Stock Price Modelling by Long-Memory Processes: Overview of the Fractional Brownian Motion Approach

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The Black-Scholes pricing model...
What's next?

In the classical Black-Scholes model the stock price S is defined by

$$dS_t = \mu S - tdt + \sigma S_t dW_t, \;\; S_0 > 0;$$

the bond price is $B_t = \mathrm{e}^{rt}$,

where parameters $\mu \in \mathbb{R},\ \sigma, r \in \mathbb{R}^+$ supposed to be known.

The randomness of the stock price S is due to Brownian motion W.

Traditionally we consider a simple B-S model without dividends, transaction costs and any limitations on short-selling of the stock, with the same interest rate r for lending and saving on the bond.

The main properties of the B-S prising model are:

- the model is **arbitrage free**
- we can find a unique price for options on the stock S, e.g. the fair price of an European call-option $(S_T - K)^+$ is
$$S_0\Phi(d_+) - K\mathrm{e}^{-rt}\Phi(d_-),$$
where
$$d_\pm = \frac{\ln(S_0/K) + rT \pm \sigma^2/2T}{\sigma T^{1/2}}$$
- we can hedge options using the Itô-Clark-Ocone formula.

Problems

The B-S model stipulates that the log-returns

$$R_{t_k} = \log \frac{S_{t_k}}{S_{t_{k-1}}} =$$

$$= \left(\mu - \frac{\sigma^2}{2}\right)(t_k - t_{k-1}) + \sigma(W_{t_k} - W_{t_{k-1}})$$

are independent normal random variables.

The dependence structure of the log-returns have been studied using the Hurst parameter H. In the uncorrelated case one should have $H = \frac{1}{2}$. However, some recent empirical studies show that $H \simeq 0.642$ (cf. W. Willinger, M. Taqqu, V. Teverovsky (1999) [1]).

To overcome with this critical point, it has been proposed that the Brownian motion W should be replaced by *fractional Brownian motion (fBm)*.

Long-range dependence, self-similarity and fBm

The stationary sequence $X = (X_k)_{k\in\mathbb{N}}$ exhibits the statistical *long-range dependency property* if its autocorrelation function ρ satisfies

$$\lim_{k\to\infty} \frac{\rho(k)}{c_\rho k^{-l}} = 1$$

for some constants c_ρ and $l \in (0, 1)$.
A centered stochastic process $X = (X_t)_{t\in[0;T]}$ is said to be *statistically self-similar* with Hurst exponent H if

$$(X_t)_{t\in[0;T]} \overset{d}{=} (a^{-H} X_{at})_{t\in[0,T]}$$

for all $a > 0$. If, in addition, the process X is square integrable with stationary increments it follows that

$$\mathrm{Cov}(X_t, X_s) = \frac{\mathrm{Var} X_1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

The increments $Y_k := X_k - X_{k-1}$ are stationary with autocorrelation function

$$\rho(k) = \frac{1}{2}((k+1)^{2H} - 2k^{2H} + (k-1)^{2H}).$$

Therefore,

$$\lim_{k\to\infty} \frac{\rho(k)}{H(2H-1)k^{2H-2}} = 1.$$

and if $H \in (1/2, 1)$ the increments Y_k exhibit the long-range dependency property with $l = 2 - 2H$ and $c_\rho = H(2H-1)$.

The Fractional Brownian Motion B_t^H is a continuous and centered Gaussian process with the covariance function

$$\mathbb{E}\left(B_t^H B_s^H\right) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) =$$

$$= H(2H-1)\int_0^t \int_0^s |t-s|^{2H-2} ds dt$$

The fractional Brownian motion is the unique centered Gaussian process with stationary increments having the self-similarity property with Hurst index $H \in (0, 1)$.

The fBm was originally defined and studied by Kolmogorov ([2]) within a Hilbert space framework where it was called a Wiener helix. The name "fractional Brownian motion" comes from Mandelbrot and Van Ness ([3]). They defined it as a stochastic integral with respect to the standard Brownian motion:

$$B_t^H = \int_{-\infty}^{t} K(t,s)dW_s$$

with a deterministic kernel K depending on H.

If $H = 1/2$ then the fBm is just a standard Brownian motion. In the case $H > 1/2$ the fBm exhibits the statistical long-range dependency property as shown above. Hence, in financial modelling we usually assume that $H \in (1/2, 1)$

fBm is not semimartingale!

Let $\tau = (0 = t_0 < t_1 < ... < t_n = T)$ be a finite partition of the interval $[0, T]$ and for a stochastic process X set

$$s_p(X, \tau) := \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}|^p$$

The p-variation v_p of a stochastic process X over an interval $[0, T]$ is defined as

$$v_p(X, [0, T]) := \sup_{\tau} s_p(X, \tau)$$

The index of p-variation v of a process X is then

$$v(X; [0; T]) = \inf\{p > 0 : v_p(X, [0, T]) < \infty\}$$

if the set above is non-empty and ∞ otherwise. For fBm B_t^H with index $H \in (1/2, 1)$ one has $v(B_t^H) = 1/H$ (see Dudley and Norvaiša [4]).

For semimartingales M one must have $v(M) \in [0,1] \bigcup \{2\}$ (see Dudley and Norvaiša [4]). Thus, the fractional Brownian motion is not a semimartingale when $H \neq 1/2$. So, we cannot use the Itô theory to define stochastic integrals with respect to it. However, we can define a pathwise integrals as a refinement of the Riemann-Stieltjes integrals using the p-variation.

Pathwise integral with respect to fBm

For a function f on $[0,t]$ we define

$$s_p(f;\pi) := \sum_{t_k \in \pi} |f(t_k) - f(t_{k-1})|^p$$

If

$$v_p := \sup_{\pi} s_p(f;\pi)$$

is finite we say that f has *bounded p-variation* and denote $f \in \mathbb{V}_p$

Young [5] proved that if $f \in \mathbb{V}_p$ and $g \in \mathbb{V}_q$ for some p and q: $1/p + 1/q > 1$ then integral

$$\int\limits_0^T g(t)df(t)$$

exists in the Riemann-Stieltjes sense.

It has been shown that paths of the fBm B_t^H belongs almost surely to the space $\mathbb{V}_p$ iff $p < 1/H$ (cf. Dudley and Norvaiša [4]). Therefore, the Riemann-Stieltjes integral

$$\int\limits_0^T g(t)dB_t^H$$

exists a.s. if $g \in \mathbb{V}_q$ for some $q < 1/(1-H)$ a.s.

Wick-Itô-Skorohod integral w.r.t. fBm

Integration is based on Malliavin calculus (cf. Nualart D. [6]).

We define $\mathcal{L}$ as a linear span of the indicator functions $\{\chi_{[0,t]} : t \in [0,T]\}$ completed w.r.t. the inner product

$$\langle \chi_{[0,t]}, \chi_{[0,s]} \rangle_{\mathcal{L}} := \mathbb{E}\left(B_t^H B_s^H \right)$$

For step functions f, g we have

$$\langle f, g \rangle_{\mathcal{L}} = H(2H-1) \int_0^T \int_0^T f(s) g(t) |t-s|^{2H-2} ds dt$$

The mapping $\chi_{[0,t]} \mapsto B_t^H$ extends to an isometry between $\mathcal{L}$ and $\mathcal{L}_1$, the linear space of B^H. We denote by $B^H(\phi)$ the image of $\phi \in \mathcal{L}$ in this isometry. Let $\mathcal{C}$ be the set of random variables of the form

$$F = f\left(B^H(\phi_1), ..., B^H(\phi_n) \right)$$

for $n \geq 1$, $\phi_k \in \mathcal{L}, k = \overline{1, ..., n}$ and $f \in C_b^\infty(\mathbb{R}^n)$.

For $F \in \mathcal{C}$ the *Malliavin derivative* of F is the $\mathcal{L}$-valued random variable

$$D_t F := \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \left(B^H(\phi_1), ..., B^H(\phi_n) \right) \phi_k(t)$$

The *divergence* operation δ is defined as the adjoint of D. The domain of δ,

$$\mathrm{Dom}\, \delta = \left\{ u \in \mathcal{L} : |\mathbb{E}\langle DF, u \rangle_{\mathcal{L}}|^2 \leq c\mathbb{E}\left(F^2 \right) \right\}$$

For $u \in \mathrm{Dom}\, \delta$ the *divergence* $\delta(u)$ is a square integrable random variable defined by

$$\mathbb{E}(\delta(u)F) = \mathbb{E}\langle DF, u \rangle_{\mathcal{L}}$$

for all F.

The *Wick-Itô-Skorohod integral* of a process u w.r.t. fBm B_t^H is the divergence

$$\int_0^T u_t \delta B_t^H := \delta(u)$$

It can be shown that using a Wick product $F \diamond G$, under some regularity assumptions on u

$$\int_0^T u_t \delta B_t^H = \lim_{|\pi| \to 0} \sum_{t_k \in \pi} u_{t_{k-1}} \diamond \left(B_{t_k}^H - B_{t_{k-1}}^H \right)$$

Theorem 1. *Let for stochastic process* $u \in \operatorname{Dom} \delta$

$$\int_0^T \int_0^T |D_s u_t| |t-s|^{2H-2} ds dt < \infty$$

Then the Riemann-Stieltjes integral exists and we have

$$\int_0^T u_t dB_t^H = \int_0^T u_t \delta B_t^H +$$

$$+ H(2H-1) \int_0^T \int_0^T |D_s u_t| |t-s|^{2H-2} ds dt$$

Option pricing in fractional B-S models

Fractional Brownian motion is a non-semimartingale, therefore martingale measures do not exist and the "first fundamental asset pricing theorem":
no arbitrage is equivalent to existence of a martingale measure
doesn't hold. This creates difficulties for finding a rational price for hedging of the contingent claim f_T using martingale techniques.
However, in the fractional B-S model there is a unique equivalent measure $\mathbb{Q}$ under which the solution to

$$dS_t = \mu(t) S_t dt + \sigma S_t dB_t^H \text{(RS equation)}$$

or

$$dS_t = \nu(t) S_t dt + \sigma S_t \delta B_t^H \text{(WIS equation)}$$

is given by a geometric fBm $S_t = S_0 \mathrm{e}^{\widetilde{B}_t^H - \frac{1}{2} t^{2H}}$

For WIS fractional B-S model we can define *self-financing* condition

$$\delta V_t^{WIS}(\pi) = \pi_t \delta S_t$$

It has been shown (cf. Hu and Øksendal [7], Elliott and van der Hoek [8], Bender [9]) that WIS fractional B-S model is arbitrage free. The absence of arbitrage follows basically from the fact that WIS integral is centered.

Let $\pi_t = \frac{\partial h}{\partial x}(t, S_t)$ for a function h introduced by Hu and Øksendal in [7]

$$h(t,x) = x\Phi\left(\frac{\ln(x/K) + 1/2\left(T^{2H} - t^{2H}\right)}{\sqrt{T^{2H} - t^{2H}}}\right) -$$

$$-K\Phi\left(\frac{\ln(x/K) - 1/2\left(T^{2H} - t^{2H}\right)}{\sqrt{T^{2H} - t^{2H}}}\right)$$

Then we can obtain that $V_T^{WIS}(\pi) = h(T, S_T) = (S_T - K)^+$ and corresponding fair price for European call option is

$$C_T\left((S_T - K)^+\right) = h(0, S_0)$$

Regularized fBm

RS fractional B-S model is not arbitrage free. Arbitrage portfolios were constructed by Rogers [10], Shiryaev [11], Dasgupta [12].

To overcome the shortcomings of the RS fractional B-S model some remedies have been proposed, namely regularization of fBm and mixed models.

The regularization procedure to the fBm was suggested by Rogers [11] and was further studied by Cheredito [13]. Using the Mandelbrot and Van Ness integral representation

$$B_t^H = \int_{-\infty}^{t} K(t,s) dW_s$$

we can regularize the fBm in the following way.

Replace the kernel K by a "regularized" kernel $\widetilde{K}$ so that the Gaussian process $\widetilde{B}_t^H = \int\limits_{-\infty}^{t} \widetilde{K}(t,s)dW_s$ has stationary increments and satisfies the following two conditions

1) $\widetilde{B}_t^H$ is close to the original fBm B_t^H in the sense that Cov$(B_t^H, B_s^H) \approx$ Cov$(\widetilde{B}_t^H, \widetilde{B}_s^H)$

2) the law of the $\widetilde{B}_t^H$ is equivalent to the law of the standard Brownian motion 1) implies that $\widetilde{B}_t^H$ is another Gaussian process with the same long range dependence as original fBm;

2) implies that the new fractional B-S model w.r.t. $\widetilde{B}_t^H$ has a unique equivalent martingale measure. Hence, it is arbitrage-free and complete.

Mixed Brownian-fBm model

Consider a mixed Brownian-fBm model

$$dS_t = S_t(\mu dt + \varepsilon dW_t + \sigma dB_t^H)$$

Recently it has been shown that the mixed process $\varepsilon dW_t + \sigma dB_t^H$ is equivalent in law to εdW_t whenever W and B^H are independent and the index $H \in (3/4, 1)$. Hence this model is similar to the regularized one considered above in the sense of existence of the unique risk neutral measure.

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