Problem Set # 7

1.

(a) Is **A** self-adjoint?

(b) Find the eigenvalues and eigenvectors of **A**.

(c) Are its eigenvectors orthogonal to one another?

(d) Solve the equation

$$\frac{dx}{dt} = -Ax, \quad x(t=0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

2. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

X = UZ
X(1) = UZ(1)
Z(1) = U (a) Determine **P** and **J** matrix such that **P**$^{-1}$ **A P** = **J**.

(b) Solve the equation

$$^{-1}\frac{dx}{dt} = -Ax, \quad x(t=0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad X(1) =$$

$$\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 3/\sqrt{5} \end{bmatrix}$$

$\therefore$ We have $\frac{d}{dt} z_1 = -5z_1 - \frac{4}{\sqrt{5}}, \quad z_1(1) = \frac{1}{\sqrt{5}}$

3. Consider the following chemical reaction system:

$$A \xrightarrow{k_1} B \underset{k_3}{\overset{k_2}{\rightleftarrows}} C$$

Assume that all steps are first-order reactions. Write the set of linear ordinary differential equations, which describe the kinetics of these reactions. The values of the kinetic rate constants are

$$k_1 = 1,\ \ k_2 = 2,\ \ k_3 = 3$$

The initial conditions for the three components are

$$A_o = 1,\ \ B_o = 0,\ \ C_o = 0$$

(a) Determine the eigenvalues and eigenvectors of the matrix of kinetic rate constants.

(b) Find C_A (t), C_B(t) and C_C(t) using eigenvalues and eigenvectors.

4. Consider the following second-order linear differential equation

$$\frac{d}{dt} z_2 = \frac{5}{\sqrt{5}\ t} + \frac{8}{\sqrt{5}} \frac{d^2 x}{dt^2} - 3\frac{dx}{dt} - 10x = 5$$

The initial conditions for this equation are, at t = 0,

$$x\ (o) = 3 \ \text{ and } x'\ (o) = 15$$

(a) Transform the above differential equation into a set of first-order linear differential equations with appropriate initial conditions.

(b) Find the solution using eigenvalues and eigenvectors method.

5. Find the solution of the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix}, \quad \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Problem Set # 7

Q 1.

(1)

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(a) Yes, A is self adjoint.
(b) To find eigenvalue,

$$f(l) = l^2 - 4l + 3 = 0 \Rightarrow l = 3,1$$

$$For \quad l = 3$$

$$(A - lI) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$For \quad l = 1,$$

$$A - lI = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$Adj(A - lI) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\text{f}v_2 = b\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Where a and ß are arbitrary values

(c) $$\langle v_1, v_2 \rangle = \sum_{i=1}^{2} v_i v_j = 1 - 1 = 0$$

Its eigenvectors are orthogonal each other.

$$= \begin{bmatrix} e^{\frac{3t}{\sqrt{2}}} & e^{\frac{t}{\sqrt{2}}} \\ e^{\frac{3t}{\sqrt{2}}} & e^{\frac{-t}{\sqrt{2}}} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2(e^{3t} + e^t) \\ 1/2(e^{3t} - e^t) \end{bmatrix}$$

Q 2.

(2) $$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

(a) Determine P & J matrix such that $P^{-1}AP=J$

To determine eigenvalues,

$$f(l) = l^2 - 4l + 4 = 0 \qquad \Rightarrow l = 2,2$$

$$For \quad l = 2$$

$$(A - lI) = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$Adj\,(A - lI) = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$v_1 = a\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(A - lI)P = v_1$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$-p_1 - p_2 = 1 \qquad \Rightarrow p_1 = 1, p_2 = -2$$

$$p_1 + p_2 = -1 \qquad \Rightarrow p = a\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$p = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$P^{-1}AP = J$$

$$\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = J$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = J$$

(b) Solve the equation

$$\frac{dx}{dt} = -Ax, \qquad x(t=0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x(t=0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To determine eigenvalues,

$$f(l) = l^2 + 4l + 4 = 0 \qquad \Rightarrow l = -2, -2$$

$$\text{f}(-A - lI) = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$Adj\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$(-A - lI)v_1 = 0$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + v_2 = 0 \qquad \Rightarrow p_1 = 1 \;\; , p_2 = 0$$

$$-p_1 - p_2 = -1$$

$$p = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$p^{-1}Ap = J$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = J$$

$$\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} = J$$

$$x(t) = pe^t p^{-1}x_0$$

$$p^{-1}x_0 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x(t) = pe^t p^{-1}x_0$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} e^{-2t} & 1 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 + e^{-2t} \\ -1 \end{bmatrix}$$

Q 3.

$$A \xrightarrow{k_1} B \underset{k_3}{\overset{k_2}{\rightleftarrows}} C$$

$$\frac{\partial C_A}{\partial t} = -k_1 C_A$$

$$\frac{\partial C_B}{\partial t} = k_1 C_A - k_2 C_B + k_3 C_C$$

$$\frac{\partial C_C}{\partial t} = k_2 C_B - k_3 C_C$$

$$\frac{d}{dt}\begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & k_3 \\ 0 & k_2 & k_3 \end{bmatrix}\begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix}, \quad \begin{bmatrix} C_{A_0} \\ C_{B_0} \\ C_{C_0} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 3 \\ - & 2 & -3 \end{bmatrix}\begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix}$$

$$\frac{dC}{dt} = KC \qquad C(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix}, \qquad K = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 3 \\ 0 & 2 & -3 \end{bmatrix}, \qquad C(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

eigenvalues of K,

$$f(l) = \det(\mathbf{K} - l\mathbf{I}) = \det\begin{bmatrix} -1-l & 0 & 0 \\ 1 & -2-l & 3 \\ 0 & 2 & -3-l \end{bmatrix}$$

$$0 = -l^3 - 6l^2 - 5l$$

$$0 = l(l+5)(l+1)$$

$$l_1 = 0, l_2 = -1, l_3 = -5$$

Eigenvector of K for $l_1 = 0$

$$adj\mathbf{K} = adj\begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 3 \\ 0 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\mathbf{v}_1 = a\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Eigenvector of K for $l_2 = -1$

$$adj(\mathbf{K}-\mathbf{I}) = adj\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 3 \\ 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Eigenvector of K for $l_3 = -5$

$$adj(\mathbf{K}+5\mathbf{I}) = adj\begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 3 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 8 & -12 \\ 2 & -8 & 12 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 0 & -2 & 0 \\ 3 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\det \mathbf{U} = 10$$

$$\mathbf{U}^{-1} = \frac{1}{10}\begin{bmatrix} 2 & 2 & 2 \\ -5 & 0 & 0 \\ 1 & -4 & 6 \end{bmatrix}$$

$$C(t) = \mathbf{U}(e^{\Lambda t})\mathbf{U}^{-1}\mathbf{C}_0$$

$$\begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 3 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} e^0 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-5t} \end{bmatrix}\frac{1}{10}\begin{bmatrix} 2 & 2 & 2 \\ -5 & 0 & 0 \\ 1 & -4 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2e^{-t} & 0 \\ 3 & e^{-t} & -e^{-5t} \\ 2 & e^{-t} & e^{-5t} \end{bmatrix}\frac{1}{10}\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$C_A = e^{-t}$

$C_B = \frac{1}{10}(6 - 5e^{-t} - e^{-5t}) = 0.6 - 0.5e^{-t} - 0.1e^{-5t}$

$C_C = \frac{1}{10}(4 + 5e^{-t} + e^{-5t}) = 0.4 + 0.5e^{-t} + 0.1e^{-5t}$

Q 4.

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} - 10x = 5$$

Let $x = x_1$

$$\frac{dx}{dt} = \frac{dx_1}{dt} = x_2 \qquad , x_1(0) = 3$$

$$\frac{d^2x}{dt^2} = \frac{dx_2}{dt} = 3x_2 + 10x_1 + 5 \qquad , x_2(0) = 15$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax} + \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$$

Eigenvalues of **A**,

$$(\mathbf{A} - l\mathbf{I}) = \begin{bmatrix} 0 - l & 1 \\ 10 & 3 - l \end{bmatrix}$$

$$(\mathbf{A} - l\mathbf{I}) = l^2 - 3l - 10$$

$$0 = (l - 5)(l + 2)$$

$l_1 = 5, l_2 = -2$

for $l_1 = 5$

$$adj(\mathbf{A} - l_1\mathbf{I}) = adj\begin{bmatrix} -5 & 1 \\ 10 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -10 & -5 \end{bmatrix} \Rightarrow \mathbf{v}_1 = a\begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$adj(\mathbf{A} - l_2\mathbf{I}) = adj\begin{bmatrix} 2 & 1 \\ 10 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -10 & 2 \end{bmatrix} \Rightarrow \mathbf{v}_2 = a\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$x = \sum_{i=1}^{n} c_i \mathbf{v}_i e^{l_i t} - \mathbf{A}^{-1} b$$

$$= c_1 \mathbf{v}_1 e^{l_1 t} + c_2 \mathbf{v}_2 e^{l_2 t} - \mathbf{A}^{-1} b$$

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t} - \frac{1}{-10} \begin{bmatrix} 3 & -1 \\ -10 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$$

Apply initial conditions at $t = 0$, $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$

$$\begin{bmatrix} 3 \\ 15 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$$

$$c_1 - c_2 - \frac{1}{2} = 3 \Rightarrow c_1 - c_2 = 3\frac{1}{2} \qquad \cdots \text{eqn I}$$

$$5c_1 + 2c_2 + 0 = 15 \Rightarrow 5c_1 + 2c_2 = 15 \qquad \cdots \text{eqn II}$$

$$\begin{aligned} \text{I} \times 2 \cdots \qquad & 2c_1 - 2c_2 = 7 \\ \text{II} \ldots \qquad & 5c_1 + 2c_2 = 15 \\ & 7c_1 = 22 \\ & c_1 = \frac{22}{7} \end{aligned}$$

$$2\left(\frac{22}{7}\right) - 2c_2 = 7$$

$$c_2 = -\frac{5}{14}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{22}{7} \begin{bmatrix} e^{5t} \\ 5e^{5t} \end{bmatrix} - \frac{5}{14} \begin{bmatrix} -e^{2t} \\ 2e^{-2t} \end{bmatrix} + \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$$

Q 5.

$$\frac{d}{dx} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/t \\ 2/t + 4 \end{bmatrix}, \quad \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}, \qquad \mathbf{g}(t) = \begin{bmatrix} 1/t \\ 2/t + 4 \end{bmatrix}$$

$$f(l) = l^2 + 5l$$
$$0 = l(l+5)$$
$$l_1 = -5, l_2 = 0$$

for $l_1 = -5$, $adj(\mathbf{A} - l_2\mathbf{I}) = adj\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \Rightarrow \mathbf{v}_2 = a\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

for $l_2 = 0$, $adj(\mathbf{A} - l_1\mathbf{I}) = adj\begin{bmatrix} -4 & 2 \\ 2 & =1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \Rightarrow \mathbf{v}_1 = a\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$U = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

Since A is symmetric, the normalize eigen vector $U^{-1} = U^T$.

By normalizing,

$$U^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$U^{-1}g(t) = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1/t \\ 2/t + 4 \end{bmatrix} = \begin{bmatrix} -4/\sqrt{5} \\ 5/\sqrt{5}\,t + 8/\sqrt{5} \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix}$$

X = UZ
X(1) = UZ(1)

$$Z(1) = U^{-1}X(1) = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 3/\sqrt{5} \end{bmatrix}$$

$\therefore$ We have $\dfrac{d}{dt}z_1 = -5z_1 - \dfrac{4}{\sqrt{5}}, \quad z_1(1) = \dfrac{1}{\sqrt{5}}$

$$\frac{d}{dt}z_2 = \frac{5}{\sqrt{5}\,t} + \frac{8}{\sqrt{5}}, \quad z_2(1) = \frac{3}{\sqrt{5}}$$

$$z_1 = c_1 e^{-5t} + e^{-5t}\int_0^t e^{5s}(-\frac{4}{\sqrt{5}})\ ds$$

$$= c_1 e^{-5t} - \frac{4}{\sqrt{5}} e^{-5t}\left[\frac{e^{5s}}{5}\right]_0^t$$

$$= c_1 e^{-5t} - \frac{4}{\sqrt{5}} e^{-5t}\left[\frac{e^{5t}}{5} - \frac{1}{5}\right]$$

$$= c_1 e^{-5t} - \frac{4}{\sqrt{5}}(\frac{1}{5} - \frac{e^{-5t}}{5})$$

$$= c_1 e^{-5t} - \frac{4}{5\sqrt{5}}(1 - e^{-5t})$$

$$z_1 = \frac{1}{\sqrt{5}},\ \ t = 1$$

$$\frac{1}{\sqrt{5}} = c_1 e^{-5} - \frac{4}{5\sqrt{5}}(1 - e^{-5})$$

$$c_1 = 119.11$$

$$z_1 = 1190.11 e^{-5t} - \frac{4}{5\sqrt{5}}(1 - e^{-5t})$$

$$z_2 = c_2 e^{0t} + e^{0t}\int_0^t (\frac{\sqrt{5}}{s} + \frac{8}{\sqrt{5}})e^{-0s}\ ds$$

$$= c_2 + \frac{1}{\sqrt{5}}\int_0^t (\frac{5}{s} + 8)\ ds$$

$$= c_2 + \left[5\ln s\Big|_0^t + 8s\Big|_0^t\right]$$

$$= c_2 + \frac{1}{\sqrt{5}}\left[5\ln s - \ln 0 + 8t\right]$$

$$= c_2 + \frac{1}{\sqrt{5}}(5\ln t + 8t)$$

$$\text{t = 1, } z_2 = \frac{3}{\sqrt{5}}$$

$$\frac{3}{\sqrt{5}} = c_2 + \frac{1}{\sqrt{5}}(5\ln t + 8t)$$

$$c_1 = -\frac{5}{\sqrt{5}}$$

$$z_2 = -\frac{5}{\sqrt{5}} + \frac{1}{\sqrt{5}}(5\ln t + 8t)$$

$$= \frac{1}{\sqrt{5}}(5\ln t + 8t - 5)$$

X = UZ

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 119.11e^{-5t} - \frac{4}{5\sqrt{5}}(1-e^{-5t}) \\ \frac{1}{\sqrt{5}}(5\ln t + 8t - 5) \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 106.86e^{-5t} + \ln t + 1.6t - 1.32 \\ -53.43e^{-5t} + 2\ln t + 3.2t - 1.84 \end{bmatrix}$$

Q 3.

$$A \xrightarrow{k_1} B \underset{k_3}{\overset{k_2}{\rightleftarrows}} C$$

$$\frac{\partial C_A}{\partial t} = -k_1 C_A$$

$$\frac{\partial C_B}{\partial t} = k_1 C_A - k_2 C_B + k_3 C_C$$

$$\frac{\partial C_C}{\partial t} = k_2 C_B - k_3 C_C$$

$$\frac{d}{dt}\begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & k_3 \\ 0 & k_2 & k_3 \end{bmatrix}\begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 3 \\ - & 2 & -3 \end{bmatrix}\begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix}$$

$$\frac{dC}{dt} = KC \qquad C(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix}, \qquad K = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 3 \\ 0 & 2 & -3 \end{bmatrix}, \qquad C(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

eigenvalues of K,

$$f(l) = \det(\mathbf{K} - l\mathbf{I}) = \det\begin{bmatrix} -1-l & 0 & 0 \\ 1 & -2-l & 3 \\ 0 & 2 & -3-l \end{bmatrix}$$

$$0 = -l^3 - 6l^2 - 5l$$

$$0 = l(l+5)(l+1)$$

$l_1 = 0,\ l_2 = -1,\ l_3 = -5$

Eigenvector of K for $l_1 = 0$

$$adj\mathbf{K} = adj\begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 3 \\ 0 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\mathbf{v}_1 = a\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Eigenvector of K for $l_2 = -1$

$$adj(\mathbf{K} - 1\mathbf{I}) = adj\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 3 \\ 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Eigenvector of K for $l_3 = -5$

$$adj(\mathbf{K} + 5\mathbf{I}) = adj\begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 3 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 8 & -12 \\ 2 & -8 & 12 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 0 & -2 & 0 \\ 3 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}$$

$\det \mathbf{U} = 10$

$$\mathbf{U}^{-1} = \frac{1}{10}\begin{bmatrix} 2 & 2 & 2 \\ -5 & 0 & 0 \\ 1 & -4 & 6 \end{bmatrix}$$

$$C(t) = \mathbf{U}(e^{\Lambda t})\mathbf{U}^{-1}\mathbf{C}_0$$

$$\begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 3 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^0 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-5t} \end{bmatrix} \frac{1}{10} \begin{bmatrix} 2 & 2 & 2 \\ -5 & 0 & 0 \\ 1 & -4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2e^{-t} & 0 \\ 3 & e^{-t} & -e^{-5t} \\ 2 & e^{-t} & e^{-5t} \end{bmatrix} \frac{1}{10} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$C_A = e^{-t}$$

$$C_B = \frac{1}{10}(6.5e^{-t} - e^{-5t}) = 0.6 - 0.5e^{-t} - 0.1e^{-5t}$$

$$C_C = \frac{1}{10}(4 + 5e^{-t} + e^{-5t}) = 0.4 + 0.5e^{-t} + 0.1e^{-5t}$$

Q 4.

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} - 10x = 5$$

Let $x = x_1$

$$\frac{dx}{dt} = \frac{dx_1}{dt} = x_2 \qquad , x_1(0) = 3$$

$$\frac{d^2x}{dt^2} = \frac{dx_2}{dt} = 3x_2 + 10x_1 + 5 \qquad , x_2(0) = 15$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax} + \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$$

Eigenvalues of **A**,

$$(\mathbf{A} - l\mathbf{I}) = \begin{bmatrix} 0 - l & 1 \\ 10 & 3 - l \end{bmatrix}$$

$$(\mathbf{A} - l\mathbf{I}) = l^2 - 3l - 10$$

$$0 = (l - 5)(l + 2)$$

$$l_1 = 5, l_2 = -2$$

Eigenvalue for $l_1 = 5$

$$adj(\mathbf{A}-l_1\mathbf{I}) = adj\begin{bmatrix}-5 & 1\\ 10 & 2\end{bmatrix} = \begin{bmatrix}-2 & -1\\ -10 & -5\end{bmatrix} \Rightarrow \mathbf{v}_1 = a\begin{bmatrix}1\\ 5\end{bmatrix}$$

$$adj(\mathbf{A}-l_2\mathbf{I}) = adj\begin{bmatrix}2 & 1\\ 10 & 5\end{bmatrix} = \begin{bmatrix}5 & -1\\ -10 & 2\end{bmatrix} \Rightarrow \mathbf{v}_2 = a\begin{bmatrix}1\\ 5\end{bmatrix}$$

$$x = \sum_{i=1}^{n} c_i \mathbf{v}_i e^{l_i t} - \mathbf{A}^{-1}b$$

$$= c_1\mathbf{v}_1 e^{l_1 t} + c_2\mathbf{v}_2 e^{l_2 t} - \mathbf{A}^{-1}b$$

$$\mathbf{x} = c_1\begin{bmatrix}1\\ 5\end{bmatrix}e^{5t} + c_2\begin{bmatrix}-1\\ 2\end{bmatrix}e^{-2t} - \frac{1}{-10}\begin{bmatrix}3 & -1\\ -10 & 0\end{bmatrix}\begin{bmatrix}0\\ 5\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = c_1\begin{bmatrix}1\\ 5\end{bmatrix} + c_2\begin{bmatrix}-1\\ 2\end{bmatrix} + \begin{bmatrix}-1/2\\ 0\end{bmatrix}$$

Apply initial conditions at $t = 0$, $\mathbf{x}(0) = \begin{bmatrix}3\\ 15\end{bmatrix}$

$$\begin{bmatrix}3\\ 15\end{bmatrix} = c_1\begin{bmatrix}1\\ 5\end{bmatrix} + c_2\begin{bmatrix}-1\\ 2\end{bmatrix} + \begin{bmatrix}-\tfrac{1}{2}\\ 0\end{bmatrix}$$

$$c_1 - c_2 - \frac{1}{2} = 3 \Rightarrow c_1 - c_2 = 3\frac{1}{2} \qquad \cdots \text{eqn I}$$

$$5c_1 + 2c_2 + 0 = 15 \Rightarrow 5c_1 + 2c_2 = 15 \qquad \cdots \text{eqn II}$$

$$\text{I} \times 2 \cdots \qquad 2c_1 - 2c_2 = 7$$

$$\text{II} \ldots \qquad 5c_1 + 2c_2 = 15$$

$$7c_1 = 22$$

$$c_1 = \frac{22}{7}$$

$$2\left(\frac{22}{7}\right) - 2c_2 = 7$$

$$c_2 = -\frac{5}{14}$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \frac{22}{7}\begin{bmatrix}e^{5t}\\ 5e^{5t}\end{bmatrix} - \frac{5}{14}\begin{bmatrix}-e^{2t}\\ 2e^{-2t}\end{bmatrix} + \begin{bmatrix}-\tfrac{1}{2}\\ 0\end{bmatrix}$$

Q 5.

$$\frac{d}{dx}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/t \\ 1/t + 4 \end{bmatrix}, \quad \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}, \qquad \mathbf{g}(t) = \begin{bmatrix} 1/t \\ 1/t + 4 \end{bmatrix}$$

$$f(l) = l^2 + 5l$$
$$0 = l(l+5)$$

$$l_1 = 0, \; l_2 = -5$$

$$adj(\mathbf{A} - l_1\mathbf{I}) = adj\begin{bmatrix} -4 & 2 \\ 2 & =1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \Rightarrow \mathbf{v}_1 = a\begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$adj(\mathbf{A} - l_2\mathbf{I}) = adj\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \Rightarrow \mathbf{v}_2 = a\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\mathbf{U} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$\mathbf{U}^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$\mathbf{x}(t) = \mathbf{U}e^{\Lambda t}\mathbf{U}^{-1}\mathbf{x}_0$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-5t} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$