/ (1-1)

ENGINEERING 5432/5435 Problem Set #1

Sample Solutions

1a) $\vec{A} = 4\hat{i} + 7\hat{j}$, $\vec{B} = -5\hat{i} + 11\hat{j}$ as vectors in $\mathbb{R}^2$

$\Rightarrow \vec{A}\cdot\vec{B} = (4\hat{i}+7\hat{j})\cdot(-5\hat{i}+11\hat{j}) = (4)(-5)+(7)(11) = -20+77$
$= 57$

$\vec{A}\times\vec{B}$ does not exist.

as vectors in $\mathbb{R}^3$

$\vec{A}\cdot\vec{B} = (4\hat{i}+7\hat{j}+0\hat{k})\cdot(-5\hat{i}+11\hat{j}+0\hat{k}) = (4)(-5)+(7)(11)+(0)(0)$
$= -20+77+0 = 57$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 7 & 0 \\ -5 & 11 & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} 7 & 0 \\ 11 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 4 & 0 \\ -5 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 4 & 7 \\ -5 & 11 \end{vmatrix}$$

$= 0\hat{i} + 0\hat{j} + [(4)(11)-(7)(-5)]\hat{k} = 0\hat{i}+0\hat{j}+79\hat{k}$

b) $\vec{A} = 8\hat{i}+2\hat{j}-\hat{k}$, $\vec{B} = -\hat{i}+7\hat{j}+4\hat{k}$

$\Rightarrow \vec{A}\cdot\vec{B} = [8\hat{i}+2\hat{j}-\hat{k}]\cdot(-\hat{i}+7\hat{j}+4\hat{k}) = -8+14-4 = 2$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & 2 & -1 \\ -1 & 7 & 4 \end{vmatrix} = 15\hat{i} - 31\hat{j} + 58\hat{k}$$

c) $\vec{A} = 3\hat{i}+4\hat{j}-5\hat{k}$, $\vec{B} = -3\hat{i}+5\hat{j}+4\hat{k}$

$\Rightarrow \vec{A}\cdot\vec{B} = -9+20-20 = -9$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & -5 \\ -3 & 5 & 4 \end{vmatrix} = 41\hat{i} + 3\hat{j} + 27\hat{k}$$

2a) $\frac{dw}{dz} - 2w = z^2 e^z$

is a 1st order linear ODE with the integrating factor

$$\mu(z) = \text{Exp}\left[\int -2\,dz\right] = e^{-2z}$$

and the general solution

$$w(z)\mu(z) = A + \int z^2 e^z \mu(z)\,dz$$

$$\therefore w(z)e^{-2z} = A + \int z^2 e^z e^{-2z} dz = A + \int z^2 e^{-z} dz$$

$$= A - \left[(z^2+2z+2)e^{-z}\right]$$

$$\therefore W(z) = Ae^{2z} - (z^2+2z+2)e^{z}$$

b) $\frac{dy}{dx} = \frac{2x(y^3-4y)}{x^2+4}$ is separable.

Rewrite in the form.

$$\frac{dy}{y^3-4y} = \frac{2x}{x^2+4}dx \Rightarrow \frac{1}{y(y-2)(y+2)}dy = \frac{2x}{x^2+4}dx$$

By Partial Fractions.

$$\frac{1}{y(y-2)(y+2)} = \frac{A}{y} + \frac{B}{y-2} + \frac{C}{y+2} \Rightarrow A(y-2)(y+2) + By(y+2) + Cy(y-2) = 1$$

$\therefore$ @ $y=0$ $-4A=1 \Rightarrow A = -1/4$

@ $y=2$ $8B=1 \Rightarrow B = 1/8$

@ $y=-2$ $8C=1 \Rightarrow C = 1/8$

Also use the substitution $x^2+4 = z$, $2x\,dx = dz$ to get

$$\frac{1}{8}\left[\frac{dy}{y+2} + \frac{dy}{y-2} - 2\frac{dy}{y}\right] = \frac{1}{z}dz$$

$$\therefore \ln|y+2| + \ln|y-2| - 2\ln|y| = \ln|A| + 8\ln|z|$$

Using properties of ln

namely $\ln(ab) = \ln(a) + \ln(b)$

$\ln(a/b) = \ln(a) - \ln(b)$

and $\ln(a^b) = b\ln(a)$

we get

$$\ln\left|\frac{(y+2)(y-2)}{y^2}\right| = \ln\left|Az^8\right| \qquad \text{But } z = x^2+4$$

$$\therefore \ln\left|\frac{y^2-4}{y^2}\right| = \ln\left|A(x^2+4)^8\right|.$$

OR $$\frac{y^2-4}{y^2} = A(x^2+4)^8$$

c) The ODE

$$(4xy + xe^{xy} + 2y)\frac{dy}{dx} = -(2y^2 + ye^{xy})$$

when rewritten in the form

$$(2y^2 + ye^{xy})\,dx + (4xy + xe^{xy} + 2y)\,dy = 0$$

is an exact ODE because

$$\frac{\partial}{\partial y}\left[2y^2 + ye^{xy}\right] = 4y + e^{xy} + xye^{xy}, \text{ and}$$

$$\frac{\partial}{\partial x}\left[4xy + xe^{xy} + 2y\right] = 4y + e^{xy} + xye^{xy}$$

$\therefore$ there exists a function $\phi(x,y) = c$ such that

(i) $$\frac{\partial\phi(x,y)}{\partial x} = 2y^2 + ye^{xy} \qquad \text{and}$$

(ii) $$\frac{\partial\phi(x,y)}{\partial y} = 4xy + xe^{xy} + 2y.$$

Integrate (i) w.r.t. "x" holding y constant to get

(iii) $\phi(x,y) = 2xy^2 + e^{xy} + f(y)$

From (iii) $\frac{\partial \phi(x,y)}{\partial y} = 4xy + xe^{xy} + \frac{df(y)}{dy}$

But from (ii) $\frac{\partial \phi(x,y)}{\partial y} = 4xy + xe^{xy} + 2y$

$\therefore \frac{df(y)}{dy} = 2y \quad \Rightarrow f(y) = y^2 \quad (iv)$

$\therefore$ The general solution from (iii) using (iv) is

$\phi(x,y) = 2xy^2 + e^{xy} + y^2 = c.$

3 Given the 2^{nd} order linear ODE with constant coefficients of the form

$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = 0$

there exists a solution of the form $y = e^{kx}$ where K satisfies the auxillary equation.

$AK^2 + BK + C = 0$

The given ODE is classified as being
(i) Elliptic when $B^2 - 4AC < 0$ (Underdamped)
(ii) Parabolic when $B^2 - 4AC = 0$ (Critically damped)
(iii) Hyperbolic when $B^2 - 4AC > 0$ (Overdamped).

a) $\frac{d^2y}{dx^2} + 4y = 0$ has auxillary equation

$K^2 + 4 = 0$

with roots $K_1 = 2j$ and $K_2 = -2j$

$\therefore$ elliptic.

The most general solution is

$$y(x) = C_1 e^{k_1 x} + C_2 e^{k_2 x}$$
$$= C_1 e^{2j x} + C_2 e^{-2j x}$$

The more common form is

$$y(x) = A \cos(2x) + B \sin(2x)$$

b) $\dfrac{d^2 z}{d\theta^2} - 9z = 3\theta^2 - 5$

The auxillary equation $k^2 - 9 = 0$ has ~~the root~~ $k_1 = -3$, $k_2 = 3$

$\therefore$ The most general solution of $\dfrac{d^2 z_c}{d\theta^2} - 9z_c = 0$ is

$$z_c(\theta) = C_1 e^{-3\theta} + C_2 e^{3\theta}$$

Let a solution of $\dfrac{d^2 z_p}{d\theta^2} - 9 z_p = 3\theta^2 - 5$ be of the form $z_p = a\theta^2 + b\theta + c$

$\therefore$ $dz_p/d\theta = 2a\theta + b$, $\dfrac{d^2 z_p}{d\theta^2} = 2a$

We want $2a - 9[a\theta^2 + b\theta + c] = 3\theta^2 - 5$

$\Rightarrow$ θ^2: $-9a = 3$ $\Rightarrow$ $a = -1/3$

θ^1: $-9b = 0$ $\Rightarrow$ $b = 0$

θ^0: $2a - 9c = -5$ $\Rightarrow$ $c = \frac{2}{9}a + \frac{5}{9} = -\frac{2}{27} + \frac{15}{27} = \frac{13}{27}$

The most general solution is

$$z(\theta) = z_c(\theta) + z_p(\theta)$$
$$= C_1 e^{-3\theta} + C_2 e^{3\theta} + \left(\frac{-\theta^2}{3} + \frac{13}{27}\right)$$

c) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

The auxillary equation $k^2+4k+4=0$ has roots $k_1 = k_2 = -2$

$\therefore$ One solution is $y_1(x) = e^{-2x}$

A second solution is $y_2(x) = xe^{-2x}$

$\therefore$ the most general solution is

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$
$$= C_1 e^{-2x} + C_2 x e^{-2x}$$

d) $\frac{d^2w}{dx^2} + 4\frac{dw}{dx} + 2w = e^{3x}$

The auxillary equation $k^2+4k+2=0$ has roots of

$k_1 = -2-\sqrt{2}$ and $k_2 = -2+\sqrt{2}$

$$\therefore W_c(x) = C_1 e^{k_1 x} + C_2 e^{k_2 x}$$
$$= C_1 e^{-(2+\sqrt{2})x} + C_2 e^{-(2-\sqrt{2})x}$$

Let a solution of $\frac{d^2w_p}{dx^2} + 4\frac{dw_p}{dx} + 2w_p = e^{3x}$ be $W_p(x) = ae^{3x}$

$$\Rightarrow \frac{dw_p}{dx} = 3ae^{3x}, \frac{d^2w_p}{dx^2} = 9ae^3$$

$\therefore$ we want $9ae^{3x} + 4(3ae^{3x}) + 2(ae^{3x}) = 23ae^{3x} = e^{3x}$

$$\Rightarrow a = 1/23$$

$$\therefore W(x) = W_c(x) + W_p(x)$$
$$= C_1 e^{-(2+\sqrt{2})x} + C_2 e^{-(2-\sqrt{2})x} + \frac{1}{23} e^{3x}$$

4a) $\int \left[\frac{y+3y^5}{y^2} - 2e^{3y} + 7\cos(2y)\right] dy$

$= \int \frac{1}{y}\,dy + 3\int y^3\,dy - 2\int e^{3y}\,dy + 7\int \cos(2y)\,dy$

$= \ln|y| + \frac{3y^4}{4} - \frac{2}{3}e^{3y} + \frac{7}{2}\sin(2y) + A$

b) $\int \left[\frac{\sin(\theta) + \sin(\theta)\tan^2(\theta)}{\sec^2(\theta)}\right] d\theta$

$= \int \frac{\sin(\theta)}{\sec^2(\theta)}\left[1 + \tan^2(\theta)\right] d\theta = \int \frac{\sin(\theta)}{\sec^2(\theta)}\left[\sec^2(\theta)\right] d\theta$

$= \int \sin(\theta)\,d\theta = -\cos(\theta) + B.$

c) $\int y^3 \cos(2y)\,dy$

D		I
y^3	+	$\cos(2y)$
$3y^2$	-	$\frac{1}{2}\sin(2y)$
$6y$	+	$-\frac{1}{4}\cos(2y)$
6	-	$-\frac{1}{8}\sin(2y)$
0	+	$\frac{1}{16}\cos(2y)$

$= \frac{y^3}{2}\sin(2y) + \frac{3y^2}{4}\cos(2y)$

$- \frac{6y}{8}\sin(2y) - \frac{6}{16}\cos(2y) + C$

$= \left[\frac{y^3}{2} - \frac{3y}{4}\right]\sin(2y) + \left[\frac{3y^2}{4} - \frac{3}{8}\right]\cos(2y) + C$

d) $\int e^{3y}\sin(2y)\,dy$

D		I
$\sin(2y)$	+	e^{3y}
$2\cos(2y)$	-	$\frac{1}{3}e^{3y}$
$-4\sin(2y)$	+	$\frac{1}{9}e^{3y}$

$= \frac{1}{3}\sin(2y)e^{3y} - \frac{2}{9}\cos(2y)e^{3y}$

$-\frac{4}{9}\int e^{3y}\sin(2y)\,dy$

$\therefore \frac{13}{9}\int e^{3y}\sin(2y)\,dy = \frac{1}{3}\sin(2y)e^{3y} - \frac{2}{9}\cos(2y)e^{3y} + \frac{13D}{9}$

$\therefore \int \sin(2y)\, e^{3y} dy = \frac{3}{13} \sin(2y)\, e^{3y} - \frac{2}{13} \cos(2y) e^{3y} + D$

e) $\int [\cos(5\theta)\sin(2\theta)]\, d\theta$

Use the trigonometric multiple angle formula.

$\sin(a)\cos(b) = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$

with $a = 2\theta,\ b = 5\theta$

$\therefore \sin(2\theta)\cos(5\theta) = \frac{1}{2}[\sin(7\theta) + \sin(-3\theta)]$

$\therefore \int [\cos(5\theta)\sin(2\theta)]\, d\theta = \frac{1}{2}\int \sin(7\theta)\, d\theta - \frac{1}{2}\int \sin(3\theta)\, d\theta$

$= -\frac{1}{14}\cos(7\theta) + \frac{1}{6}\cos(3\theta) + E$

5a) $f(x,y) = x^2 y - x^3 y^2$

$\frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x}[x^2 y - x^3 y^2] = 2xy - 3x^2 y^2$

$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}[x^2 y - x^3 y^2] = x^2 - 2x^3 y$

$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f(x,y)}{\partial y}\right] = \frac{\partial}{\partial x}[x^2 - 2x^3 y] = 2x - 6x^2 y$

NOTE:

$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f(x,y)}{\partial x}\right] = \frac{\partial}{\partial y}[2xy - 3x^2 y^2] = 2x - 6x^2 y = \frac{\partial^2 f(x,}{\partial x \partial y}$

5 b) $f(x,y) = \tan(x^3 - y^2) = \tan(\phi)$ where $\phi = x^3 - y^2$

$$\frac{\partial f(x,y)}{\partial x} = \frac{df(\phi)}{d\phi}\frac{\partial \phi}{\partial x} = \frac{d[\tan(\phi)]}{d\phi}\frac{\partial}{\partial x}(x^3-y^2)$$
$$= [\sec^2(\phi)]\cdot(3x^2) = 3x^2\sec^2(x^3-y^2)$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{df(\phi)}{d\phi}\frac{\partial \phi}{\partial y} = \frac{d[\tan(\phi)]}{d\phi}\frac{\partial}{\partial y}(x^3-y^2)$$
$$= [\sec^2(\phi)](-2y) = -2y\sec^2(x^3-y^2)$$

$$\frac{\partial^2 f(x,y)}{\partial x\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f(x,y)}{\partial y}\right] = \frac{\partial}{\partial x}\left[-2y\sec^2(x^3-y^2)\right]$$

$$= -2y\frac{\partial}{\partial x}\left[\sec^2(x^3-y^2)\right] = -2y\left[\frac{d}{d\phi}\{\sec^2(\phi)\}\right]\frac{\partial \phi}{\partial x}$$
$$= -2y\left[2\sec(\phi)\cdot\sec(\phi)\tan(\phi)\right]\left\{\frac{\partial}{\partial x}(x^3-y^2)\right\}$$
$$= -4y\left[\sec^2(\phi)\tan(\phi)\right]\{3x^2\}$$
$$= -12x^2y\sec^2(x^3-y^2)\tan(x^3-y^2)$$

c) $f(\vec{r}) = xy^2z - 2xz^3 + y^{-1}z^2$

$$\frac{\partial f(\vec{r})}{\partial x} = \frac{\partial}{\partial x}\left[xy^2z - 2xz^3 + y^{-1}z^2\right] = y^2z - 2z^3$$
$$\frac{\partial f(\vec{r})}{\partial y} = \frac{\partial}{\partial y}\left[xy^2z - 2xz^3 + y^{-1}z^2\right] = 2xyz - y^{-2}z^2$$
$$\frac{\partial f(\vec{r})}{\partial z} = \frac{\partial}{\partial z}\left[xy^2z - 2xz^3 + y^{-1}z^2\right] = xy^2 - 6xz^2 + 2y^{-1}z$$

(d) $f(\vec{r}) = \exp(x^{-1}y^2z^3) = e^u$ with $u = x^{-1}y^2z^3$

$$\frac{\partial f(\vec{r})}{\partial x} = \frac{df(u)}{du}\frac{\partial u}{\partial x} = \frac{de^u}{du}\frac{\partial}{\partial x}(x^{-1}y^2z^3)$$

$$= e^u\left[-x^{-2}y^2z^3\right] = -\frac{y^2z^3}{x^2}e^{x^{-1}y^2z^3}$$

$$\frac{\partial f(\vec{r})}{\partial y} = \frac{df(u)}{du}\frac{\partial u}{\partial y} = \frac{de^u}{du}\frac{\partial}{\partial y}(x^{-1}y^2z^3)$$

$$= e^u\left[2x^{-1}yz^3\right] = \frac{2yz^3}{x}e^{(x^{-1}y^2z^3)}$$

$$\frac{\partial f(\vec{r})}{\partial z} = \frac{df(u)}{du}\frac{\partial u}{\partial z} = \frac{de^u}{du}\frac{\partial}{\partial z}(x^{-1}y^2z^3)$$

$$= e^u\left[3x^{-1}y^2z^2\right] = \frac{3y^2z^2}{x}e^{(x^{-1}y^2z^3)}$$

$$\frac{\partial^2 f(\vec{r})}{\partial y\partial z} = \frac{\partial}{\partial y}\left[\frac{\partial f(\vec{r})}{\partial z}\right] = \frac{\partial}{\partial y}\left[\frac{3y^2z^2}{x}e^{(x^{-1}y^2z^3)}\right]$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{3y^2z^2}{x}\right)\right]e^{(x^{-1}y^2z^3)} + \frac{3y^2z^2}{x}\frac{\partial e^{(x^{-1}y^2z^3)}}{\partial y}$$

$$= \frac{6yz^2}{x}e^{(x^{-1}y^2z^3)} + \frac{3y^2z^2}{x}\frac{\partial f(\vec{r})}{\partial y}$$

$$= \frac{6yz^2}{x}e^{(x^{-1}y^2z^3)} + \frac{3y^2z^2}{x}\left[\frac{2yz^3}{x}e^{(x^{-1}y^2z^3)}\right]$$

$$= \left[\frac{6yz^2}{x} + \frac{6y^3z^5}{x^2}\right]e^{(x^{-1}y^2z^3)}$$