# Calculus Chapter 2

**Calculus—Chapter 2-1**

The Derivative and the Tangent Line Problem

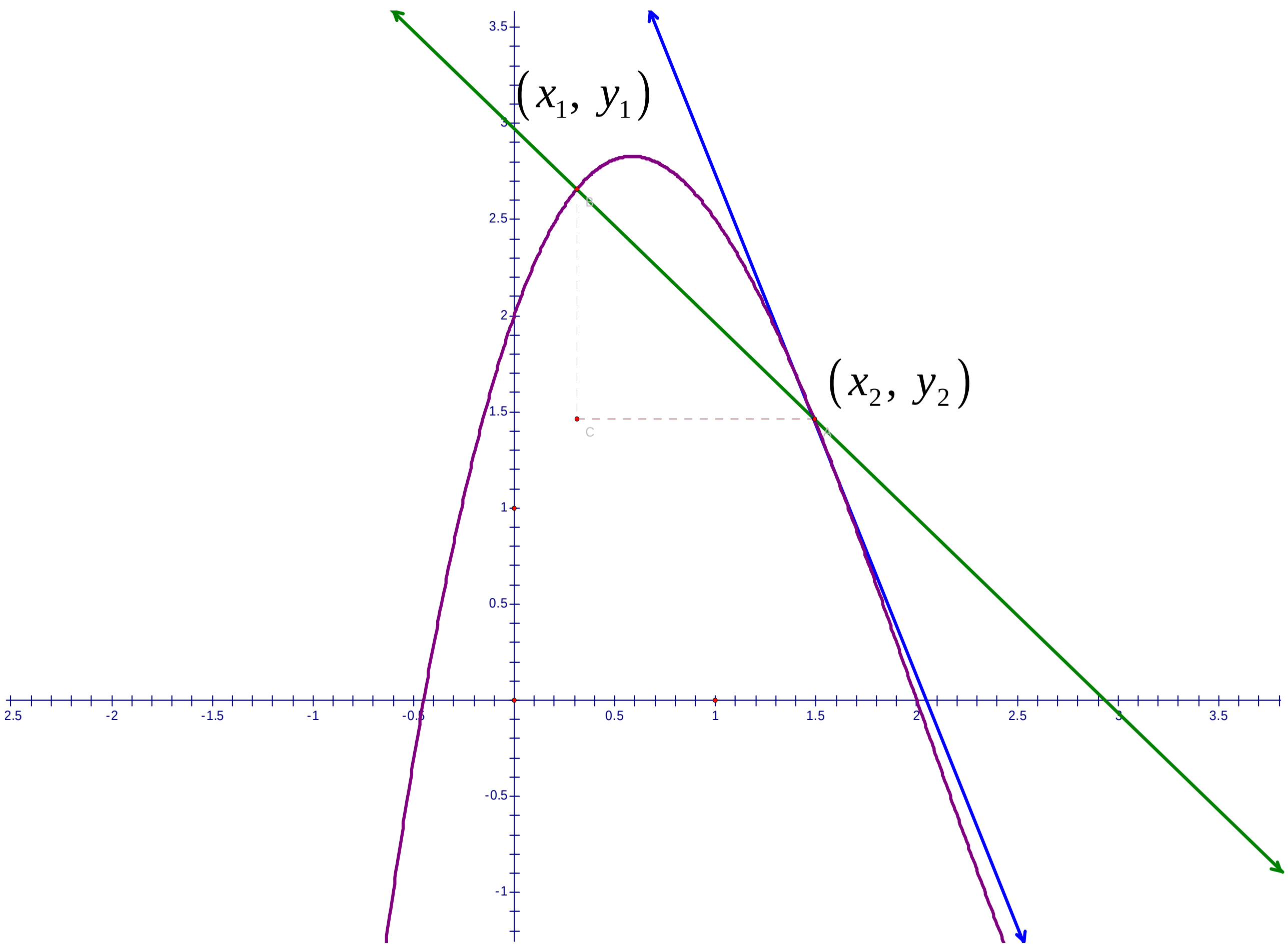
Remember???? The tangent line and the circle? Importantly remember that the tangent line touches the circle only at one point and is perpendicular to the radius.

Now the tangent line with respect to a curve is the line that touches a curve (graph) at exactly one point.

Determining the tangent line at a point $P$ is the problem of finding the slope of the tangent line at that point $P$. Not easy since you have only one point and you need 2 to find the slope of a line.

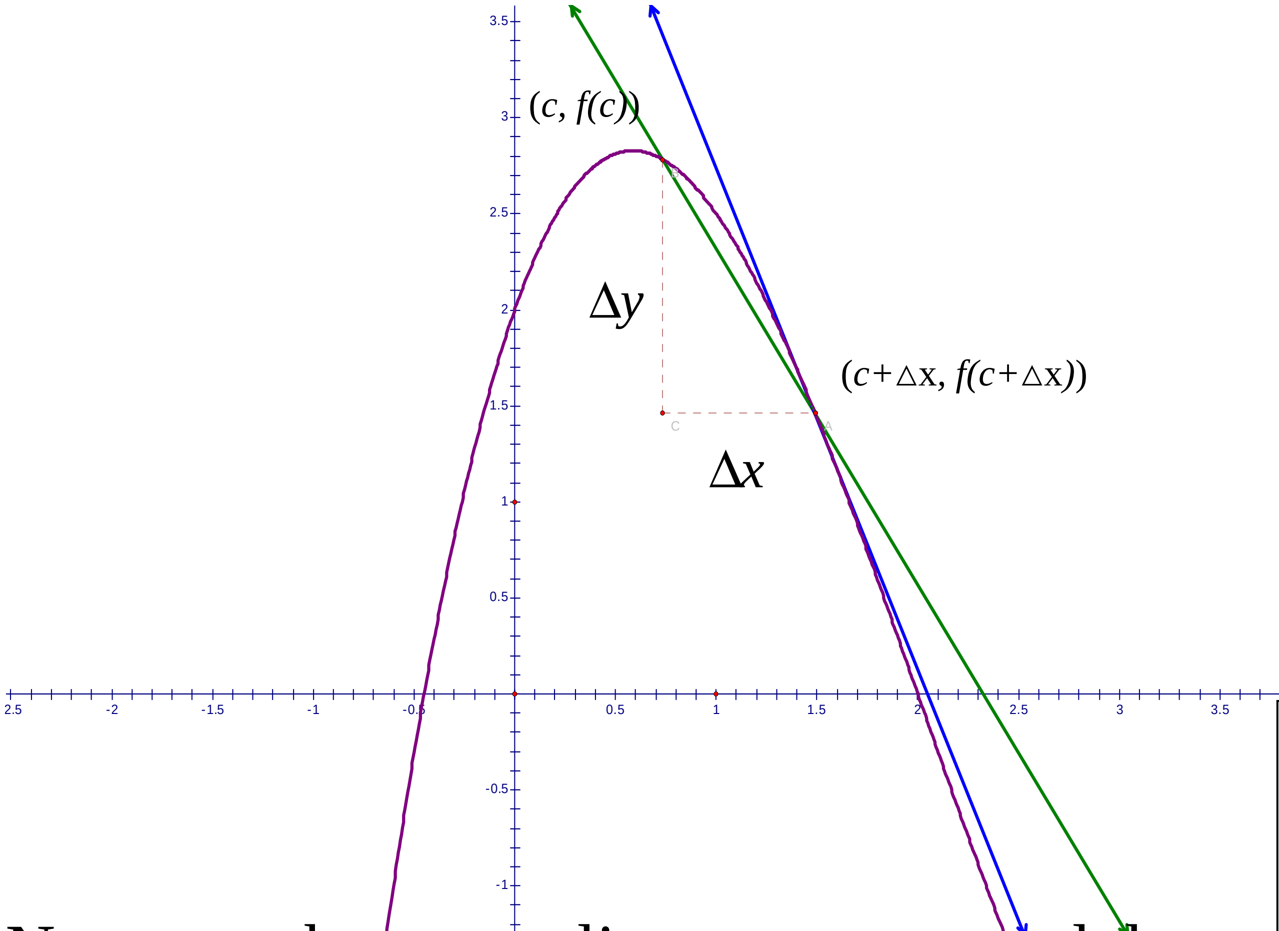
Start by approximating this slope by using a secant line.

Secant Line Graph




1. Find the slope of the secant line through the tangent point and another point on the curve.

$$\text{Slope} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$




Now—as the secant line moves toward the point of tangency the

$$\text{Slope of the Secant Line} = \frac{f(c+\Delta x)-f(c)}{(c+\Delta x)-c}$$

$$= \frac{f(c+\Delta x)-f(c)}{\Delta x}$$

The right-hand side of this equation is a ***different quotient.***

Thus to find the slope of the tangent you must

$$\lim_{\Delta x\to 0}\frac{f(c+\Delta x)-f(c)}{\Delta x}=m$$

WAIT!! Why Limit??

Ex.: Find the slope of the tangent line at the point ( 2, 1 ) of the function $f(x) = 2x - 3$ .

Wait!! Find the equation of the tangent line……
Easy------Now another example…

Find the slope of the tangent lines at the points ( 0, 1 ) and ( -1, 2 ) of the function $f(x) = x^2 + 1$.

Wait!!! Answer is in terms of *x*……What does that mean……try substituting in the *x*-value of the point under consideration.

Therefore the slope of any point ( *x*, *f(x)* ) on the above graph $f(x) = x^2 + 1$ is *2x*.

The definition of a tangent line to a curve *does not cover* the possibility of a **vertical** tangent line.

All of this leads us to the definition of the

# Derivative

Definition: The derivative of $f$ at $x$ is given by $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ provided the limit exists.

Operation/Process is called **differentiation.**

A function is **differentiable** at $\boldsymbol{x}$ if its derivative exists at $x$.

A function is **differentiable** on an open interval $(a, b)$ if it is differentiable at <u>every</u> point in the interval.

# Derivative Notation:

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y]$$

this notation is read as "the derivative of *y* with respect to *x*"

# Find the Derivative (slope of the tangent line) using these examples:

$$f(x) = x^3 + 2x \qquad f(x) = \sqrt{x} \qquad y = \frac{2}{t}$$

# Check on the calculator…….

Differentiability and Continuity
Alternative form of the derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Note: the function is differentiable on the closed interval $[a, b]$ if it is differentiable on $(a, b)$ and if the derivative from the right at $a$ and the derivative from the left at $b$ both exist.

Greatest integer function is not continuous and not differentiable where it is discontinuous.

Note: Absolute value function-----the function is **not** differentiable at the "vertex" because the derivatives from the left and from the right are not equal.

Also a function is **not** differentiable wherever there is a vertical tangent.

Thus—differentiability implies continuity.
**But** it is possible for a function to be continuous and not differentiable.

Theorem:
If $f$ is differentiable at $x = c$, then $f$ is continuous at $x = c$.

Remember—if $f$ is continuous at $x = c$ then it may or may not be differentiable at $x = c$.

# Calculus—Chapter 2-2

Differentiation Rules:

Constant Rule:

$$\frac{d}{dx}[\text{constant}] = 0$$

Power Rule:

$$\frac{d}{dx}\left[\mathrm{x}^{\mathrm{n}}\right] = n\,x^{n-1}$$

Constant Multiple Rule:

$$\frac{d}{dx}[c\,f(x)] = c\,f'(x)$$

Sum and Difference Rules:

$$\frac{d}{dx}[\,f(x)+g(x)] = f'(x)+g'(x)$$

$$\frac{d}{dx}[\,f(x)-g(x)] = f'(x)-g'(x)$$

Derivatives of Sine and Cosine:

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

*Remember—sometimes *rewriting* a function in a different form makes differentiating much easier.

ex: $$y = \frac{1}{x^3} = (1)\left(x^{-3}\right);$$

$$\frac{dy}{dx} = -3x^{-4} = \frac{-3}{x^4}$$

Knowing the slope of the tangent line and a point on the tangent line will enable you to state the equation of the tangent line *at that point* using the point-slope form.

$$y - y_1 = m\left(x - x_1\right)$$

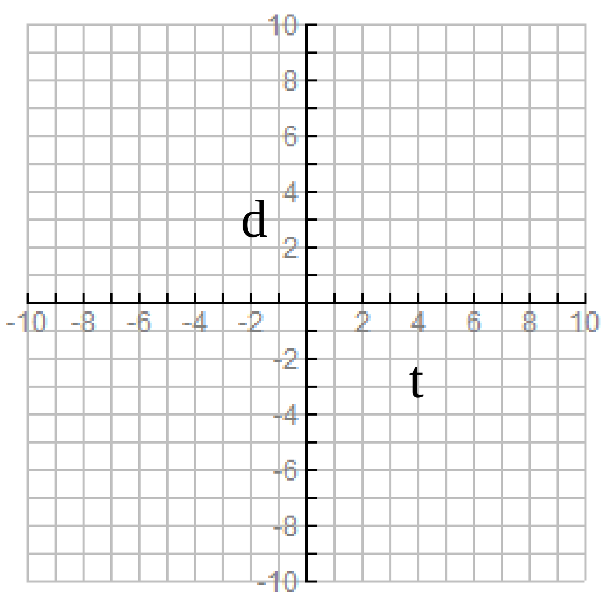
# Rates of Change

*slope* → *derivative* →*Rate of change*
of a function at a point

We all know $\text{Average Rate} = \dfrac{\Delta\text{distance}}{\Delta\text{time}}$

Thus: $\text{Average Velocity} = \dfrac{\Delta\text{distance}}{\Delta\text{time}}$

*Average Rate/Velocity* is NOT the *instantaneous* rate of change/velocity at a point but the average rate over the entire interval thus is the slope of the secant line $t_1$ to $t_2$.



# Remember the Position Function???

$$h(t) = s(t) = \frac{1}{2}gt^2 + v_0 t + s_0$$

On Earth → g= -32 ft/sec/sec or -9.8 m/sec/sec

This is just the position *s(t)* of an object at time *t* ……

Now the *instantaneous* velocity of an object is the derivative of the position function…..

$$s'(t) = v(t) = gt + v_0$$

Ex: If a billiard ball is dropped from a height of 100 feet, its height $s$ at time $t$ is given by the position function $s(t) = -16t^2 + 100$. Why??

Find the average velocity ---- over the interval [ 1, 2 ].

| t | 1 | 2 |
|---|---|---|
| s(t) | 84 | 36 |

Find the instantaneous velocity when $t = 1.5$ .

$$s'(t) = v(t) = -32t;$$
$$v(1.5) = -32(1.5) = -48\,\text{ft/sec}$$

IMPORTANT—the average velocity and the instantaneous velocity are not the same.

Usually when a problem asks for just the velocity, assume that you need to find the instantaneous velocity.

# Calculus—Chapter 2-3

Another Rule to memorize…..

Product Rule:

$$\frac{d}{dx}[\,f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2},\ \ g(x) \neq 0$$

More Trig derivatives to memorize---

$$\frac{d}{dx}[\tan x] = \sec^2 x;$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x;$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x;$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x;$$

# Higher-Order Derivatives

Just as the velocity function can be obtain by differentiating a position function, the acceleration function can be obtain by differentiating a velocity function.

Thus: $s(t)$

$$s'(t) = v(t)$$

$$s''(t) = v'(t) = a(t)$$

# **Calculus—Chapter 2-4**

Even MORE Rules…….

Chain Rule:

$$\frac{d}{dx}\left[\, f\left(g(x)\right)\right] = f'(g(x))g'(x) = g'(x)\, f'(g(x))$$

Note: *f(x)* is the outer (outside) function
*g(x)* is the inner (inside) function

**Important:** you should know the Differentiation Rules on page 130.

**Calculus—Chapter 2-5**

Explicit & Implicit functions

- Explicit Form→ $y = \frac{1}{x}$
- Implicit Form→ $x \cdot y = 1; \quad x^2 - 2y^3 + 4y = 2$

Remember ????? $\frac{dy}{dx}$ means differentiate the function $y$ with respect to $x$. In other words—find the general slope of the tangent line, $\frac{\Delta y}{\Delta x}$. Finding the general slope is a straight forward procedure with an explicit function.

WHY?? When you differentiate both sides you end up with $\frac{dy}{dx}$ which is the derivative, which is the slope……which is $\frac{\Delta y}{\Delta x}$.

Ex: $y = 2x^3 + x$

Differentiate: $\frac{d}{dx} y^1 = \frac{d}{dx} 2x^3 + \frac{d}{dx} x^1$ **Note the variables**

Using the Chain Rule:

$$(1) \frac{dy}{dx} = 6x^2 + 1$$

Where $\frac{dy}{dx}$ is the derivative of the function…

Suppose you have an implicit function?????

$$x^2 - 2y^3 + 4y = 2$$

What happens if you use the standard procedure to differentiate an implicit function? Can you solve for $y$ as a function of $x$? NO!

What happens?

Thus, function must be differentiate using the Chain Rule-----

$$\frac{d}{dx}x^2 - 2\frac{d}{dx}y^3 + 4\frac{d}{dx}y = 0$$

$$2x - 2(3)y^2\frac{dy}{dx} + 4(1)\frac{dy}{dx} = 0$$

Now solve for $\frac{dy}{dx}$ -----

$$\frac{dy}{dx}\left(-6y^2 + 4\right) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{\left(-6y^2 + 4\right)}$$

Finally, ☺ Substitute for $y$ if possible.

How can you enter this into your calculator?

$$x^2 - 2\left(y(x)\right) \wedge 3 + 4y(x) = 0$$

$y(x)$ because the calculator needs to know that "*y*" is a function of *x*.

Next differentiate by entering this

$$d\left(ans(1), \mathrm{x}\right) \qquad \langle \text{2nd} \rangle \langle (\text{-}) \rangle$$

Now you need to take the new *ans*(1) and substitute all of the $\frac{d}{dy}\left(y(x)\right)$ with *yp*

stands for *y* Prime

$$ans(1) \mid d\left(y(x), x\right) = yp$$

Next—solve *ans*(1) for *yp*

solve(*ans*(1),*yp*) which is just $\frac{d}{dy}\left(y(x)\right)$

---check out the answer---

*****Not easy on the calculator *****

**Calculus—Chapter 2-6**

Related Rates→Problems in which you try to find the rate at which some quantity is changing by relating it to some other quantities whose rates of change are known……

Ex: Pebble dropped into a pond…..

------the radius of the outer ripple is increasing at a *constant* rate of 1 ft/min

------At what rate is the total area, $A$, created by the ripple changing when the radius is 4 ft.

In other works—find the rate of change of the Area when the radius is 4 ft.

Note: What variable(s) are changing???

STEPS:

1. Sketch & label – ID given quantities & quantities to be determined IMP: determine what variable(s) are changing and what variable(s) are held constant.
2. Write an equation that includes the variables that are given/to be determined
3. Differentiate both sides of the equation with respect to **TIME** → ($t$)
4. Finally – substitute known values and solve

Follow the steps with the Pebble example

Ex: Speed of an airplane -- distance to station

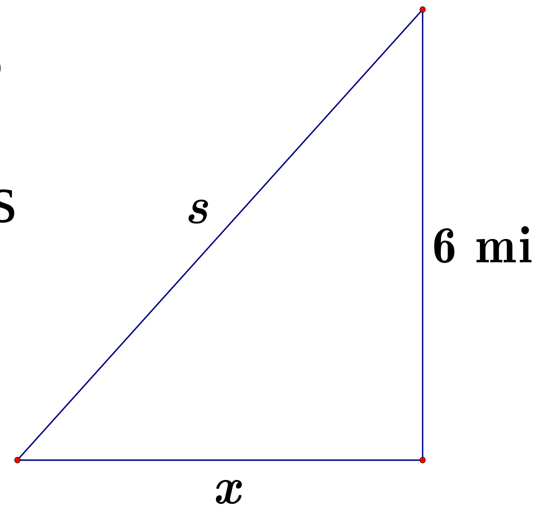
decreasing at rate of 400 m/hr when *s=10* miles

What is the speed of the Plane?

Variable(s) changing???

What variable represents

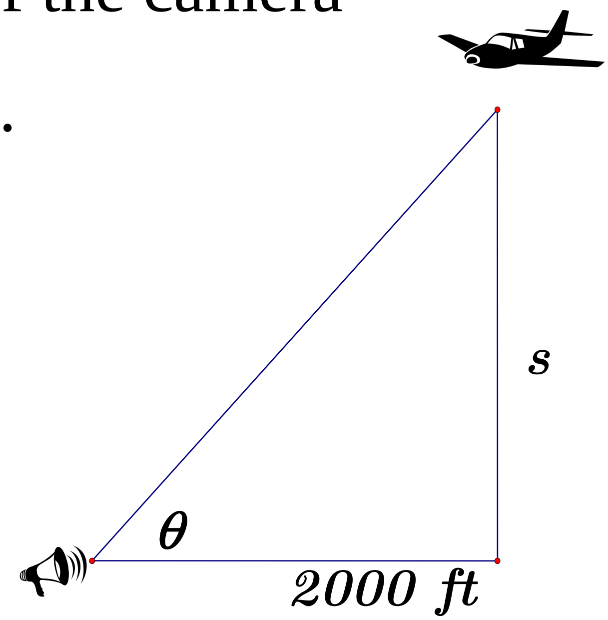
the speed?




Differentiate with respect to is TIME…..WHY?

Follow the steps-------

Ex: Angle of elevation—Find the rate of change in the angle of elevation of the camera at 10 seconds after lift-off.




What variables are changing?? ***s*** & ***θ***

Is there an equation that relates ***s*** & ***θ*** **????** $\tan\theta = \frac{s}{2000\,ft}$

**Again differentiate with respect to *time*.**

**Something to remember—do not substitute values until you have differentiated.**