

Problem set # 2

(No need to submit)

- (a) If \mathbf{I} is the (3×3) identity matrix and k is an unspecified real number, what is the determinant of the matrix $k\mathbf{I}$?
(b) If the following matrix \mathbf{A} satisfies $\mathbf{A}^3 = k\mathbf{I}$, find k .

- Find the rank of the following matrix.

$$\begin{bmatrix} 1 & 3 & 0 & 5 \\ -1 & 1 & 1 & 0 \\ 2 & -1 & 2 & 5 \\ -1 & 2 & -1 & -1 \end{bmatrix}$$

Show that the fourth column, for example, is a linear combination of the first three columns and that the fourth row is a linear combination of the first three rows.

- If \mathbf{u} and \mathbf{v} are two linearly independent vectors, determine whether $(\mathbf{u} + \mathbf{v})$ and $(2\mathbf{u} - \mathbf{v})$ are linearly independent.
- Consider the linear system

$$x + y + z = 2$$

$$2x + 3y + 3z = 5$$

$$2x + 3y + (p^2 - 1)z = p + 3$$

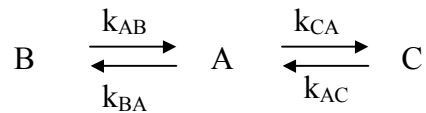
Find all values of p for which the resulting linear system has

- no solutions
 - a unique solution
 - infinite solutions.
- The matrix equation $\mathbf{A} \mathbf{x} = \mathbf{b}$ can only be solved for \mathbf{x} when \mathbf{b} belongs to a certain plane.

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Find an equation for this plane.

- Consider the three-component isomerization reaction occurring in a CSTR:



where the residence time is $\tau = 1$ min, the inlet concentrations are

$$[C_{AO} \quad C_{BO} \quad C_{CO}]^T = [2 \quad 1 \quad 0]$$

and the rate constants are:

$$k_{AB} = 10 \text{ min}^{-1}$$

$$k_{BA} = 10 \text{ min}^{-1}$$

$$k_{CA} = 3 \text{ min}^{-1}$$

$$k_{AC} = 1 \text{ min}^{-1}$$

Find the concentrations C_A , C_B and C_C .

7. Solve the following system of equations:

$$8x_1 + 11x_2 + 3x_3 - 2x_4 = 0$$

$$3x_1 + 2x_2 + 2x_3 - x_4 = 0$$

$$-x_1 + 7x_2 - 3x_3 + x_4 = 0$$

$$-11x_1 + 4x_2 - 12x_3 + 5x_4 = 0$$

How many linearly independent solutions are there in the above set of equations? Find the general solution.

8. For what values of p does the system

$$x - y + z = 0$$

$$2x - y + 5z = 0$$

$$x - y + pz = 0$$

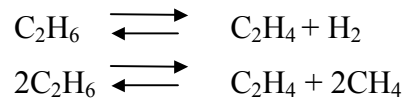
have (a) no solution, (b) one solution, and (c) infinitely many solutions?

Give reasons.

9. Find the general solution of the following

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

10. In the study of chemical reaction, Aris developed a technique of writing simultaneous chemical reactions in the form of linear algebraic equations. For example, the following two simultaneous chemical equations



can be rearranged in the form

$$\begin{array}{l} \text{C}_2\text{H}_4 + \text{H}_2 - \text{C}_2\text{H}_6 = 0 \\ \text{C}_2\text{H}_4 + 2\text{CH}_4 - 2\text{C}_2\text{H}_6 = 0 \end{array}$$

If we identify A_1 with C_2H_4 , A_2 with H_2 , A_3 with CH_4 and A_4 with C_2H_6 , the set of equations becomes

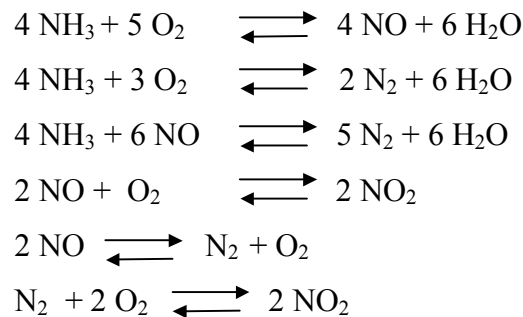
$$\begin{array}{l} A_1 + A_2 - A_4 = 0 \\ A_1 + 2 A_3 - 2 A_4 = 0 \end{array}$$

This can be generalized to a system of R reactions between S chemical species by the set of equations represented by

$$\sum_{j=1}^S \alpha_{ij} A_j = 0 \quad i = 1, 2, \dots, R$$

where α_{ij} are the stoichiometric coefficients of each species A_j in each reaction i .

Aris demonstrated that the number of independent chemical reactions in a set of R reactions is equal to the rank of the matrix of stoichiometric coefficient α_{ij} . Determine the number of independent chemical reactions in the following reaction systems:



Problem Set #2

2. Find the rank of the following matrix. Show that the fourth column is a linear combination of the first three columns and that the fourth row is a linear combination of the first three rows.

$$\begin{bmatrix} 1 & 3 & 0 & 5 \\ -1 & 1 & 1 & 0 \\ 2 & -1 & 2 & 5 \\ -1 & 2 & -1 & -1 \end{bmatrix}$$

$$\begin{array}{l} s_1 = -1 \\ \det s_1 = -1 \neq 0 \end{array}$$

$$s_2 = \begin{bmatrix} 2 & 5 \\ -1 & -1 \end{bmatrix}$$

$$\det s_2 = -2 - (-5) = 3 \neq 0$$

$$s_3 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 5 \\ 2 & -1 & -1 \end{bmatrix}$$

$$\det s_3 = 1(-2+5) - 1(1-10) + 0(1-4) = 12 \neq 0$$

$$s_4 = \begin{bmatrix} 1 & 3 & 0 & 5 \\ -1 & 1 & 1 & 0 \\ 2 & -1 & 2 & 5 \\ -1 & 2 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 3 & 1 & 5 \\ 0 & -7 & 2 & 5 \\ 0 & 5 & -1 & 4 \end{bmatrix}$$

$$= 1 \begin{vmatrix} 4 & 1 & 5 \\ -7 & 2 & -5 \\ 5 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 5 \\ -7 & 2 & -5 \\ 5 & -1 & 4 \end{vmatrix}$$

$$= 4(8-5) - 1(-28+25) + 5(7-10)$$

$$= 4(3) - 1(-3) + 5(-3)$$

$$= 0$$

$$\therefore s_4 = 0$$

$$\therefore \text{Rank} = 3 \leftarrow$$

Columns

$$\alpha \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 1 \\ -1 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 5 \\ -1 \end{pmatrix}$$

$$\alpha + 3\beta = 5 \quad (1)$$

$$-\alpha + \beta + \gamma = 0 \quad (2)$$

$$2\alpha - \beta + 2\gamma = 5 \quad (3)$$

$$-\alpha + 2\beta - \gamma = -1 \quad (4)$$

$$\therefore 2 \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \\ -1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 5 \\ -1 \end{pmatrix}$$

\therefore Forth column is linear combination of the first three columns.

Rows

$$\alpha[1 \ 3 \ 0 \ 5] + \beta[-1 \ 1 \ 1 \ 0] + \gamma[2 \ -1 \ 2 \ 5] = [-1 \ 2 \ -1 \ -1]$$

$$\alpha - \beta + 2\gamma = -1 \quad (1)$$

$$3\alpha + \beta - \gamma = 2 \quad (2)$$

$$\beta + 2\gamma = -1 \quad (3)$$

$$5\alpha + 5\gamma = -1 \quad (4)$$

$$\beta + 2\gamma = -1$$

$$\beta + 2\left(-\frac{3}{5}\right) = -1$$

$$\beta = -1 + \frac{6}{5} = \frac{1}{5} \leftarrow$$

\therefore The fourth row is a linear combination of the first three rows.

3.

$$\text{Let } u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$u + v = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \quad 2u - v = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\alpha + 3\beta = 0 \quad (1)$$

$$2\alpha + 4\beta = 0 \quad (2)$$

By solving the equations:

$$\alpha = 0, \beta = 0$$

\therefore u and v are linearly independent.

$$\alpha + 3\beta = 4 \quad (1)$$

$$2\alpha + 4\beta = 6 \quad (2)$$

By solving equations:

$$\alpha = 1, \beta = 1$$

4.

$$x + y + z = 2$$

$$2x + 3y + 3z = 5$$

$$2x + 3y + (p^2 + 1)z = p + 3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 2 & 2 & p^2 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ p + 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 3 & 5 \\ 2 & 3 & p^2-1 & p+3 \end{array} \right] \quad m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2$$

$$\xrightarrow[\substack{R_2 \neq R_2 - m_{21}R_1 \\ R_3 = R_2 + R_3}]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & p^2-4 & p-2 \end{array} \right]$$

(a) $p = -2$, no solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -4 \end{array} \right]$$

(b) $p \neq \pm 2$, unit solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & p^2-4 & p-2 \end{array} \right]$$

(c) $p = 2$, infinite solutions.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

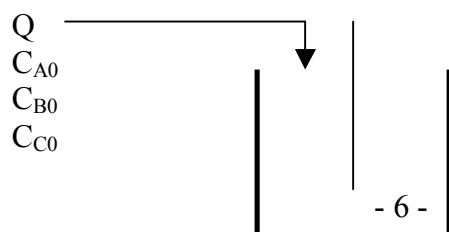
$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

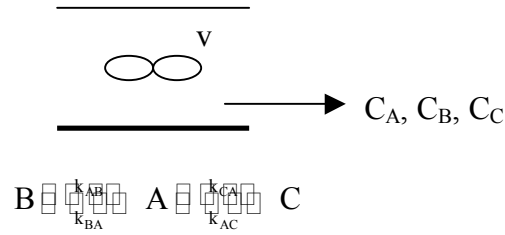
For particular solution, $Ax = b$
 $x = A^{-1}b$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & b_1 \\ 1 & 0 & -1 & b_2 \\ 0 & 1 & -1 & b_3 \end{array} \right] \xrightarrow{R_2 = R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & b_1 \\ 0 & 1 & -1 & b_2 - b_1 \\ 0 & 1 & -1 & b_3 \end{array} \right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & b_1 \\ 0 & 1 & -1 & b_2 \\ 0 & 1 & -1 & b_3 \end{array} \right]$$

6.





$$r_A = k_{AB}C_B - k_{CA}C_A + k_{AC}C_C - k_{BA}C_A$$

$$r_B = -k_{AB}C_B + k_{BA}C_A$$

$$r_C = k_{CA}C_A - k_{AC}C_C$$

Mass balance

$$A: \quad QC_{A0} + vk_{AB}C_B + vk_{AC}C_C - vk_{CA}C_A - vk_{BA}C_A - QC_A = 0$$

$$B: \quad QC_{B0} - vk_{AB}C_B + vk_{BA}C_A - QC_B = 0$$

$$C: \quad QC_{C0} + vk_{CA}C_A - vk_{AC}C_C - QC_C = 0$$

Dividing by Q and substituting:

$$k_{AB} = 10 \quad v/Q = \tau = 1$$

$$k_{BA} = 10$$

$$k_{CA} = 3$$

$$k_{AC} = 1$$

$$A: \quad C_{A0} + 10C_B + C_C - 3C_A - 10C_A - C_A = 0$$

$$-14C_A + 10C_B + C_C = -2$$

$$14C_A - 10C_B - C_C = 2$$

$$B: \quad 1 - 10C_B + 10C_A - C_B = 0$$

$$-10C_A + 11C_B = 1$$

$$C: \quad 0 + 3C_A - C_C - C_C = 0$$

$$3C_A - 2C_C = 0$$

$$\Rightarrow \begin{bmatrix} 14 & -10 & -1 \\ -10 & 11 & 0 \\ 3 & 0 & -2 \end{bmatrix} \begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 14 & -10 & -1 \\ -10 & 11 & 0 \\ 3 & 0 & -2 \end{bmatrix}$$

$$\det A = 14(-22) + 10(20) - 1(0 - 33) = -75$$

$$\text{adj } A = \begin{bmatrix} -22 & -20 & 11 \\ -20 & -25 & 10 \\ -33 & -30 & 54 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix} &= \frac{1}{\det A} (\text{adj } A) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{-75} \begin{bmatrix} -22 & -20 & 11 \\ -20 & -25 & 10 \\ -33 & -30 & 54 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{-75} \begin{bmatrix} -64 \\ -65 \\ -96 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} C_A \\ C_B \\ C_C \end{bmatrix} = \begin{bmatrix} 0.8533 \\ 0.8667 \\ 1.2800 \end{bmatrix} \leftarrow$$

7.

$$8x_1 + 11x_2 + 3x_3 - 2x_4 = 0$$

$$3x_1 + 2x_2 + 2x_3 - x_4 = 0$$

$$-x_1 + 7x_2 - 3x_3 + x_4 = 0$$

$$-11x_1 + 4x_2 - 12x_3 + 5x_4 = 0$$

no. of linearly independent solutions?

Rank = no. of linearly independent solution

$$\begin{bmatrix} 8 & 11 & 3 & -2 \\ 3 & 2 & 2 & -1 \\ -1 & 7 & 3 & 1 \\ -11 & 4 & -12 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$s_1 = 5$$

$$s_2 = \begin{bmatrix} -3 & 1 \\ -12 & 5 \end{bmatrix}, \det s_2 = 3$$

$$s_2 = \begin{bmatrix} 7 & 4 \\ 1 & 5 \end{bmatrix}, \det s_2 = 31$$

$$s_3 = \begin{bmatrix} 2 & 2 & -1 \\ 7 & -3 & 1 \\ 4 & -12 & 5 \end{bmatrix}$$

$$\det s_3 = 2(-15+12)+2(-1)(35-4)+(-1)(-84+12) = 4$$

$$s_4 = \begin{bmatrix} 8 & 11 & 3 & -2 \\ 3 & 2 & 2 & -1 \\ -1 & 7 & 3 & 1 \\ -11 & 4 & -12 & 5 \end{bmatrix}$$

$$\begin{aligned} \det s_4 &= 8[(2)(-15+12)=2(35-4)(-1)+(-1)(-84+12)] \\ &+ (-11)[3(-15+12)+(-1)(2)(-5+11)+(-1)(12-33)] \\ &+ 3[3(35-4)+(-1)(2)(-5+11)+((-1)(-4+77)] \\ &+ 2[3(-84+12)+(-1)(2)(12+33)+(2)(-4+77)] \\ &= 0 \end{aligned}$$

∴ no of rank = 3

There are three linearly independent solutions in the above set of equations.

8.

$$\begin{aligned} x - y + z &= 0 \\ 2x - y + 5z &= 0 \\ x - y + pz &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 5 \\ 1 & -1 & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 2 & -1 & 5 & | & 0 \\ 1 & -1 & p & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & p-1 & | & 0 \end{bmatrix}$$

(a) For this problem, no solution is not possible for any p.

(b) When $p \neq 1$, one solution

$$\begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & p-1 & | & 0 \end{bmatrix}$$

(c) when $p = 1$, many solutions

$$\begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

*****END*****

9.

Find the general solution of the following.

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

we have three equations, four unknown
more unknown, less equation (under defined problem)

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & 3 & 3 & 0 \end{array} \right] 5$$

$$\frac{A}{B}$$

$$s_1 = [3], \quad \det s_1 = 3$$

$$s_2 = \begin{bmatrix} 6 & 9 \\ 3 & 3 \end{bmatrix}, \quad \det s_2 = 18 - 27 = -9$$

$$s_3 = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & 3 & 3 \end{bmatrix}$$

$$\begin{aligned} \det s_3 &= 1(-1)^{1+1}(18-27) + 3(-1)^{1+2}(6+9) + 3(-1)^{1+3}(6+6) \\ &= -9 - 45 + 36 \\ &= -18 \end{aligned}$$

\therefore rank of A = 3

$$S_B = \begin{bmatrix} 3 & 3 & 1 \\ 6 & 9 & 5 \\ 3 & 3 & 5 \end{bmatrix}$$

$$\begin{aligned} \det s_B &= 3(-1)^{1+1}(45-15) + 3(-1)^{1+2}(30-15) + 1(-1)^{1+3}(18-27) \\ &= 90 - 45 - 9 \\ &= 36 \end{aligned}$$

\therefore rank of B = 3

rank of A = rank of B \neq n

\therefore The problem has infinite many solutions.

For a particular solution, let $y = 0$

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & 3 & 3 & 5 \end{array} \right] \Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & 3 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

$$Ax_p = b$$

$$x_p = A^{-1}b$$

$$= \frac{1}{\det A} [\text{adj } A] b$$

$$x_p = \frac{1}{-18} \begin{bmatrix} -9 & 0 & 9 \\ -15 & 6 & -3 \\ 12 & -6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

$$= \frac{1}{-18} \begin{bmatrix} -9+0+45 \\ -15+30+15 \\ 12-30+0 \end{bmatrix} = \frac{1}{-18} \begin{bmatrix} 36 \\ 0 \\ -18 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore x_p = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

For homogeneous solution, $Ax_h = 0$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 3 \\ -1 & 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $y = 1$, we have

$$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & 3 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix}$$

$$Ax = b$$

$$x = A^{-1}b$$

$$A^{-1} = \frac{1}{\det A} (\text{adj } A) b$$

$$\det A = -18$$

$$x = \frac{1}{-18} \begin{bmatrix} -9 & 0 & 9 \\ -15 & 6 & 3 \\ 12 & -6 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix}$$

$$= \frac{1}{-18} \begin{bmatrix} 18+0+0 \\ 30-30+0 \\ -24+30+0 \end{bmatrix} = \frac{1}{-18} \begin{bmatrix} 18 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

$$x_h = \begin{bmatrix} -1 \\ 0 \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

$\therefore x = x_p + \alpha x_h$ satisfies $Ax = b$.

$$\therefore x = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \alpha = \begin{bmatrix} -1 \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

We have many solutions as α could have many values.
