Problem Set # 9

(No need to submit)

1. Consider the operator

$$Lu = u'' + u, \quad 0 < x < 2p$$

with boundary conditions established by the linear combinations

$$B_1u = u(0) + u'(0), \qquad B_2u = u'(2\pi) - u(2\pi)$$

(a) Find the adjoint $L^\dagger$ of L and adjoint boundary functional $B_1^\dagger$ and $B_2^\dagger$ for the boundary conditions $B_1u = B_2u = 0$. Is the given problem self-adjoint?

(b) Does L have a homogeneous solution with the boundary conditions $B_1u = 0$ and $B_2u = 0$?

(c) What is the solvability condition for the problem $Lu = f$, $B_1u = B_2u = 0$?

(d) What is the solvability condition for the problem $Lu = f$, $B_1u = \gamma_1$, $B_2u = \gamma_2$?

(e) Find the solution to $Lu = 2$; $B_1u = 0$, $B_2u = 0$.

2. Consider the operator

$$Lu = u''' = \frac{d^3u}{dx^3}, \quad 0 < x < 1$$

with boundary functions given by

$$B_1u = u(0), \qquad B_2u = u'(1) - u(1), \qquad B_3u = u''(0)$$

(a) Find the adjoint $L^\dagger$ of L and adjoint boundary functional $B_1^\dagger$, $B_2^\dagger$ and $B_3^\dagger$ for the boundary conditions $B_1u = B_2u = B_3u = 0$. Is the given problem self-adjoint?

(b) Does L have a homogeneous solution with the boundary conditions $B_1u = 0$, $B_2u = 0$, $B_3u = 0$?

(c) What is the solvability condition for the problem $Lu = f$, $B_1u = B_2u = B_3u = 0$?

(d) Find the necessary conditions on f(x) for which the equation Lu = f can have a solution if

$$Lu = u''' = f, \; B_1u = u(0) = \alpha, \; B_2u = u'(0) = \beta, \; B_3u = u''(0) = \gamma$$

3. Find the necessary conditions on f for which the equation Lu = f can have a solution if

$$Lu = -u'' - u,\ u(\pi) - u(-\pi) = \alpha,\ u'(\pi) - u'(-\pi) = \beta.$$

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Q 1.

1. $$Lu = u'' + u, \quad 0 < x < 2p$$

$$B_1 u = u(0) + u'(0)$$
$$B_2 u = u'(2p) - u(2p)$$

(a) $$< v, Lu > = < L^+ v, u >$$

$$< v, Lu > = \int_0^{2p} (u'' + u)\, v \ dx$$

$$= \int_0^{2p} u'' v \ dx + \int_0^{2p} uv \ dx$$

$$= u'v \Big|_0^{2p} - \int_0^{2p} u'v' \ dx + \int_0^{2p} uv \ dx$$

$$= u'v \Big|_0^{2p} - uv' \Big|_0^{2p} + \int_0^{2p} v'' u \ dx + \int_0^{2p} uv \ dx$$

$$= (vu' - v'u) \Big|_0^{2p} + \int_0^{2p} (v'' + v) u \ dx$$

$$= J(u,v) + < L^+ v, u >$$

$$\therefore\ L^+ = v'' + v$$

$$J(u,v) = (vu' - v'u) \Big|_0^{2p}$$

$$J(u,v) = v(2p)\,u'(2p) - v'(2p)\,u(2p) - v(0)\,u'(0) = 0$$

$$= [u(0) + u'(0)]\,v'(0) + [u'(2p) + u(2p)]\,v(2p) - [v'(2p) + v(2p)]\,u(2p)$$

$$-[v'(0) + v(0)]\,u'(0) = 0$$

$$= [B_1 u][B_4^+ v] + [B_2 u][B_3^+ v] + [B_3 u][B_2^+ v] + [B_4 u][B_1^+ v] = 0$$

$$B_1 u = u(0) + u'(0) = 0 \quad (given)$$

$$B_2 u = u'(2p) - u(2p) = 0 \quad (given)$$

$$B_2^+ v = 0 = v'(2p) - v(2p)$$

$$B_1^+ v = 0 = v(0) + v'(0)$$

Adjoint boundary condition

$\therefore L = L^+ ,\; B_1 u = B_1^+ v \;\; and \;\; B_2 u = B_2^+ v$

$\therefore$ Problem is self adjoint.

(b) Does L have a homogeneous solution for

$$B_1 u = 0, \; B_2 u = 0$$

$$Lu = u'' + u = 0$$

Solution :

$$u = c_1 \sin x + c_2 \cos x$$

$$u = c_1 \cos x - c_2 \sin x$$

$$u'' = -c_1 \sin x - c_2 \cos x$$

Apply BC $\Rightarrow$

$$c_1 \sin 0 + c_2 \cos 0 + c_1 \cos 0 - c_2 \sin 0 = 0$$

$$\Rightarrow c_1 + c_2 = 0$$

$$c_1 \cos 2p - c_2 \sin 2p - c_1 \sin 2p - c_2 \cos 2p = 0$$

$$\Rightarrow c_1 - c_2 = 0$$

$$\Rightarrow c_1 = c_2 = 0 \leftarrow contradiction$$

It has only trivial solution.

(c) Lu = f , B_1u = 0 , B_2 u = 0 have unique solution.

< f , v > = 0 will be always satisfied . Since v = 0 is the only solution for the adjoint problem.

(d) Lu = f , B_1u = g_1 , B_2 u = g_2

Solvability Condition;

$$\int_0^{2p} f(x)v(x)dx = [B_1 u][B_4^+ v] + [B_2 u][B_3^+ v]$$

$$= g_1(0) + g_2(0)$$

$$= 0$$

Since $B_4^+ v = v'(0) = 0$

$B_3^+ v = v'(0) = 0$

$\therefore$ Always unique solution.

(e) $Lu = 2$, $B_1 u = 0$, $B_2 u = 0$

General Solution:

$u_{ih}(x) = c_1 u_1(x) + c_2 u_2(x) + u_i(x)$

Where $u_1(x) = \sin x$, $u_2(x) = \cos x$ and $u_i(x)$ is solution of

$u'' + u = 2$, $u(0) = 0$, $u'(0) = 0$

$u_i(x) = c_1(x)\sin x + c_2(x)\cos(x)$

Find c_1 and c_2 from $c_1' \sin x + c_2' \cos x = 0$ and

$$c_1' \cos x - c_2' \sin x = 2$$

$$c_1' = \frac{\begin{vmatrix} 0 & \cos x \\ 2 & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & 0 \\ \cos x & -\sin x \end{vmatrix}} = \frac{-2\cos x}{-1} = 2\cos x$$

$$c_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & 2 \end{vmatrix}}{-1} = -2\sin x$$

$\therefore u_{ih}(x)$

$\therefore u_{ih}(x) = = c_1 \sin x + c_2 \cos x + 2 - 2\cos x$

Apply Boundary Condition 1:

$u(0) + u'(0) = 0$

or $c_1 \sin 0 + c_2 \cos 0 + 2 - 2 \cos 0 + c_1 \cos 0 - c_2 \sin 0 + 2 \sin 0 = 0$

or $c_2 + 2 - 2 + c_1 = 0$

or $c_2 + c_1 = 0$

Apply Boundary Condition 2:

$u'(2p) - u(2p) = 0$

$c_1 \cos 2p - c_2 \sin 2p + 2\sin 2p - c_1 \sin 2p - c_2 \cos 2p - 2 + 2 \cos 2p = 0$

$c_1 - c_2 = 0$

$c_1 = c_2 = 0$

$u_{ih}(x) = 2 - 2 \cos x = 2(1 - \cos x)$ is the solution of $Lu = u'' + u = 2$ and

$$B_1 u = u(0) + u'(0)$$
$$B_2 u = u'(2p) - u(2p)$$

Q 2.

$$Lu = u'''$$
$$B_1 u = u(0) = 0$$
$$B_2 u = u'(1) - u(1) = 0$$
$$B_2 u = u''(0) = 0$$
$$< v, Lu > = < L^+ v, u >$$

$$< v, Lu >= \int_0^1 v u''' dx$$

$$= v u'' \Big|_0^1 - \int_0^1 v' u'' dx$$

$$= \left[v u'' - v' u' \right]_0^1 + \int_0^1 v'' u' dx$$

$$= \left[v u'' - v' u' + v'' u \right]_0^1 - \int_0^1 v''' u dx$$

$$= J(u,v) + < L^+ v, u >$$

Where

$$L^+ = -v'''$$

and

$$J(u,v) = \left[v u'' - v' u' + v'' u \right]_0^1$$

$$J(u,v) = 0 \Rightarrow$$

$$v(1)u''(1) - v'(1)u'(1) + v''(1)u(1) - v(0)u''(0) + v'(0)u'(0) - v''(0)u(0) = 0$$

or

$$-u(0)v''(0) - \left[u'(1) - u(1) \right] v'(1) - v(0)u''(0) - u(1)v'(1) + v(1)u''(1) + v''(1)u(1) + v'(0)u'(0) = 0$$

or

$$-(B_1 u)v''(0) - (B_2 u)v(1) - (B_3 u)v(0) + u(1)\left[-v'(1) + v''(1) \right] + u''(1)v(1) + u'(0)v'(0) = 0$$

$$\therefore v'(1) - v''(1) = 0$$

$$v(1) = 0, and$$

$$v'(0) = 0$$

$$\therefore L^+ = -v'''$$

$$B_1^+ v = v'(0) = 0, B_2^+ v = v(1) = 0$$

$$B_3^+ v = v'(1) - v''(1) = 0 \Leftarrow$$

Q 3.

$$Lu = -u'' - u = f$$

$$B_1 u = u(p) - u(-p) = a$$

$$B_2 u = u'(p) - u'(-p) = b$$

The sufficient conditions for the above to have solutions are:

$$\int_{-p}^{p} fvdx = a(B_4^+ v) + b(B_3^+ v)$$

Where v is the solution of

$$L^+ v = 0$$

$$B_1^+ v = 0$$

$$B_2^+ v = 0$$

$$< v, Lu > = < L^+ v, u >$$

$$< v, Lu > = \int_{-p}^{p} (-u'' - u) vdx$$

$$or$$

$$-\int_{-p}^{p} u'' vdx - \int_{-p}^{p} uvdx$$

But, $\int_{-p}^{p} u'' vdx = (vu' - v'u)\Big|_{-p}^{p} + \int_{-p}^{p} uv'' dx$

$$or,$$

$$< v, Lu > = (v'u - vu')\Big|_{-p}^{p} + \int_{-p}^{p} -(v'' + v)udx$$

$$or,$$

$$L^+ = -v'' - v$$

and $J(u,v) = (v'u - vu')\Big|_{-p}^{p} = 0$

$$v'(p) - u(p) - v(p)u'(p) - v'(-p)u(p) + v(-p)u'(-p) = 0$$

$$[u(p) - u(-p)]v'(p) + [u'(p) - u'(-p)](-v(p)) + u(-p)[v'(p) - v'(-p)] + u'(-p)[-v(p) + v(-p)] = 0$$

$$or,$$

$$(B_1 u)(B_4^+ v) + (B_2 u)(B_3^+ v) + (B_3 u)(B_2^+ v) + (B_4 u)(B_1^+ v) = 0$$

$$B_4^+ v = v'(p)$$

$$B_3^+ v = -v(p)$$

$$B_2^+ v = v'(p) - v'(-p) = 0$$

$$B_1^+ v = -v(p) + v(-p) = 0$$

$$Now,$$

$$L^+ v = -v'' - v = 0$$

General solution is

$v(x) = A \cos x + B \sin x$

Applying the homogeneous boundary conditions;

$v' = -A \sin x + B \cos x$

Boundary condition 1

$$-A \sin p + B \cos p + A \sin(-p) - B \cos(-p) = 0$$
$$or, \quad -B + B = 0$$

Boundary condition 2

$$-A \cos p - B \sin p + A \cos(-p) + B \sin(-p) = 0$$
$$or, \quad A - A = 0$$

Therefore,

$u(x) = A \cos x + B \sin x$ is the solution with A and B being arbitrary.

$\therefore$ Solvability condition is

$$\int_{-p}^{p} (A \cos x + B \sin x) f(x) dx = -aB + bA$$

Where A and B are arbitrary constants and are not interdependent.

Note that

$$v(x) = A\cos x + B\sin x$$
$$v'(x) = -A\sin x + B\cos x$$

$$B_4^+ v = v'(p) = -A\sin p + B\cos p$$
$$= B$$
$$B_3^+ v = -v(p) = -A\cos p + B\sin p$$
$$= A$$