Problem set # 1

(No need to submit)

1. Evaluate the determinant

$$\begin{vmatrix} 3 & 5 & 2 & 4 \\ 1 & 1 & -1 & 6 \\ 2 & 3 & 5 & 1 \\ 2 & 1 & 4 & 8 \end{vmatrix}$$

(a) By Laplace's development using two rows, by Laplace's development using two columns, and by elements of a single row or column.

(b) By using the elementary operations to produce a number of zeros in rows and columns and then expanding.

2. A matrix is symmetric if $\mathbf{A} = \mathbf{A}^T$ and skew symmetric if $\mathbf{A} = -\mathbf{A}^T$.
Show that for every square matrix **A**, the matrices $\mathbf{A} + \mathbf{A}^T$, $\mathbf{A}\mathbf{A}^T$, $\mathbf{A}^T\mathbf{A}$ are symmetric, and $\mathbf{A} - \mathbf{A}^T$ is skew symmetric.

3. Consider a determinant Δ partitioned into four blocks:

$$\Delta = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

where **A** and **D** are square matrices.

(a) Show that

$$\Delta = |A| \; \left|D - CA^{-1}B\right| \quad \text{if } |A| \neq 0$$

(b) Use the above formulae to compute the determinant:

$$\begin{vmatrix} -2 & 1 & 1 & 3 \\ -4 & -3 & 7 & -1 \\ 2 & -4 & 2 & 3 \\ 5 & 2 & -7 & -2 \end{vmatrix}$$

4. Given the matrix

$$\begin{bmatrix} 3 & -2 & 1 \\ 2 & 0 & -4 \\ 1 & 1 & 1 \end{bmatrix}$$

(a) Compute A^2

(b) Compute A^{-1}

5. Let the following vectors, [**x**, **y**, **z**], map the matrix **A** into the corresponding vectors [**u**, **v**, **w**].

$$\mathbf{x} \rightarrow \mathbf{u}: \quad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

$$\mathbf{y} \rightarrow \mathbf{v}: \quad \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

$$\mathbf{z} \rightarrow \mathbf{w}: \quad \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}$$

(a) What vector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ mapped into by **A**?

(b) What vector is mapped into $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ by **A**?

6. Consider two matrices **P** and **Q** such that **PQ** = **0**, where **0** is the zero matrix. Does this imply **P** = **0** or **Q** = **0**? If not, construct a counter example. Use this to construct an example for three matrices **A**, **B**, and **C** such that **AC** = **BC** but the matrices **A** and **B** are not equal.

Problem Set #1

Evaluate the determinant

(a) By Laplace's development using two rows (row 1 & row 2)

$$\begin{vmatrix} 3 & 5 & 2 & 4 \\ 1 & 1 & -1 & 6 \\ 2 & 3 & 5 & 1 \\ 2 & 1 & 4 & 8 \end{vmatrix}$$

= (-2) (1) (36) + (-5) (-1) (23) + (14) (1) (7) + (-7) (1) (14)
+ (26) (-1) (-2) + (16) (1) (-4)
= -72 + 115 + 98 – 98 + 52 – 64
= 31

(b) By using two columns (column 1 & column 2)

$$\begin{vmatrix} 3 & 5 & 2 & 4 \\ 1 & 1 & -1 & 6 \\ 2 & 3 & 5 & 1 \\ 2 & 1 & 4 & 8 \end{vmatrix}$$

= (-2) (1) (36) + (-1) (-1) (-32) + (-7) (1) (-31) + (1) (1) (0)
+ (-1) (-1) (-18) + (-4) (1) (16)
= - 72 – 32 + 217 + 0 – 18 – 64
= 31

(c) By elements of a single row or column (by column) (cofactor expansion)

$$\begin{vmatrix} 3 & 5 & 2 & 4 \\ 1 & 1 & -1 & 6 \\ 2 & 3 & 5 & 1 \\ 2 & 1 & 4 & 8 \end{vmatrix}$$

= 3[(-1) (36) + (3) (-1) (-32) + (1) (1) (-31)]
+ (-1) [(5) (36) + (3) (-1) (0) + (1) (1) (-18)]
+ (2) [(5) (-32) + (1) (-1) (0) + (1) (1) (16)]
-(2) [(5) (-31) + (-1) (-18) + (3) (16)]
= 31

(d) By elements of a single row (row1)

$$\begin{vmatrix} 3 & 5 & 2 & 4 \\ 1 & 1 & -1 & 6 \\ 2 & 3 & 5 & 1 \\ 2 & 1 & 4 & 8 \end{vmatrix}$$

= 3 [(1) $(-1)^{1+1}$ (36) + (-1) $(-1)^{1+2}$ (23) + (6) $(-1)^{1+3}$ (7)]
  + (-5) [(1) $(-1)^{1+1}$ (36) + (-1) $(-1)^{1+2}$ (14) + (6) $(-1)^{1+3}$ (-2)]
  + (2) [(1) $(-1)^{1+1}$ (23) + (1) $(-1)^{1+2}$ (14) + (6) $(-1)^{1+3}$ (-4)]
  + (-4) [(1) $(-1)^{1+1}$ (7) + (1) $(-1)^{1+2}$ (-2) + (-1) $(-1)^{1+3}$ (-4)]
= 3 [36 + 23 + 42] + (-5) [36 + 14 – 12] + [23 – 14 – 24]
  + (-4) [7 + 2 + 4]
= 31

1(b). Evaluate the determinant by using the elementary operations to produce a number of zeros in rows and columns and then expanding.

$$\begin{bmatrix} 3 & 5 & 2 & 4 \\ 1 & 1 & -1 & 6 \\ 2 & 3 & 5 & 1 \\ 2 & 1 & 4 & 8 \end{bmatrix} \xrightarrow[r_4=r_4-2r_2]{r_3=r_3-2r_1} \begin{bmatrix} 3 & 5 & 2 & 4 \\ 1 & 1 & -1 & 6 \\ 0 & 1 & 7 & -11 \\ 0 & -1 & 6 & -4 \end{bmatrix} \xrightarrow{r_2=r_2-r_3} \begin{bmatrix} 3 & 5 & 2 & 4 \\ 1 & 0 & -8 & 17 \\ 0 & 1 & 7 & -11 \\ 0 & -1 & 6 & -4 \end{bmatrix}$$

$$\xrightarrow{r_1=r_1-3r_2} \begin{bmatrix} 0 & 5 & 26 & -47 \\ 1 & 0 & -8 & 17 \\ 0 & 1 & 7 & -11 \\ 0 & -1 & 6 & -4 \end{bmatrix}$$

$$= (-1)\begin{vmatrix} 5 & 26 & -47 \\ 1 & 7 & -11 \\ -1 & 6 & -4 \end{vmatrix}$$

= (-1) [(5) (-28 + 66) – 26 (- 4 – 11) + (- 47) (6 +7)]
= (-1) [190 + 390 + 611]
= 31

2.

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$$

$a_{12} = a_{21} = 5$

$\therefore$ This matrix is symmetric ($\because a_{ij} = a_{ji}$)

$$A A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1+4 & 3+8 \\ 3+8 & 9+16 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$$

Since $a_{21} = a_{12}$

This matrix is symmetric.

$$A^T A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+9 & 2+12 \\ 2+12 & 4+16 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$

Since $a_{21} = a_{12}$

This matrix is symmetric.

$$A - A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

This matrix is skew-symmetric.

($\because$ all principal diagonal value = 0
off-diagonal value $a_{ij} = a_{ji}$)

3.(a)

$$\Delta = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} A & 0 \\ 0 & I \end{vmatrix} \begin{vmatrix} I & A^{-1}B \\ C & D \end{vmatrix}$$

$$= |AI - 0| \, |ID - CA^{-1}B|$$

$$= |A| \, |D - CA^{-1}B|$$

3.(b)

$$\begin{vmatrix} -2 & 1 & 1 & 3 \\ -4 & -3 & 4 & -1 \\ 2 & -4 & 2 & 3 \\ 5 & 2 & -7 & -2 \end{vmatrix}$$

$$A = \begin{bmatrix} -2 & 1 \\ -4 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6+4} \begin{bmatrix} -3 & -1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} -0.3 & -0.1 \\ 0.4 & -0.2 \end{bmatrix}$$

$$D - CA^{-1}B = \begin{bmatrix} 2 & 3 \\ -7 & -2 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -0.3 & -0.1 \\ 0.4 & -0.2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 7 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & 3 \\ -7 & -2 \end{bmatrix} - \begin{bmatrix} 2 & -7.2 \\ -7 & -1.2 \end{bmatrix} = \begin{bmatrix} 0 & 10.2 \\ 0 & -0.8 \end{bmatrix}$$

$$|A| = 10$$

$$\begin{vmatrix} 0 & 10.2 \\ 0 & -0.8 \end{vmatrix} = |D - CA^{-1}B| = 0$$

$$\therefore \Delta = |A|\,|D - CA^{-1}B| = 0 \;\leftarrow$$

4.(a)

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 0 & -4 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 0 & -4 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 1 \\ 2 & 0 & -4 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9-4+1 & -6+0+1 & 3+8+1 \\ 6+0+-4 & -4+0-4 & 2+0-4 \\ 3+2+1 & -2+0+1 & 1-4+1 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 12 \\ 2 & -8 & -2 \\ 6 & -1 & -2 \end{bmatrix}$$

4.(b)

D = 3 (0 +4) – (-2) (2 + 4) + (1) (2 - 0) = 12 + 12 + 2 = 26 $\neq$ 0

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 0 & -4 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} 0 & -4 \\ 1 & 1 \end{vmatrix} = 4 \qquad A_{12} = -\begin{vmatrix} 2 & -4 \\ 1 & 1 \end{vmatrix} = -(2+4) = -6$$

$$A_{13} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2 \qquad A_{21} = -\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -(-2-1) = 3$$

$$A_{22} = \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 3-1 = 2 \qquad A_{23} = -\begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} = -(3+2) = -5$$

$$A_{31} = \begin{vmatrix} -2 & 1 \\ 0 & -4 \end{vmatrix} = 8-0 = 8 \qquad A_{32} = \begin{vmatrix} 3 & 1 \\ 2 & -4 \end{vmatrix} = -(-12-2) = 14$$

$$A_{33} = \begin{vmatrix} 3 & -2 \\ 0 & 2 \end{vmatrix} = 0+4 = 4$$

$$A^{-1} = \left[\frac{1}{D}\right][adjA] = \frac{1}{26}\begin{bmatrix} 4 & 3 & 8 \\ -6 & 2 & 14 \\ 2 & -5 & 4 \end{bmatrix}$$

5.

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}$$

a + 2b = -1 (1)
d + 2e = 3 (2)
g + 2h = 2 (3)

2b + c = 3 (4)
2e + f = 4 (5)
2h + i = -2 (6)

- a + b + c = 3 (7)
- d + e + f = 1 (8)
- g + h + i = - 3 (9)

By solving these equations:

a = 1	b = -1	c = 5
d = 3	e = 0	f = 4
g = 0	h = 1	i = -4

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 0 & 4 \\ 0 & 1 & -4 \end{bmatrix}$$

(a) $$\begin{bmatrix} 1 & -1 & 5 \\ 3 & 0 & 4 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ -3 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & -1 & 5 \\ 3 & 0 & 4 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$D = \begin{vmatrix} 1 & -1 & 5 \\ 3 & 0 & 4 \\ 0 & 1 & -4 \end{vmatrix} = \text{(1) (-4) + (1) (-12) + (5) (3) = -1}$$

By Creamer's rule

$$D_1 = \begin{vmatrix} 1 & -1 & 5 \\ 1 & 0 & 4 \\ 1 & 1 & -4 \end{vmatrix} = \text{(1) (-4) + (1) (-4-4) + (5) (1) = -7}$$

$$D_2 = \begin{vmatrix} 1 & 1 & 5 \\ 3 & 1 & 4 \\ 0 & 1 & -4 \end{vmatrix} = \text{(1) (-4-4) – (1) (-12) + (5) (3) = 19}$$

$$D_3 = \begin{vmatrix} 1 & -1 & 1 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \text{(1) (0-1) – (-1) (3) + (1) (3) = 5}$$

$$x_1 = x = \frac{D_1}{D} = \frac{-7}{-1} = 7$$

$$x_2 = y = \frac{D_2}{D} = \frac{19}{-1} = -19$$

$$x_3 = z = \frac{D_3}{D} = \frac{5}{-1} = -5$$

$$\begin{bmatrix} 1 & -1 & 5 \\ 3 & 0 & 4 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} 7 \\ -19 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \leftarrow$$

6.
PQ = 0 where 0 is the zero matrix.
It does not imply P = 0 or Q = 0.

Because,

Let $P=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}$ and $Q=\begin{bmatrix}-3 & 6 & -3\\ 6 & -12 & 6\\ -3 & 6 & -3\end{bmatrix}$

Then $PQ=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$