Calculus Chapter 3

Chapter 3.1

Behavior of a function is important in real-life situations. What is a function's maximum and/or minimum value(s)? Where is the function increasing or decreasing? In other words—*What is the Behavior of a function????*

Extrema→the extreme "*y*" -value(s) of a function.

- Absolute Extrema→Extreme behavior which occurs only with an Open or Closed Interval
- Relative Extrema→Extreme behavior which occurs in the "hills" and "valleys" of a function

Definition of Absolute Extrema:

A Function is defined on an <u>interval</u> containing *c*.

- *f(c)* is the **<u>minimum</u>** of f if $f(c) \le f(x)$ for all *x* in the interval
- *f(c)* is the **<u>maximum</u>** of f if $f(c) \ge f(x)$ for all *x* in the interval

The Extreme Value Theorem

Note: only for an interval

If a function is continuous on a *closed* interval [a,b], then the function has both a minimum and a maximum on the interval.

Definition of Relative Extrema:

- Open interval containing *c* on which *f(c)* is the **maximum** of the function—hill
- Open interval containing *c* on which *f(c)* is the **minimum** of the function—valley

Relative means that the max or min is relative to the area and is the "hill" or "valley".

Note: the "hill" or "valley" can occur in two ways:

- the graph is smooth and rounded and has a horizontal tangent line at the high/low point
- the graph is sharp and peaked in which the function is not differentiable at the high/low point

Critical Number

Given a function, f , and f is defined at c.
If $f'(c) = 0$ or if f' is undefined at c,
then c is a critical number of the function f .

Thm:
If f has a relative minimum/maximum at $x=c$,
then c is a **critical number** of f.

Guidelines for finding Extrema on a Closed Interval

1. Find the critical numbers of f in (a,b)
2. Evaluate f at each critical number
3. Evaluate f at each endpt of $[a,b]$
4. Least→minimum Greatest→maximum

Chapter 3.2

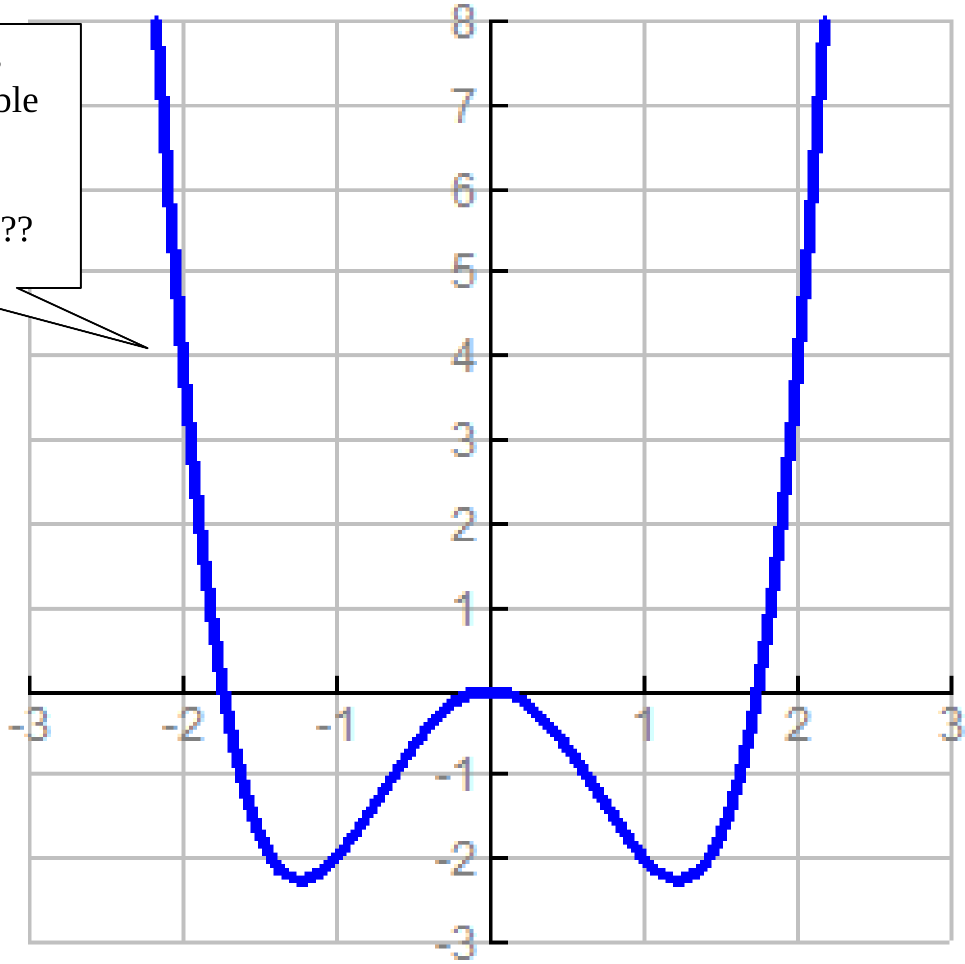
Rolle's Thm

A function, *f*, is continuous on the closed interval [*a*,*b*] and differentiable on the open interval (*a*,*b*).

NOW: If $f(a) = f(b)$

then there is at least one number c in (a,b) such that $f'(c) = 0$

[-2,2]→continuous
(-2,2)→differentiable
f(-2)=*f*(2)
At least 1 number such that $f'(c) = 0$??



The Mean Value Theorem

A function, *f*, is continuous on the closed interval [*a*,*b*] and differentiable on the open interval (*a*,*b*), then there exists a number *c* in (*a*,*b*) such that

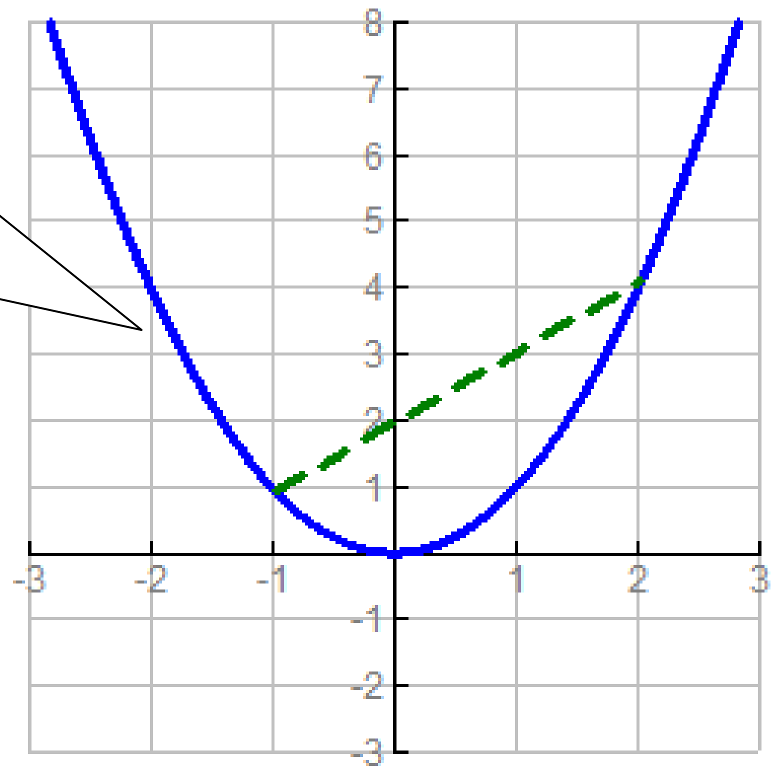
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Whoa!!! What does that mean????

[-1,2]→continuous
(-1,2)→differentiable
Is there a tangent line such that

$$f'(c) = \frac{f(a) - f(b)}{a - b}$$

and what is the value of *c*?????



In other words—for what value of *c* will the slope of the tangent line = the slope of the secant line.

Ex: $f(x) = 5 - \frac{4}{x}$

Find all values of c in the open interval (1,4) such that the slope of the tangent line at c equals the slope of the secant line thru the endpts.

Ex: Finding an Instantaneous Rate of Change

Two stationary patrol cars equipped with radar are 5 miles apart on a highway. As a truck passes the first patrol car, its speed is clocked at 55 mi/hr. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 mi/hr. Show that the truck must have exceeded the speed limit of 55 mi/hr at some time during the four minutes.

OKAY! What are the variables that you would use if you were to graph this? Is it a function?

NOW!! Remember that the slope is the rate of change—miles per hour…….
Remember that the Average Velocity =slope of the secant line thru the endpts.

$$\text{Average speed} = \frac{f(b) - f(a)}{b - a}$$

$$\text{Average speed} = \left| \frac{5 - 0}{4/60 - 0} \right| = 75\ {}^{mi}/_{hr}$$

By the Mean Value Thm there must be some time in which the truck driver exceeded the speed limit.

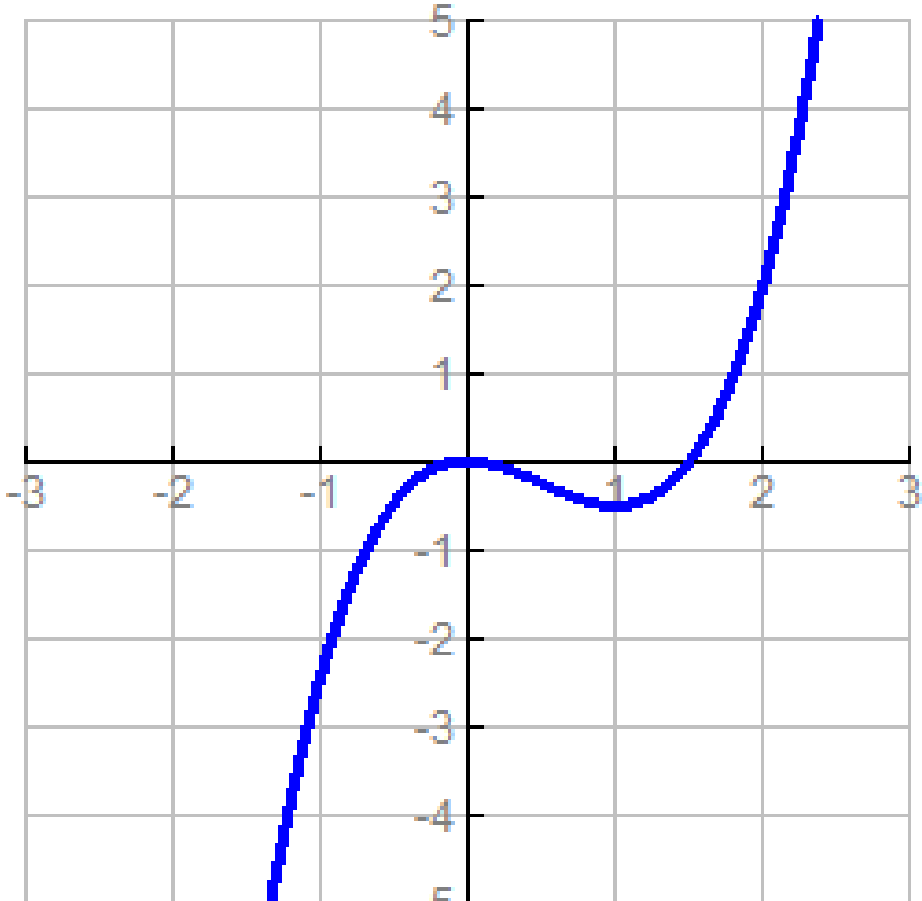
How would you find this time? Given a function for the velocity, it would be easy. ☺

✿

Chapter 3.3

- Increasing and decreasing functions
- 1st Derivative Test

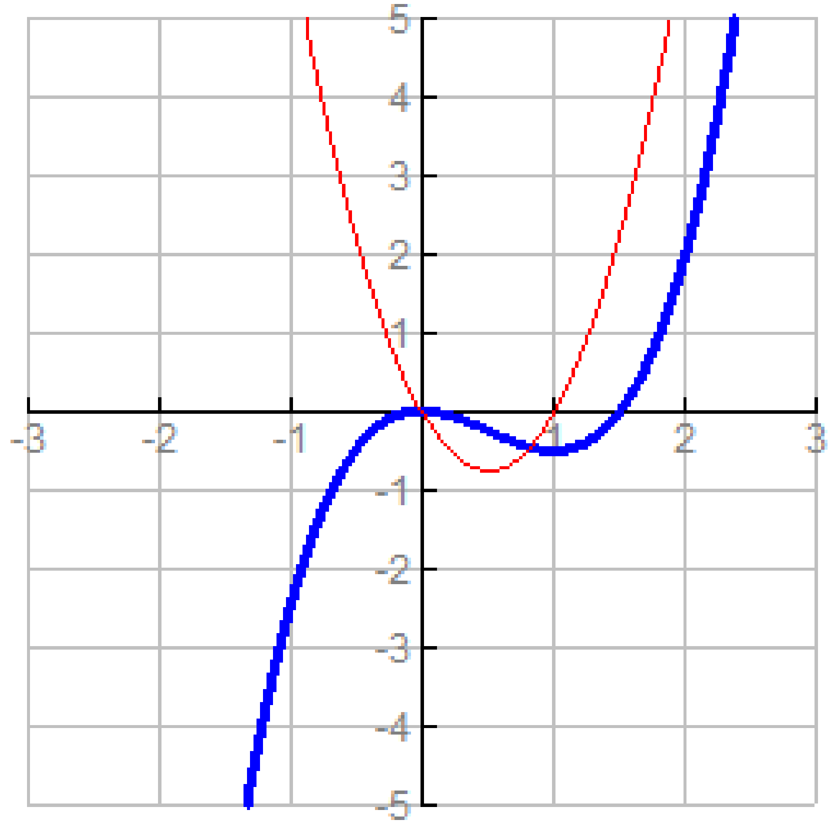
Consider function behavior over its domain not just on an interval…….



$$f(x) = x^3 - \frac{3}{2}x^2$$

As the function takes on different x-values from left to right what is the slope of the tangent line? Is it:

- Increasing?
- Decreasing?
- Zero?



$$f'(x) = 3x^2 - 3x$$

Visually—the function's behavior is clear. What if you can not produce a visual representation of a function? How would you determine the behavior of the function at particular points? Can you determine intervals in which the function is increasing and decreasing? Hint: note the derivative…

Where the derivative is positive the function is *increasing*…..
Where the derivative is negative the function is *decreasing*….
Where the derivative is zero the function has an *Inflection Point*—Chapter 3.4

Guidelines for finding intervals where the function is increasing or decreasing

- Find the critical numbers
- Set up the chart choosing x-values on the left and on the right of the critical numbers (close to the critical number)

Example: $f(x) = x^3 - \frac{3}{2}x^2$

Critical numbers are 0, 1

		0	1
Interval	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Val	x = -1	x = 1/2	x = 2
Sign $f'(x)$	$f'(-1) = 6$	$f'(\frac{1}{2}) = -\frac{3}{4}$	$f'(2) = 6$
Conclusion			

Are there any maximums & minimums?

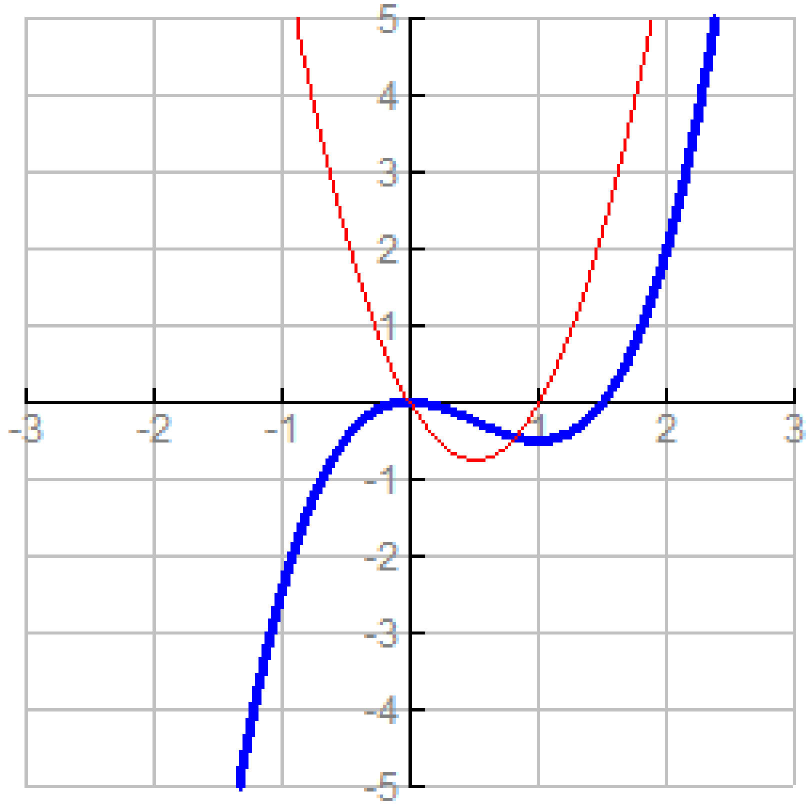
This method is called the 1st Derivative Test.

Important: Note Example 4, pg 174
Look at the **function** and determine if the function is continuous. (whether it has any asymptotes) Take that into consideration when using the 1st Derivative Test.

Chapter 3.4

Concavity & the Second Derivative Test

Again—using the function from Chapter 3.3



Where does the function curve up? curve down?

The point of inflection is where concavity changes—note where this point is located on the graph of the derivative.

Cool!!   In other words—the point of inflection is where the graph of the derivative has a max or min.

This *x*-value can be determined by taking the derivative of the 1st derivative of the function, i.e. the second derivative.

Always   Always   Remember that we are finding the behavior of a *function* not of its derivatives.

Always analyze the function first-----

Examples: Determine concavity, max(s) & min(s)

$$f(x) = \frac{x^2+1}{x^2-4} \qquad\qquad f(x) = x^4 - 4x^3$$

PS: + +   -- --

Chapter 3.5

Horizontal Asymptotes→Limits at ∞

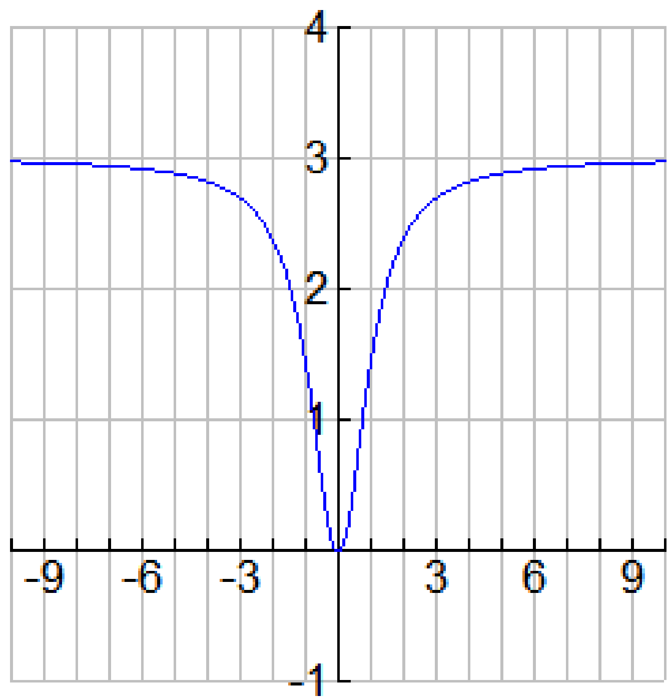
Function's Behavior—
We have described/analyzed the behavior of functions in intervals and in the hills and valleys. We have described/analyzed the behavior of functions near their vertical asymptotes.

Now we are going to look at the **END** behavior of a function. In other words, we are going to observe/describe the function's behavior (y's behavioral) when the *x*-values are going to $-\infty$ and when the *x*-values are going to $+\infty$.

$$f(x) = \frac{3x^2}{x^2+1}$$

To accomplish this we use ***LIMITS***.

$$\lim_{x \to \infty} f(x) = ?$$



Important Theorem

$$\lim_{x\to\infty}\frac{c}{x^r}=0 \quad ,$$

where r = positive rational number
c = and real number

Remember the Indeterminate form??   0/0
Also ∞/∞ is an indeterminate form……

Guidelines: page 190

- Degree of the numerator is less than the degree of the denominator→limit is 0
- Degree of the numerator is equal to the degree of the denominator→limit is the ratio of the leading coefficients
- Degree of the numerator is greater than the degree of the denominator→limit does not exist

Remember

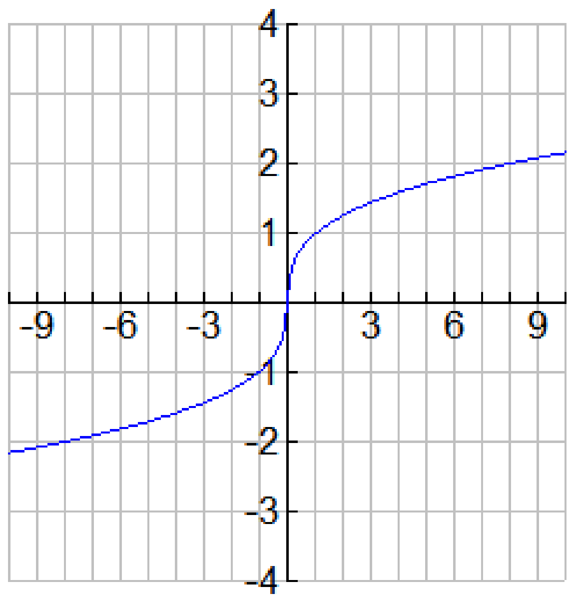
$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad ; \quad \lim_{x\to 0}\frac{1-\cos x}{x}=0$$

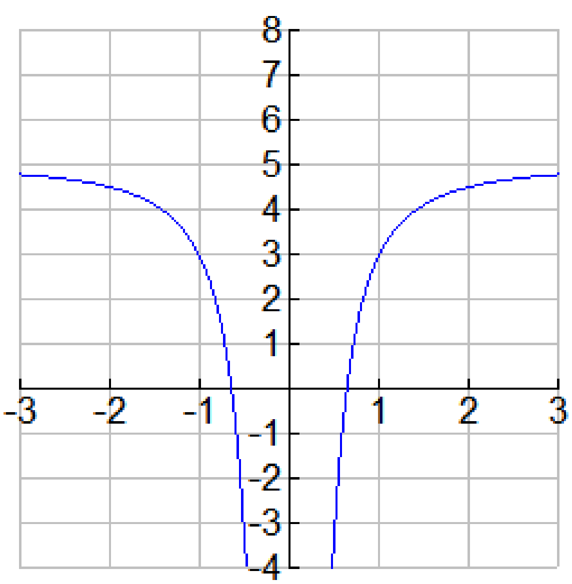
Need to remember---

$$\lim_{x\to\infty}\frac{\sin x}{x}=0$$

Consider these graphs—

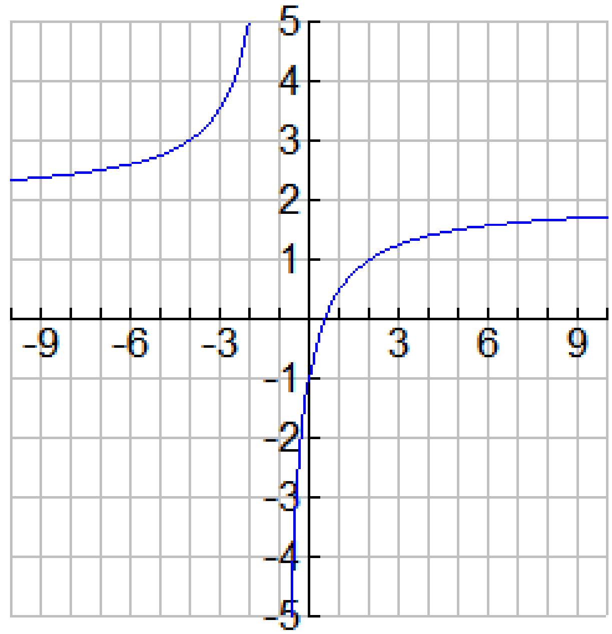
$$f(x)=x^{1/3}$$





$$f(x)=5-\frac{2}{x^2}$$

$$f(x) = \frac{2x-1}{x+1}$$



Examples:

$$\lim_{x \to \infty} = \frac{3x-2}{\sqrt{2x^2+1}}$$

$$\lim_{x \to -\infty} = \frac{3x-2}{\sqrt{2x^2+1}}$$

Chapter 3.6

Curve Sketching—

- given a function, sketch its graph
- given a function's graph, sketch its 1st & 2nd derivatives (or even derivatives of higher
- orders)
- given a derivative, sketch the function

Sketching a function—find the:

- *x* and *y* intercepts, if any
- asymptotes, vertical, horizontal & slant, if any
- possible Domain
- increasing & decreasing intervals, if any
- maximums & minimums, if any
- inflection points, if any

WAIT!! Slant Asymptotes? Remember those? A *Slant* asymptote occurs when the function is a rational function in its simplest form and the degree of the numerator exceeds the degree of the denominator by **1**.

Divide the numerator by the denominator, disregard the remainder, and the slant asymptote is y = the result.

Example: $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$

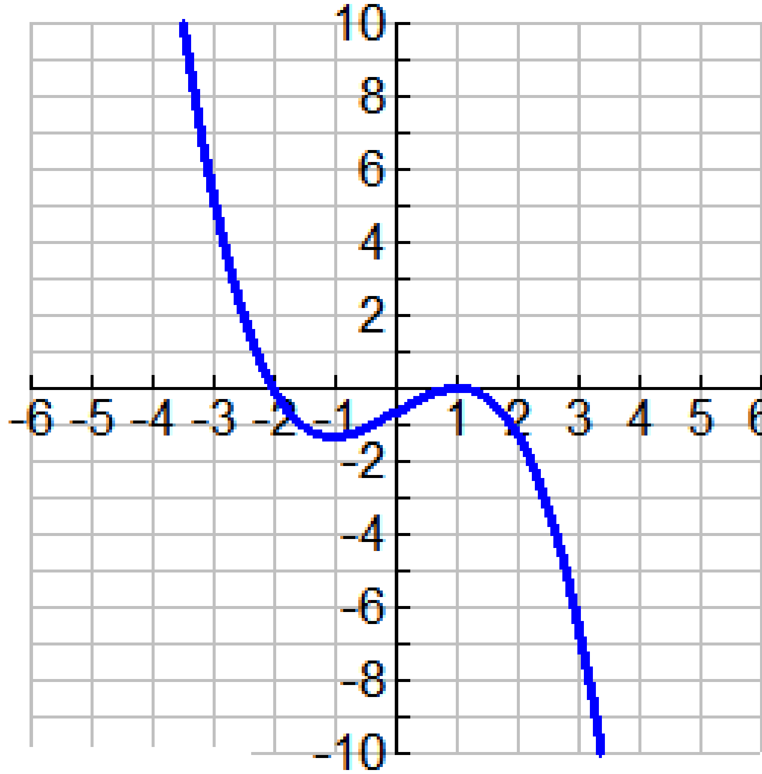
Important—Polynomial Functions
Note the 2 last paragraphs at the bottom of page 200.

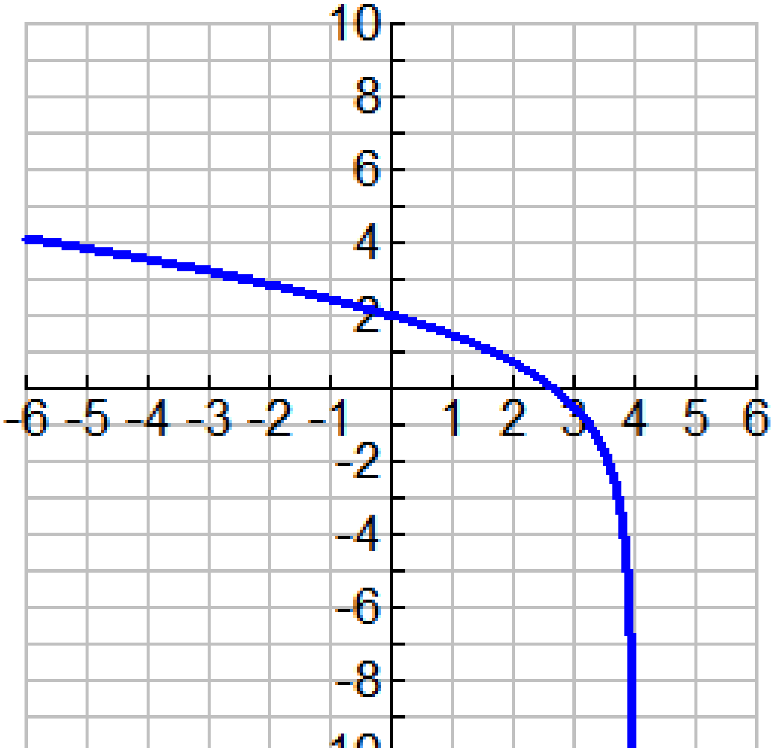
Given a graph of the derivative of a function, sketch the graph of the function and vice-versa.

Remember that the graph of a derivative is just a graph of the slopes at each point of the function. Looking at graph of the derivative as a collection of slopes is called a Slope Field. More on this later.

Sketching the derivative:

$$f(x) = -\frac{1}{3}\left(x^3 - 3x + 2\right)$$



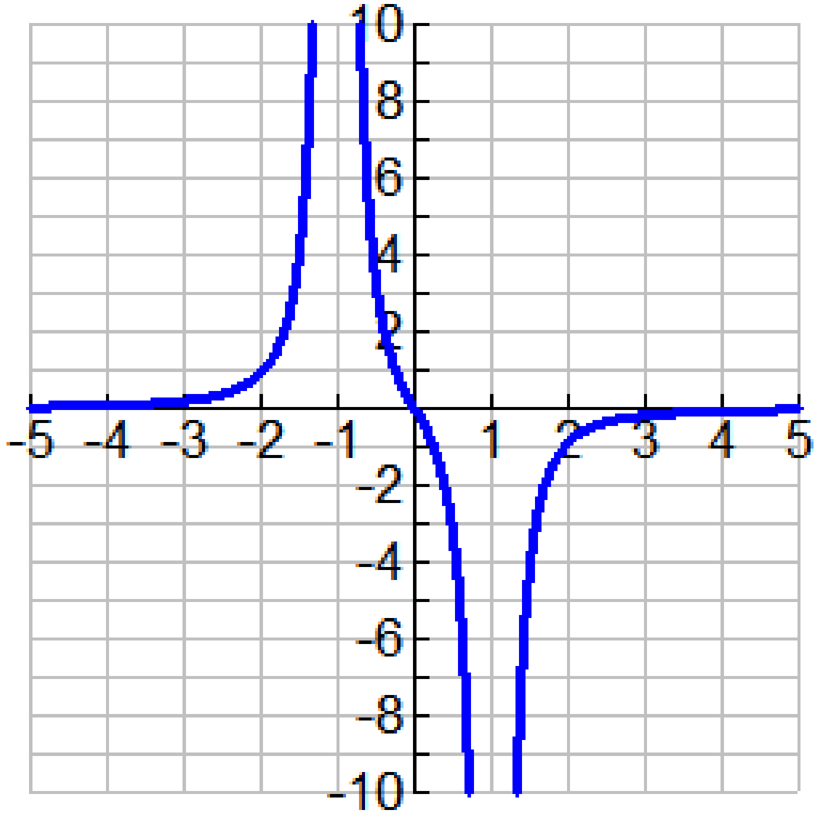


$$f'(x) = \sqrt{4-x} - \frac{x}{2\sqrt{4-x}}$$

Sketching the function

$$f(x) = x\sqrt{4-x}$$

*function is #5 on page 184



Chapter 3.7

Optimization Problems

Sort of like Related Rates

Guidelines:

1. Make a sketch.
2. Determine the Primary Equation (equation in which the max or min is being found).
3. Consider the Primary Equation—do you need a secondary equation?
4. If using a Secondary Equation, solve for needed variable & substitute into the Primary Equation
5. Simplify the Primary Equation.
6. Differentiate the Primary Equation and find the critical point (point where function is max/min).
7. Determine whether the point is a max or min.

Example: Which points on the graph of the function $y = 4 - x^2$ are closest to the point (0, 2)?

Follow guidelines……

Example: Find 2 positive numbers whose sum is $\boldsymbol{S}$ and the product is a maximum.

Chapter 3.8

Newton's Method—used for approximating the real zeros of a function

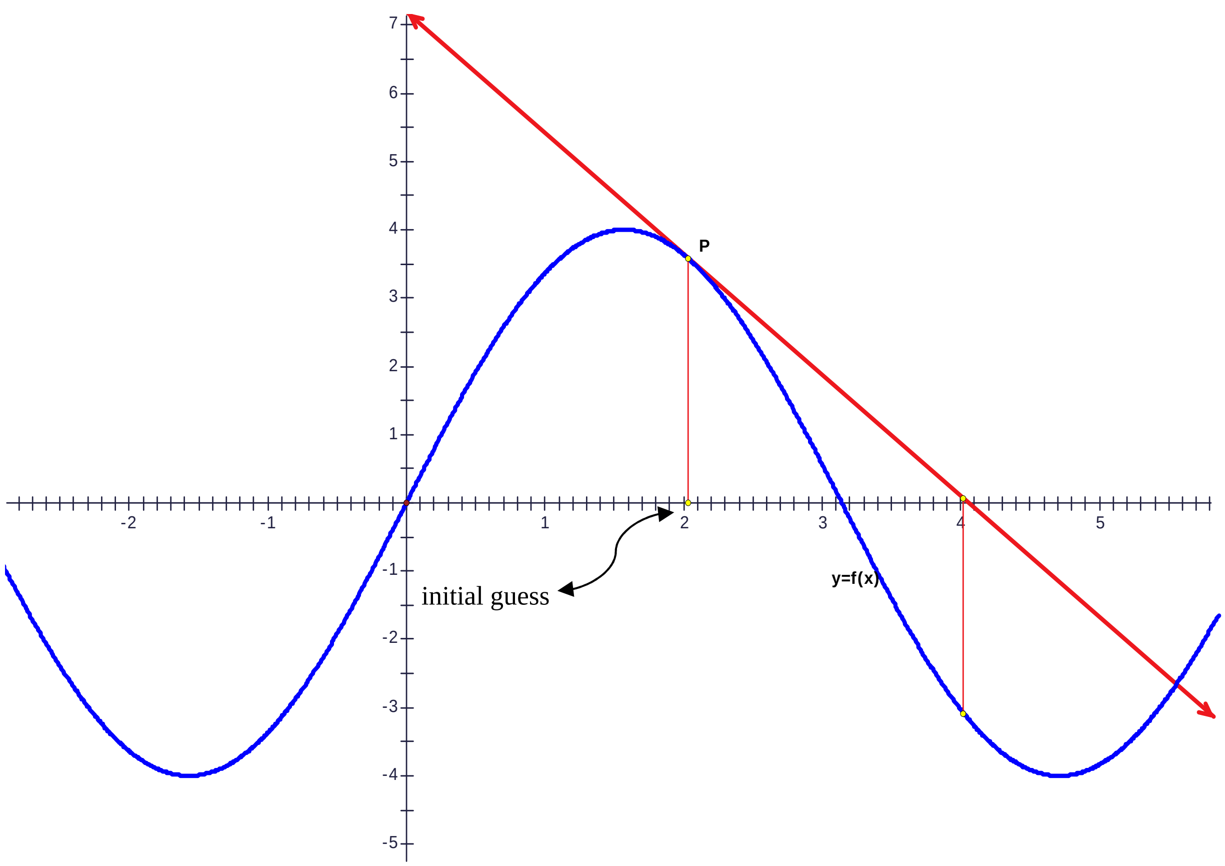
What are the ZEROS of a function? (*x-values when y = 0*)
Why are they called REAL ?

Zeros of a function are not always easy to determine. Newton's method has been around since----well, Newton (1671).

The following procedure only works for functions that are continuous on the indicated interval [*a*,*b*] and differentiable on the interval (*a*,*b*). If *f(a)* and *f(b)* differ in sign, then, by the IVT, the function must have at least one zero in the interval.

Procedure:

1. Choose an *x*-value where you think a zero might occur.
2. Using that *x*-value, determine the point (*x*, *y*) on the graph.
3. Draw a tangent at that point.
4. Using the *x*-value of the zero of the tangent line, determine another point on the graph.
5. Repeat steps 3 & 4 until……



Question?? When do you know to stop the process?

Graphically it is easy to tell but we need to duplicate this process analytically.

Equation of the tangent line:

$$y - y_1 = m\,(x - x_1)$$
$$y = m\,(x - x_1) + y_1$$

Rewriting:

$$y - f(x_1) = f'(x_1)\,(x - x_1)$$
$$y = f'(x_1)\,(x - x_1) + f(x_1)$$

Setting $y = 0$ and solving for x:

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Now: When $|x_n - x_{n+1}|$ is within the desired accuracy—an approximation within a very small value (ex: 0.00003).

Setting up a table is the best method to approach Newton's Method.

See page 216.

Chapter 3.8

Linear Approximations—graph of a function is approximated by a straight line.

In other words: Find the approximate *y*-value of a function knowing the *x*-value and using the equation of the tangent line at that *x*-value.

Similar to Newton's Method……….

Again using the Point-Slope Form:

$$y - y_1 = m\ (x - x_1)$$
$$y = m\ (x - x_1) + y_1$$

Equation of the tangent line:

$$y - f(x_1) = f'(x_1)\ (x - x_1)$$
$$y = f'(x_1)\ (x - x_1) + f(x_1)$$

Example: Find the tangent line approximation of

$$f(x) = 1 + \sin x$$, at the point