STOCHASTIC AND DETERMINISTIC MODELS OF NOISES

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Abstract The paper describes several models which may be of use for explanation of the origin and properties of experimentally observed types of noises. The stochastic models are analyzed first, including the autoregressive scheme and the moving averages on the *n*-th order. Galton board model and generalized baker map represent deterministic systems with rich set of properties. The results presented in the second half of the paper are intimately connected to the theory of deterministic chaos, and the crucial role is played there by the attractors in phase spaces of the systems.

Keywords: Models of noises, stochastic and deterministic models, generalized baker map

1. Introduction

It is generally admitted that understanding the origin and properties of variety of noises may be aided by mathematical models which are able to generate many different types of noises by changing only a few model parameters. Even if such models will not yield *immediate* physical interpretation, they can help us in our search for physical mechanisms by identifying model components, which critically influence relevant properties of generated processes, such as spectrum.

The models can be subdivided into stochastic and deterministic ones. The former have an appeal of simplicity and plausibility, the latter seem to promise better representation of the physical essence of the problem. One should evidently expect that both approaches will ultimately converge to the same results because conceptually both should represent one and the same physical phenomenon.

Noises observed in physical systems represent paradigmatic stochastic processes and therefore the most natural approach to them relies on their simulation by various types of stochastic mathematical models. Only two such models are analyzed in this context closely, namely the nonlinear generalization of autoregressive scheme of the first order, and the sequence of moving averages with variable order.

Deterministic models of noise may originate in the investigations of irreversibility of simple model systems. At first sight it might seem that this property is not related to the problem of noise, but the contrary is true. Irreversibility is found in systems with sufficiently chaotic dynamics and the latter is at the same time a necessary condition for observing noise in deterministic systems. Preliminary results seem to point to the conclusion that the joint treatment of irreversibility and noise represents the correct direction in which one has to go to arrive at the solution of both problems: the origin and essence of irreversibility, and the characterization of noises appearing in deterministic, microscopically reversible systems, with seemingly irreversible macroscopic behavior.

The paper reflects the author's preferences and experiences with different models and could be considered as a review of his work. The contribution begins with the analysis of stochastic models. One of the simplest is the one based on nonlinear generalization of autoregressive scheme of the first order.

2. Autoregressive scheme

The scheme is described by the iterative prescription

$$y_{k+1} = g(y_k) + \sigma\nu_{k+1}, \tag{1}$$

where $\{\nu_k\}$ denotes the sequence of uncorrelated normally distributed random variables with zero expectation and unit variance. The value σ represents standard deviation of a white noise source. It is evident that for $g(y) \equiv 0$ the scheme (1) will generate the white noise sequence and for $g(y) \equiv y$ the brownian one [1].

For a well defined set of functions $g(y)$ the above scheme was shown to generate noises with $1/f^{k}$-like spectra with $0 \le k \le 2$. However, the generation of $1/f$ noise using this model did not seem too convincing. Nevertheless, it was able to generate very satisfactory bistable burst noise [2]. To this effect one had to choose $g(y)$ to be a step function $g(y) = a\,\mathrm{sgn}(y)$ with positive constant a. With such a choice of $g(y)$, the waveform generated by the scheme (1) strongly resembled that of the bistable burst noise. The two values a, $-a$ represent evidently its attractors.

Numerical simulations performed for values $\sigma < a$ with samples of the length $1000 \leq N \leq 10000$ confirmed that the sequences generated by the scheme – and with eliminated white noise component – have Lorentzian spectrum which, at higher frequencies, decays as $1/f^k$ with $k \approx 2$. This result is easy to prove also analytically by demonstrating that the sequence generates random telegraph signal. The proof demonstrated that the latter may be obtained independently by similar simple iteration.

3. Moving averages

It is a well-known fact that any continuous spectrum of a stationary stochastic sequence can be arbitrarily closely approximated by the spectrum of some random sequence of moving averages of order n – or MA(n) for short. We will treat $1/f$ noise as MA(n) with n taken as a *variable* parameter.

Consider a stationary sequence of random variables $\{y_t\}$ having the spectral density $S(\lambda)$ of that of $1/f$ noise. The covariance function $C_n(h)$ of such a sequence is related to its spectrum via the Wiener-Khintchine theorem, which enables to find it as a function of n. Consider now MA(n) generated by coefficients $\{a_k\}$ ($k = 0 \ldots n$) and defined as

$$y_t = \sum_{k=0}^{n} a_k \nu_{t-k}, \qquad (2)$$

where $\{\nu_t\}$ is a sequence of totally uncorrelated random variables with unit variance. The covariance function of the sequence $\{y_t\}$ equals

$$R_n(h) = \sum_{k=0}^{n-h} a_k a_{k+h}, \quad h = 0, 1, 2, \ldots, n \qquad (3)$$

Choosing now a_k in such a way that $R_n(h) = C_n(h)$, the discretized spectrum of MA(n) will be $1/|\lambda_s|$ ($s = 0\ldots n$). We are thus facing the problem of finding, for any finite n, the corresponding set $\{a_k\}$. The solution will give us a MA(n) sequence with $1/f$ spectrum for any sample length n.

We have solved the problem numerically [3] using the Seidel iteration method for 12 values of n, ranging from 100 to 10000. The results show that $a_k < 1/\sqrt{k}$ (when counting k from 1), the difference between the left and right hand values being very small for all k except the small vicinity of $k = n$, where a_k falls off quickly.

The results allow the following physical interpretation. The coefficients a_k may be given the meaning of memory terms, or (decaying) correlations. Realistic correlations are nonzero for any k, so that realistic processes should be faithfully simulated by MA(n) with n tending to infinity. Since the infinite sum of a_k^α is defined only for α > 1/2, this result is amenable to the interpretation that 1/*f* noise can be viewed as an undefined limit of well defined processes. This behavior represents a phenomenon not unfamiliar in physics and seems to be deeply related to the divergence of 1/*f* spectrum. This may also have some relation to the Fourier invariance of the Fourier (not power) spectrum of 1/*f* noise.

4. Noise in deterministic systems

The deterministic approach to noise is based on the well known fact that many deterministic systems manifest all the features of random processes, so that one can expect various kinds of noise in them. The paradigmatic explanation of this observation relies on the theory of deterministic chaos. The hard problem is, however, how to explain the specific character of fluctuations observed in different systems, expressed usually in the form of spectra, describing the distribution of amplitudes of constituting partial waves over their frequencies.

There are in principle two great groups of dynamical systems which exhibit chaotic behavior – equilibrium and nonequilibrium ones. Archetypal equilibrium systems are conservative and the state of equilibrium is considered a consequence of chaotic motion of phase point representing the state of the system in phase space. There is no problem to analyze equilibrium in deterministic systems if they are sufficiently chaotic. The situation is essentially different with nonequilibrium systems, where the state of nonequilibrium has to be supported by outer influences and the system cannot be therefore isolated: there must be a flow of energy through the system and since this energy should not accumulate in the system, it must be dissipated. The dissipation, however, was until recently considered quite naturally an irreversible process and thus a process not obeying deterministic laws. Only recently a new treatment of dissipation appeared, based on the use of "reversible thermostatting", which enables to study also non-equilibrium reversible deterministic systems.

In the following we will study two very simple models. The first one is the model of a particle moving in a plane among regularly arranged scatterers, the second one is a still simpler model generalizing the standard well-known baker map. The former of the models has a spectrum with power law dependence, the latter one exhibits a different type of noise, but definitely not the power law. Both models have strange attractors in their phase spaces so that their comparison does not confirm our originally favorite hypothesis [4], that the presence of such fractal objects *always* implies 1/*f* noise. We will try to demonstrate under what conditions one can expect such implication to hold.

5. Galton board, or driven Lorentz gas

In an attempt to reconcile time-reversibility of microscopic equations of motion with irreversibility of macroscopic flows, Nosé-Hoover equations of motion began to be studied recently [5]. The aim is a better understanding of explicitly *nonequilibrium* systems driven away from equilibrium by boundary conditions. The latter impose velocity (temperature) gradients on the systems and in such a way cause steady nonequilibrium states. In such systems, which can undergo corresponding *reversible* momentum and energy exchanges with their surroundings, the phase-space states are no more visited with equal probabilities.

The basis of the approach is to introduce heat reservoirs into mechanics implicitly, rather than explicitly, this being done by constraining the kinetic energy of selected "thermostatted" degrees of freedom of a many-body system. This can be achieved with the help of Gauss' principle of least constraint, satisfying also nonholonomic constraints. As a result, additional time-reversible "friction" force appears, which causes the phase flow to collapse onto a strange attractor. Theoretical analysis shows that the systems obeying Nosé-Hoover equations of motion are reversible, ergodic and have negative characteristic Lyapunov exponents.

The typical evolution of such systems is then the following. In the forward direction of time the phase-space hypervolumes shrink very quickly to a fractal object which is stable. Within the zero-volume fractal subspace the motion is chaotic and formally reversible, but the time-reversed trajectory is not stable. The reversed repellor object – differing from attractor just by that the signs of all the momenta and friction coefficients are reversed – is unstable. The lowered fractal dimensionality of the attractors implies the effective loss of degrees of freedom.

Among the models studied in the context of Nosé-Hoover mechanics, there is one which seems to be highly relevant to understanding the properties of some noises. It is the model of externally driven 2-dimensional Lorentz gas, or Galton board [6]. In this model a point mass is driven through an infinite planar lattice of circular elastic scatterers (of radius R) by an external field E (pointing in x direction). The particle would be accelerated by the field, so in order to achieve a stationary nonequilibrium state it is necessary to constrain the kinetic energy which, using the Gauss' principle, leads to the following equations of motion

$$\begin{aligned} \dot{x} &= p_x / m, & \dot{y} &= p_y / m, \\ \dot{p}_x &= E - \zeta\, p_x, & \dot{p}_y &= -\zeta\, p_y, \end{aligned} \tag{4}$$

with the thermostat (friction) variable of the form $\zeta = p_x E/\mathbf{p}^2$. Observing the dynamics only at collisions of the particle with the scatterer, one obtains a Poincaré map in a two dimensional collision space. For non-zero field, measured by the value of $EmR/\mathbf{p}^2$, one observes trajectories approaching strange attractors with fractal structure and with Hausdorff dimension D, for which one finds $1 < D < 2$. The dependence of D on E is not simple but in general D falls with growing E and for $E = 0$ we have, as expected, $D = 2$.

Since the Lorentz gas model has some features distantly reminding motion of charge carriers through lattice of atoms, we suspected that it could exhibit $1/f^k$ spectrum. To verify this conjecture, we analyzed 2000 samples of simulated particle trajectories (suffering 10 000 collisions each) as represented by position vectors and vectors of momenta. Spectra of individual trajectories were analyzed using the standard FFT and subsequently averaged. It turned out that the power spectrum of trajectories in the subspace of positions reminded brownian spectrum, whereas the spectrum of momenta was surprisingly smooth and close to $1/f$. In fact it was of $1/f^k$ type with k exceeding very slightly the value of $k = 1$.

6. Generalized baker map

We conclude by the analysis of generalized baker maps, which are partly contracting and partly dilating, representing thus simple models of reversible and at the same time dissipative evolution [7]. The analysis of the maps has shown that the spectrum of their orbits is white. This seems to demonstrate that the presence of strange attractor alone is not sufficient for appearance of $1/f$ noise, as is sometimes suggested. The example further demonstrates that nonequilibrium states are not necessarily results of global dissipation – local dissipation is quite sufficient to have nonequilibrium steady state.

The model is defined by the transformation B_w denoted as the “generalized baker map” (GBM for short) which preserves the measure globally. The map is piecewise linear and it acts differently to the left and to the right of the vertical line $x = (w - 1)/w$ (dividing line). To the left, i. e. for $0 < x \le (w-1)/w$, we have $B_w(x,y) \equiv L_w(x,y)$, with

$$L_w(x,y) = (xw/(w-1),\ y/w), \tag{5}$$

and to the right, we have $B_w(x,y) \equiv R_w(x,y)$, with

$$R_w(x,y) = (xw - w + 1,\ y + (1-y)/w). \tag{6}$$

Being piecewise linear, the map represents the simplest nonlinear transformation. The contraction ratio is defined by the parameter $w \ge 1$ such that the area $(w - 1)/w$ of E, to the left of dividing line, is being contracted to $1/w$. For $w = 2$, the map becomes standard baker map, which is measure preserving also locally, and for $w = 1$ it is the linear identity map. For $w \neq 2$, the local measure preservation does not hold and with growing $w > 2$ the local contraction (of the left part of E) grows. This is the sense in which the model generalizes the classical baker map (hence the name). The consequence of the local contraction is the appearance of an attractor in E: for $w > 2$, any point, or any subset of E approaches (in the forward direction of time) a multifractal object consisting of infinite set of lines parallel to the x-axis. This attractor is self-similar and can be generated by successive applications of B_w to the line $y = 0$ (primary line). Further analysis disclosed that with growing w the information dimension D_1 of the attractor falls from $D_1 = 2$, for $w = 2$, to $D_1 \rightarrow 1$, for $w \rightarrow \infty$.

The reversibility of the map implies that the analogue to time (velocity) reversal in this model is "rotation" of the "phase" point around the "second diagonal", i.e. around the line $y = 1 - x$. Thus, to any attractor there corresponds a repellor, consisting of lines orthogonal to the lines of the attractor. The specific case of $w = 2$ has also the attractor-repellor pair. The attractor (repellor) consists in this specific case of equidistant horizontal (vertical) lines so that it is "hidden", i. e. macroscopically invisible. The analysis of B_w for arbitrary w has therefore disclosed this hitherto unknown (or at least underestimated) property of the classical baker map.

If we consider E to be a 2-dimensional phase space, then contraction of its subset, induced by B_w, might be interpreted as "cooling", and expansion of the rest as "heating". Then B_w may be thought of as representing a "dynamics" of a system being cooled and heated at opposite sides. Evidently, B_w does not have any causal connection to physically relevant thermostatting – it is just its resulting behavior, which mathematically simulates thermostatting. We can, nevertheless, view this map as an extremely simple model of thermostatting, exactly because under its action one part of E is being contracted and another one expanded, and also because the presence of an attractor demonstrates the existence of a stable nonequilibrium state.

That is why we have analyzed the spectrum of orbits generated by this map. Evidently we could not expect to find $1/f$ noise, if for no other reason, then because the orbits are bounded. Equally bounded are distances between neighboring points on orbits which could be regarded as discrete analogue of velocities, and therefore neither these increments can have $1/f$ noise. If however the model would be endowed with inherent properties able to generate such noise, we should be able to observe at least some signs of such noise. Closer analysis however showed that the spectrum of y-components of orbits was unambiguously white and the spectrum of increments between two neighboring points on an orbit was also white except in a small neighborhood of $f = 0$.

White noise of y-coordinates is quite understandable because the x-coordinates of points on an orbit move chaotically and cover the interval (0 ; 1] uniformly. We can regard them as “random” values. Their position with respect to dividing line decides whether in the next iteration the y-coordinate of a point will be submitted to the action of L_y or R_y. This means that the sequence of applications of operators L_y, R_y is “random” as well, which immediately implies that the noise of x- and y-coordinates is white. Similar reasonings apply to increments of y between iterations.

To summarize this section we may state that the example of generalized baker map reveals that not every thermostatted system with strange attractor has $1/f^{k}$ (k>0) spectrum. It demonstrates further that nonequilibrium states are not necessarily results of global dissipation – local dissipation is quite sufficient to have nonequilibrium steady state. The behavior of spectrum of Galton board then does not represent a situation to be expected everywhere. How typical the Galton board model is for a class of similar systems, deserves further investigation.

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