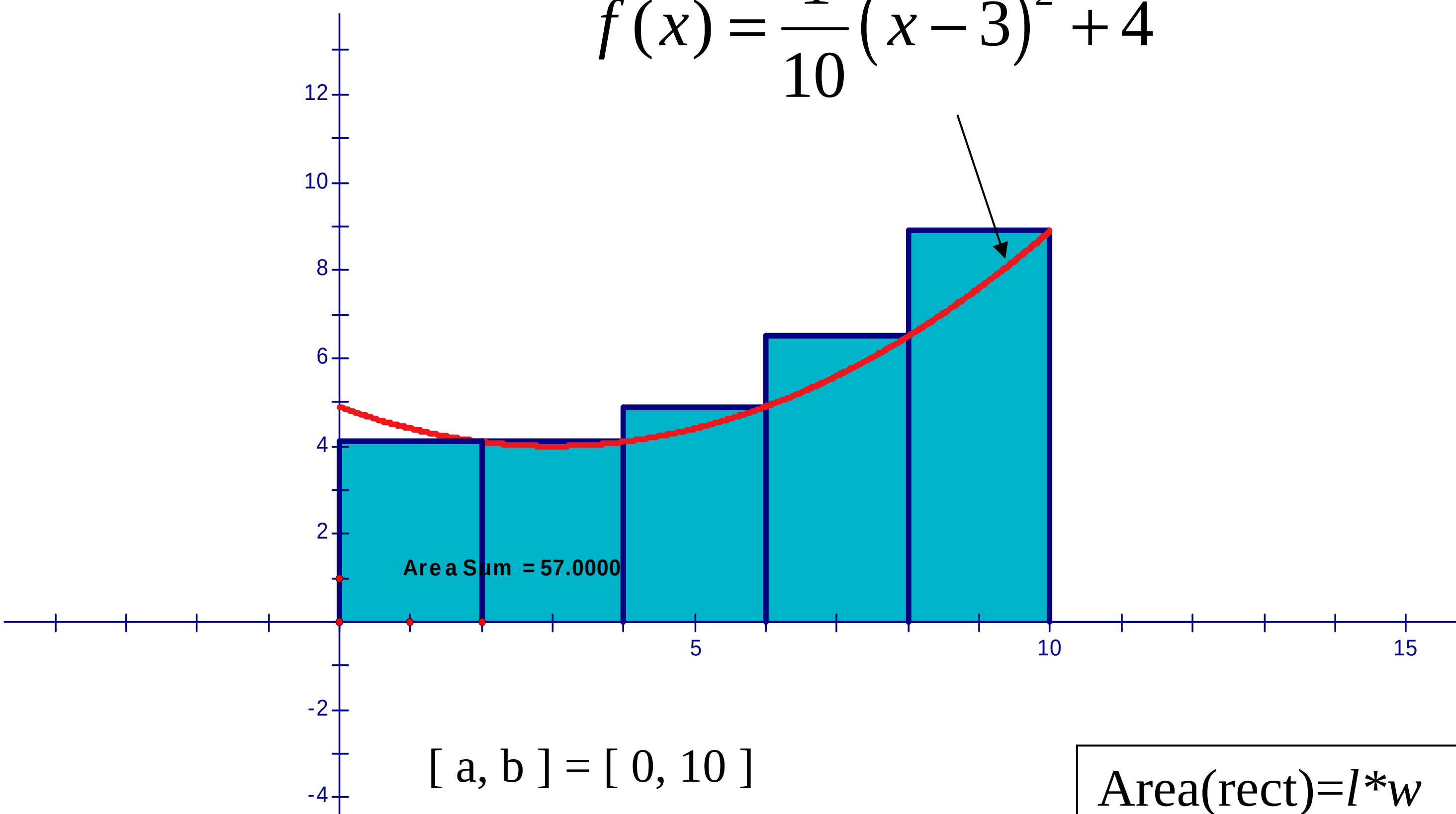
$$\sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x, \qquad \text{where } x_i = a + \Delta x\, i$$

$$x_{i-1} = a + \Delta x\left(i - 1\right)$$

Example showing the Area formed by the Right and Left Sum.

Left and Right sums.

$$f(x) = \frac{1}{10}(x-3)^2 + 4$$



[a, b] = [0, 10]

Area(rect)=*l***w*
=*x***f(x)*

A better estimate would be the average of the left sum and the right sum…….

Thus: $$\text{right sum} = \sum_{i=1}^{n}\left[\frac{1}{10}\left(x_i - 3\right)^2 + 4\right]\Delta x$$

$$\text{left sum} = \sum_{i=0}^{n-1}\left[\frac{1}{10}\left(x_i - 3\right)^2 + 4\right]\Delta x$$

Calculating the sum when n is small is easy.

$$\text{Rt Sum} = \left(.1(2-3)^2 + 4\right)2 + \left(.1(4-3)^2 + 4\right)2 + \ldots$$
$$+\left(.1(10-3)^2 + 4\right)2 = 8.2 + 8.2 + 9.8 + 13.0 + 17.8 = 57$$

$$\text{Lt Sum} = \left(.1(0-3)^2 + 4\right)2 + \left(.1(2-3)^2 + 4\right)2 + \ldots$$
$$+\left(.1(8-3)^2 + 4\right)2 = 9.8 + 8.2 + 8.2 + 9.8 + 13.0 = 49$$

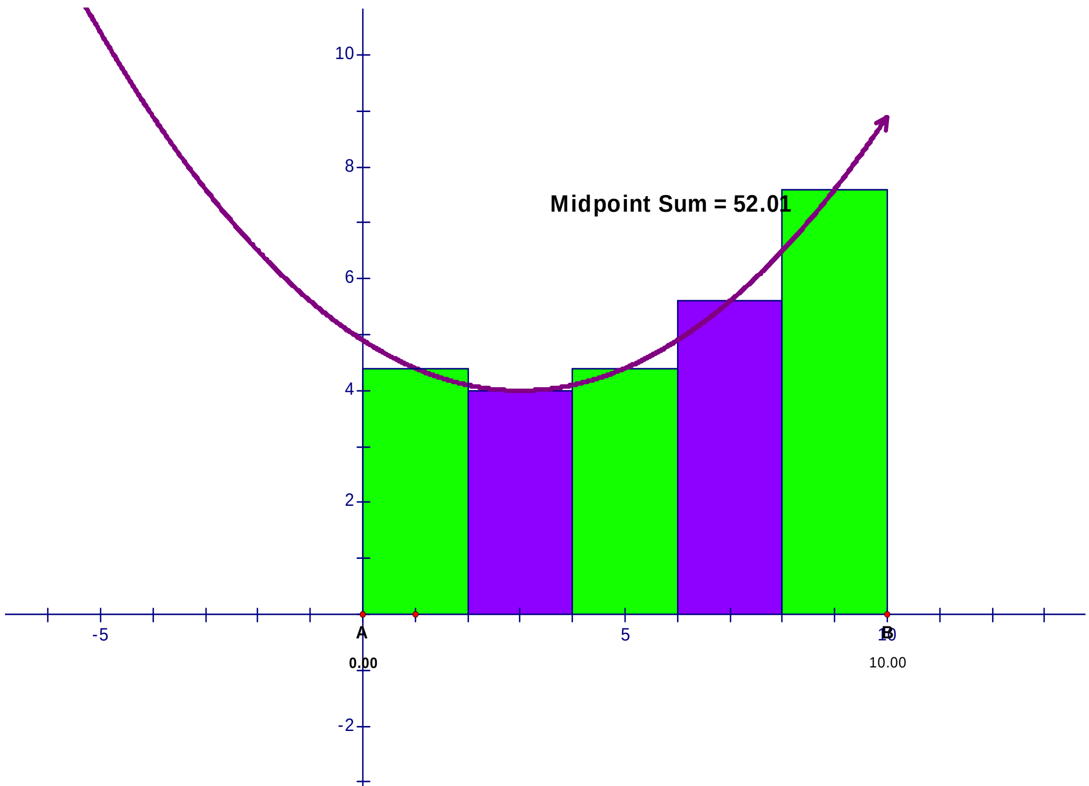
Now—Average = ?

Still not so great—really just an approximation. What would happen to our area if we increased the number of regions………

Well……Check out the function as *n* increases.

Okay, the area might be better if we use a point on the graph that is the midpoint of the rectangle.

Suppose we did just that------



This is called the Midpoint Sum or Midpoint Riemann Sum

There is a big problem when *n* becomes large. The answer is to use the summing formulas. Why?

n is number of partitions

$$\text{right sum} = \sum_{i=1}^{n}\left[\frac{1}{10}\left(0+\frac{10}{5}i-3\right)^2+4\right]\frac{10}{5}$$

Thus:

$$\text{rt sum} = \sum_{i=1}^{n}\left[\frac{1}{10}(2i-3)^2+4\right]2 = \sum_{i=1}^{n}\left[\frac{1}{10}\left(4i^2-12i+9\right)+4\right]2 =$$

$$\left[\frac{1}{10}\left(4\frac{n(n+1)(2n+1)}{6}-12\frac{n(n+1)}{2}+9n\right)+4n\right]2$$

Substituting $n = 5$: Sum = 57

We could do the same with the left sum.

Note: As n increases the difference between the right and left sums decreases. Now as n approaches ∞, both of the sums approach the same value. This can be seen graphically in any of the graphs above.

Thus, analytically you can find the area by using the limit as n approaches ∞.

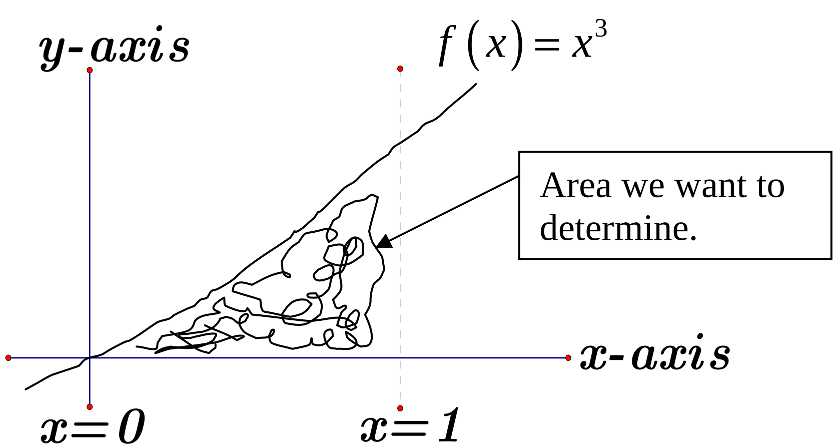
$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\ \Delta x$$

Now the summing formulas are combined with finding the limit.

Ex: Find the area of the region bounded by the graph $f(x) = x^3$, the x-axis, and the vertical lines $x = 0$, and $x = 1$.

Steps:

1. Sketch the graph—



2. Set up the interval-- [0, 1]

3. Calculate $\triangle$x $\qquad \Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{n} = \dfrac{1}{n}$

4. Set up the Limit/Sum—then evaluate…

Remember the function is $f(x) = x^3$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{n}i\right)^3\left(\frac{1}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{n^4}i^3\right) = \lim_{n\to\infty}\left(\frac{1}{n^4}\right)\sum_{i=1}^{n}\left(i^3\right)$$

$$\lim_{n\to\infty}\left(\frac{1}{n^4}\right)\left(\frac{n^2(n+1)^2}{4}\right) = \lim_{n\to\infty}\left(\frac{n^2(n+1)^2}{4n^4}\right) = \frac{1}{4}\text{units}^2$$

Note: Suppose that the area does not start at $x = 0$…….
What would x_i be equal to????

$$x_i = a + (\Delta x)i$$

Chapter 4-3

Riemann Sums
The area of the partitions in Left, Right, and Midpoint Sum is calculated with all △x's (subintervals) the same length. Now in this section the subintervals do not all have to be the same length→Riemann Sum.

Riemann Sum:
Function is defined on the closed interval $[a,b]$
△ is a partition of $[a,b]$
Δx_i is the length of the i^{th} subinterval
c_i is any point in the i^{th} subinterval

then the sum

$$\sum_{i=1}^{n} f\left(c_i\right)\Delta x_i, \qquad x_{i-1} \le c_i \le x_i$$

is called the Riemann Sum.

Definition: Definite Integral

$$\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f\left(c_i\right)\Delta x_i = \int_a^b f\left(x\right)\,dx$$

$a \to$ lower limit
$b \to$ upper limit

Definite Integral as the Area Under a Curve

$$\text{Area} = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f\left(c_i\right)\Delta x_i = \int_a^b f\left(x\right)\,dx$$

Remember **Area** is ***always*** positive……

Note properties of Definite Integrals on pgs 269 & 270

Chapter 4-4

The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a,b]$ and F is an antiderivative of f on the interval $[a,b]$ then,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

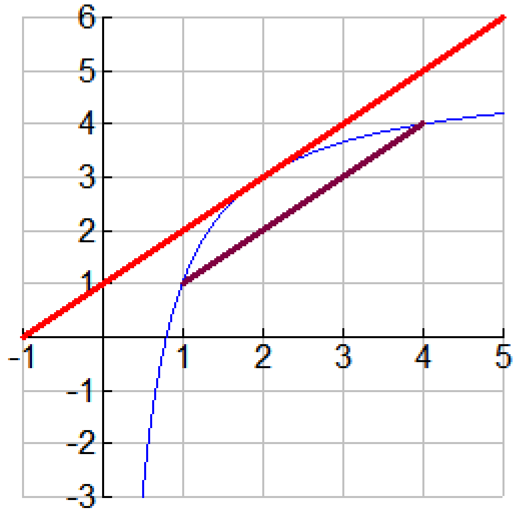
Note: Guidelines for using the Fundamental Theorem of Calculus on pg 275

Example: Evaluate

$$\int_1^3 x^2 - 3\,dx = \left[\frac{1}{3}x^3 - 3x\right]_1^3$$

Remember the Mean Value Theorem for derivatives?

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

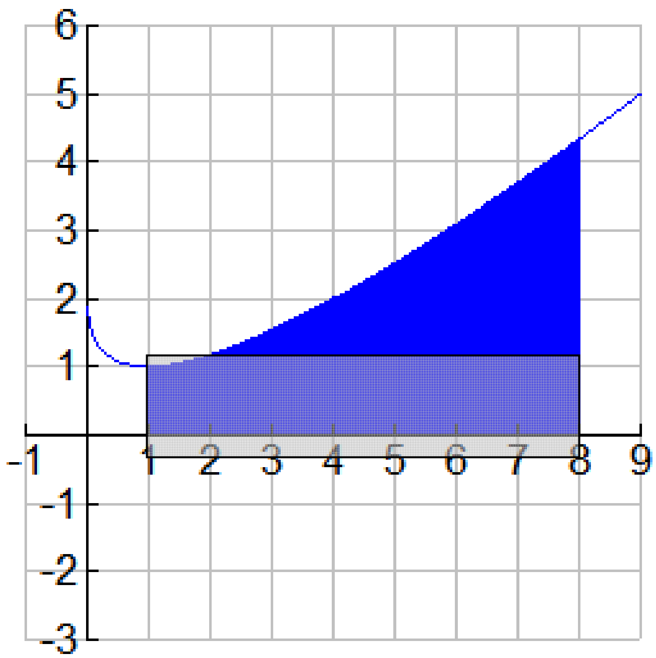


Mean Value Theorem for Integrals:

If f is continuous on the closed interval $[a,b]$, then there exists a number c in the closed interval $[a,b]$ such that

$$\int_a^b f(x)\,dx = \underbrace{f(c)}_{\text{height}}\underbrace{(b-a)}_{\text{width}}$$

In other words—there is a (one) rectangle whose area is precisely equal to the area of the region under the curve.



Average Value of a Function on an Interval

Average Value = $f(c) = \dfrac{1}{(b-a)} \underbrace{\int_a^b f(x)\,dx}_{\text{Area}}$

Ex: Find the average value of $f(x) = 3x^2 - 2x$ on the interval [1,4].

Definite Integral as a Function of Time

$$F(x) = \int_a^x f(t)\,dt$$

Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

Ex: Evaluate:

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right] = ?$$

Ex: Find the derivative of

$$\int_{\pi/2}^{x^3} \cos t\,dt$$

$$\frac{d}{dx}\left[\int_{\pi/2}^{x^3} \cos t\,dt\right] = ?$$

Chapter 4-5

Integration by Substitution……..is a method in which we change variables that often turns an unfamiliar integral into one that we can evaluate.

Usually follows a particular ***pattern***----the integrand can be divided into two expressions: one is the derivative of the other (or of the *inside expression*).

$$2x, \quad \underbrace{\left(x^2+1\right)}$$

the derivative of this expression is the other expression…….

Many functions form composite functions—Remember Composite Functions?????

$$h(x) = f\left(g\left(x\right)\right)$$

Example: $\int 2x\left(x^2+1\right)^2 dx = ?$

Ex: $g(x) = x^2 + 1$

$f(x) = x^2$, thus

$f\left(g(x)\right) = \left(x^2+1\right)^2$

Now: the derivative of the inside expression…..

$$\frac{d}{dx}\left(x^2+1\right) = 2x$$

Thus the example follows the ***pattern*** of the integrand being divided into two expressions: one is the derivative of the other (or of the inside expression).

Method to integrate this integrand:

1. Let u = inner expression.

$$u = x^2 + 1$$

2. Differentiate both sides of the u-equation with respect to x.

$$\frac{d}{dx}u = \frac{d}{dx}\left(x^2+1\right)$$

$$\frac{du}{dx} = 2x$$

3. Solve for dx .

$$du = 2x\,dx$$

$$\frac{1}{2x}du = dx$$

4. Substitute u and dx into the integrand.

$$\int 2x\,(u)^2\ \frac{1}{2x}\ du$$

5. Simplify.

$$\int 2x\,(u)^2\ \frac{1}{2x}\ du$$

$$\int\ (u)^2\ du$$

6. Integrate

$$\int\ (u)^2\ du = \frac{1}{3}u^3 + C$$

7. Switch back to the original variables.

$$\int\ (u)^2\ du = \frac{1}{3}u^3 + C$$

$$= \frac{1}{3}\left(x^2+1\right)^3 + C$$ ← **Answer**

8. If there are limits, evaluate **Answer** using the Fundamental Theorem of Calculus. *Of course, do not include the "C" in the final answer if there are limits.

Note: Change of Variables Process

When evaluating a *u*-substituting integral sometimes it is convenient to find the limits of integration for the variable u rather than to convert the antiderivative (in terms of u) back to the variable x (the antiderivative in terms of x) and then evaluating. Important to remember that the limits for the x variable are not the same limits for the u variable.

Ex: $$\int_0^1 x\left(x^2+3\right)dx = \frac{1}{2}\int_0^1 u^3\,dx$$

$$u = x^2+1: \text{ Thus: } u(0)=1,\ u(1)=2$$

Change of variables

$$\int_0^1 x\left(x^2+3\right)dx = \frac{1}{2}\int_1^2 u^3\,dx$$

Even and Odd Functions--polys

1. If function is an **even** function, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

2. If function is an **odd** function, then

$$\int_{-a}^{a} f(x)\,dx = 0$$