THE STOCHASTIC FUBINI THEOREM FOR INTEGRALS CONTAINING RANDOM INTEGRAND AND FRACTIONAL BROWNIAN MOTION AS INTEGRATOR.

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Abstract. In the present paper the stochastic Fubini theorem for integrals containing random integrand and fractional Brownian motion as integrator has been proved. The special form of integrand is considered for the purpose to change the measure in the stochastic integral containing fractional Brownian motion.

1. Introduction.

Let (Ω, F, P) be complete probability space with filtration $(F_t, t \geq 0)$. This composition we will denote by $(\Omega, F, (F_t)_{t\geq 0}, P)$. Throughout this paper, we denote by $\left(B_t^H, (F_t)_{t\geq 0}, P\right)$ a normalised fractional Brownian motion (FBM) with Hurst parameter $H \in (1/2, 1)$, characterised by the following properties:

(P1) B_t^H has stationary increments;
(P2) $B_0^H = 0$, and $\mathbb{E}B_t^H = 0$ for all t;
(P3) $\mathbb{E}\left(B_t^H\right)^2 = |t|^{2H}$for all t;
(P4) B_t^H is Gaussian;
(P5) B_t^H has continuous sample paths.

Denote $\mathcal{H}^\alpha_{[a,b]}$ the space of Hölder continuous function with index α on a segment $[a,b]$ with the norm $\|x\|_{\mathcal{H}^\alpha_{[a,b]}} = \sup\limits_{t\in[a,b]} |x(t)| + \sup\limits_{a\leq t<t'\leq b} \frac{|x(t)-x(t')|}{|t-t'|^\alpha}$.

We shall use the Hölder continuity of the trajectories of fBm with any index $0 < \alpha < H$ (i.e. $B_t^H \in \mathcal{H}^\alpha_{[a,b]}$ for any $0 < \alpha < H$ and $0 < a < b$.) According to the papers [1], [2] the Hölder continuity of B_t^H ensures the pathwise existence of the integral

$$\int\limits_a^b f(s)\, dB_s^H, \;\; 0 < a < b$$

with probability 1 for any measurable random function $f \in \mathcal{H}^\beta_{[a,b]}$, with $\beta > 1 - H$ and without any restrictions on the adaptedness of f.

As it proved in [1] this integral is a limit in probability of Riemann-Stieltjes sums.

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Let the Hölder function x on the semi-interval $[a,b)$ has $\lim\limits_{t\to b-} x(t)$. Denote its Hölder norm as

$$\|x\|_{\mathcal{H}^{\alpha}_{[a,b)}} = \sup_{t\in[a,b)} |x(t)| + \sup_{a\le t<t'<b} \frac{|x(t)-x(t')|}{|t-t'|^{\alpha}}.$$

Definition 1. *The function $f:\mathbb{R}\to\mathbb{R}$ is called piecewise Hölder with index α, $0<\alpha<1$ on the segment $[T_1,T_2]\subset\mathbb{R}$, if there exists a finite set of semi-intervals that don't intersect $\{\mathcal{R}_1,\mathcal{R}_2,....\mathcal{R}_N \mid \bigcup\limits_{i=1}^{N}\mathcal{R}_i\bigcup\{b\}=[T_1,T_2]\}$ and such that the function f is Hölder with index α on any of them.*

Definition 2. *The norm of piecewise Hölder function is denoted as $C^{\alpha}_{f}([T_1,T_2]) = \max\limits_{i=\overline{1,N}}\|f\|_{\mathcal{H}^{\alpha}_{\mathcal{R}_i}}$.*

2. Auxiliary results.

Let real functions $f\in\mathcal{H}^{\alpha}_{[a,b]}$, $g\in\mathcal{H}^{\beta}_{[a,b]}$, where $\alpha+\beta>1$. Then the following statements hold:

(1) according to the theorem 4.2.1 [1], there exists Riemann-Stieltjes integral

$$\int\limits_a^b f(x)\,dg(x) := \lim_{|\pi^{(l)}|\to 0}\sum_{k=1}^{l} f(x_k)\left[g(x_{k+1})-g(x_k)\right]; \tag{1}$$

(2) according to the theorem 21 [2], there exists sequences $\{f_n,g_n\}\in\mathbb{C}^1[a,b]$ such that $\|f_n-f\|_{\mathcal{H}^{\alpha}_{[a,b]}}\to 0$, $\|g_n-g\|_{\mathcal{H}^{\alpha}_{[a,b]}}\to 0$, $n\to\infty$.

Consider a sequence of partitions of a segment $[a,b]$:

$$\pi^{(l)} = \pi^{(l)}_{[a,b]} = \left\{a=t^0_l<t^1_l<...<t^l_l=b\right\},$$

such that $\pi^{(l)}\subset\pi^{(l+1)}$ and $\left|\pi^{(l)}\right|\to 0$, $l\to\infty$.

Denote $\triangle g(t^k_l) = g(t^k_l)-g(t^{k-1}_l)$, $1\le k\le l$.

Further we shall use the following estimations for integrals containing Hölder functions.

Lemma 1. *Let $f\in\mathcal{H}^{\alpha}_{[a,b]}$, $g\in\mathcal{H}^{\beta}_{[a,b]}$, $\alpha+\beta>1$, $\{f_n,g_n\}$, $n\ge 1$ are sequences from $\mathbb{C}^1[a,b]$ such that $\|f_n-f\|_{\mathcal{H}^{\alpha}_{[a,b]}}\to 0$, $\|g_n-g\|_{\mathcal{H}^{\beta}_{[a,b]}}\to 0$, $n\to\infty$.*

Then: 1) $\int\limits_a^b f(t)\,dg(t) = \lim\limits_{n\to\infty}\int\limits_a^b f_n(t)g_n'(t)\,dt$;

2) the following estimation holds

$$\left|\int\limits_a^b f(t)\,dg(t)\right| \le C\|f\|_{\mathcal{H}^{\alpha}_{[a,b]}}\|g\|_{\mathcal{H}^{\beta}_{[a,b]}}\max\left\{(b-a)^{1+\varepsilon},(b-a)^{\beta}\right\}$$

3) if $f(a)=0$, then

$$\left|\int\limits_a^b f(t)\,dg(t)\right| \le C\|f\|_{\mathcal{H}^{\alpha}_{[a,b]}}\|g\|_{\mathcal{H}^{\beta}_{[a,b]}}(b-a)^{1+\varepsilon},$$

$0 < \varepsilon < \alpha + \beta + 1$, $C > 0$ *is constant, that doesn't depend on* f *and* g.

Proof. 1) According to our previous notations we have

$$\left|\int\limits_a^b f(t)\,dg(t) - \int\limits_a^b f_n(t)g_n'(t)\,dt\right| \le \left|\int\limits_a^b f(t)\,dg(t) - \sum_{k=1}^{l} f(t_l^k)\triangle g(t_l^k)\right| +$$

$$+\left|\int\limits_a^b f_n(t)g_n'(t)\,dt - \sum_{k=1}^{l} f_n(t_l^k)\triangle g_n(t_l^k)\right| + \left|\sum_{k=1}^{l} f(t_l^k)\triangle g(t_l^k) - \sum_{k=1}^{l} f_n(t_l^k)\triangle g_n(t_l^k)\right|.$$

According to (1), for $\delta > 0$ we can choose partition $\pi^{(l)}$ such that

$$\left|\int\limits_a^b f(t)\,dg(t) - \sum_{k=1}^{l} f(t_l^k)\triangle g(t_l^k)\right| < \delta \tag{2}$$

According to Corollary 20 ([2]) we obtain

$$\left|\int\limits_a^b f_n(t)g_n'(t)\,dt - \sum_{k=1}^{l} f_n(t_l^k)\triangle g_n(t_l^k)\right| \le C|\pi^{(l)}|^{\varepsilon}\|f_n\|_{\mathcal{H}^{\alpha'}_{[a,b]}}\|g_n\|_{\mathcal{H}^{\beta'}_{[a,b]}}, \tag{3}$$

where $0 < \alpha' < \alpha$, $0 < \beta' < \beta$, $\alpha' + \beta' = 1 + \varepsilon$. If $\|f_n - f\|_{\mathcal{H}^{\alpha}_{[a,b]}} \to 0$, $n \to \infty$, then, $\|f_n - f\|_{\mathcal{H}^{\alpha'}_{[a,b]}} \to 0$, $n \to \infty$ for $0 < \alpha' < \alpha$, and then $\|f_n\|_{\mathcal{H}^{\alpha'}_{[a,b]}} \le C_1$, where $C_1 > 0$ doesn't depend on $n \ge 1$.

Analogously, $\|g_n\|_{\mathcal{H}^{\beta'}_{[a,b]}} \le C_2$. From here and (3) we have estimation

$$\left|\int\limits_a^b f_n(t)g_n'(t)\,dt - \sum_{k=1}^{l} f_n(t_l^k)\triangle g_n(t_l^k)\right| \le C_3|\pi^{(l)}|^{\varepsilon}.$$

Choose l in such a way that $C_3|\pi^{(l)}|^{\varepsilon} < \delta$. If now under $\pi^{(l)}$ the relation (2) and last inequality hold, then for fixed l put n such that

$$\left|\sum_{k=1}^{l} f(t_l^k)\triangle g(t_l^k) - \sum_{k=1}^{l} f_n(t_l^k)\triangle g_n(t_l^k)\right| < \delta. \tag{4}$$

It is available, because $\sup\limits_{t\in[a,b]}|g_n(t) - g(t)| \le \|g_n - g\|_{\mathcal{H}^{\beta'}_{[a,b]}} \to 0$, as $n \to \infty$ analogously for f_n.

From (2)–(4) we obtain the proof of 1).

Statement 3) follows from 1) and Lemma 19 [2], according to which the estimation 3) hold for any $f \in \mathbb{C}_0^1[a,b]$, $g \in \mathbb{C}^1[a,b]$.

Statement 2) follows from 1) and theorem 22 ([2]). Really, according to 3),

$$\left|\int\limits_a^b f(t)\,dg(t)\right| \le C\|f\|_{\mathcal{H}^{\alpha}_{[a,b]}}\|g\|_{\mathcal{H}^{\beta}_{[a,b]}}(b-a)^{1+\varepsilon},$$

and then

$$\left|\int\limits_a^b f(t)\,dg(t)\right| \le C\|f\|_{\mathcal{H}^{\alpha}_{[a,b]}}\|g\|_{\mathcal{H}^{\beta}_{[a,b]}}\max\left\{(b-a)^{1+\varepsilon}, (b-a)^{\beta}\right\}.\triangleleft$$

Let f is piecewise Hölder function on $[a,b]$. It is evident that in this case there exists the sum $\sum\limits_{i=1}^{N}\int\limits_{a_i}^{b_i} f(t)\,dB_t^H$. The next result demostrates that this sum can be considered as a unique integral.

Lemma 2. *Let f is piecewise Hölder with index $\alpha > 1-H$ on segment $[a,b]$. Then there exists Riemann-Stieltjes integral $\int\limits_a^b f(s)\,dB_s^H = \sum\limits_{i=1}^{N}\int\limits_{a_i}^{b_i} f(s)\,dB_s^H$ and for any sequences of partitions of $[a,b]$ it can be represented in such a way*

$$\int\limits_a^b f(s)\,dB_s^H = \lim_{|\pi^{(l)}|\to 0}\sum_{k=1}^{l} f(s_l^k)\,\triangle B_{s_l^k}^H$$

Proof. As f is piecewise Hölder function there exists a finite set of semi-intervals that don't intersect $\left\{[a_i,b_i)|\bigcup\limits_{i=1}^{N}[a_i,b_i)\bigcup\{b\}=[a,b]\right\}$ Put $\pi_i^{(l)} = ([a_i,b_i)\bigcap\pi^{(l)})$, then $\left|\pi_i^{(l)})\right| \le \left|\pi^{(l)}\right|$. It follows evidently from boundedness of f and a continuity of B^H that

$$\sum_{j:t_l^j\in\pi_i^{(l)}} f(t_l^j)\triangle B_{t_l^j}^H \to \int\limits_{a_i}^{b_i} f(t)\,dB_t^H$$

even if $\pi^{(l)}$ does not contain a_i and b_i. Thus

$$\sum_{k:t_l^k\in\pi^{(l)}} f(t_l^k)\triangle B_{t_l^k}^H = \sum_{i=1}^{N}\sum_{j:t_l^j\in\pi_i^{(l)}} f(t_l^j)\triangle B_{t_l^j}^H \to$$

$$\to \sum_{i=1}^{N}\int\limits_{a_i}^{b_i} f(t)\,dB_t^H = \int\limits_a^b f(t)\,dB_t^H, \text{ as } \left|\pi^{(l)}\right| \to 0. \ \triangleleft$$

3. The stochastic Fubini theorem.

Let $0 < T_1 < T_2$, $\Phi = \Phi(s,u,\omega) : [T_1,T_2]^2 \times \Omega \to \mathbb{R}$ is random, measurable function in all its variables (s,u,ω).

Stochastic Fubini theorem. *Let there exists the set $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$, and under each $\omega \in \Omega'$ function $\Phi(s,u,\omega)$ satisfies the following conditions:*

1) $\forall\, s \in (T_1,T_2),:$ $\Phi(s,\cdot,\omega)$ is piecewise Hölder function on $u \in [T_1,T_2]$ with index $\alpha_1 > \frac{1}{2}$, and moreover its piecewise Hölder norm is bounded on s, i.e. there exists $C = C(\omega)$ that $C^{\alpha}_{\Phi(s,\cdot,\omega)}([T_1,T_2]) < C$;

2) function $\int\limits_{T_1}^{T_2}\Phi(s,u,\omega)\,dB_u^H$ is Riemann integrable on segment $[T_1,T_2]$.

Then there exist iterated integrals $I_1 := \int\limits_{T_1}^{T_2}\left(\int\limits_{T_1}^{T_2}\Phi(s,u,\omega)\,dB_u^H\right) ds$ and

$I_2 := \int\limits_{T_1}^{T_2} \left(\int\limits_{T_1}^{T_2} \Phi(s,u,\omega)\, ds \right) dB_u^H$, *and a.s. they coincide,* $I_1 = I_2$.

Proof. Further, everywhere in this proof we shall consider that $\omega \in \Omega'$ is fixed and the argument ω will be omitted. According to 1) and Lemma 2 there exists an iterated integral $\int\limits_{T_1}^{T_2} \Phi(s,u)\, dB_u^H$, then, taking in account the condition 2) of our theorem, we satisfy oneself of the existence of integral I_1. Since Φ is piecewise Hölder function, then, taking in account the following inequality

$$\int\limits_{T_1}^{T_2} |\Phi(s,u_1) - \Phi(s,u_2)|\; ds \le c \int\limits_{T_1}^{T_2} |u_1 - u_2|^{\alpha_1}\, ds = c(T_2 - T_1)|u_1 - u_2|^{\alpha_1}$$

we satisfy oneself that $\int\limits_{T_1}^{T_2} \Phi(s,u)\, ds$ is piecewise Hölder function on $u \in [T_1, T_2]$ with index α_1. Further, as B_u^H is Hölder with index $\beta > \frac{1}{2}$, $\alpha_1 + \beta > 1$, according to the theorem 4.2.1 ([1]) there exists iterated integral I_2. Integral I_1 can be represented as:

$$I_1 = \lim_{|\pi^{(l)}| \to 0} \sum_{k=0}^{l-1} \int\limits_{T_1}^{T_2} \Phi(s_l^k, u)\, dB_u^H \,\triangle s_l^k, \tag{5}$$

where $\pi^{(l)} = \{T_1 = s_l^0 < s_l^1 < .... < s_l^l = T_2\}$, $|\pi^{(l)}| \to 0$ under $l \to \infty$. Now for every fixed s_l^k of partition $\pi^{(l)}$ according to 1) there exists a finite set of points $\{u_{1,k} < u_{2,k} < ... < u_{l(k),k}\}$ of Hölder discontinuity of function $\Phi(\cdot, u)$. Further $\{T_1 = u_{0,l} < u_{1,l} < u_{2,l} < ... < u_{L(l)-1,l} < u_{L(l),l} = T_2\} := \bigcup\limits_{k=1}^{n} \{\{u_{1,k} < u_{2,k} < ... < u_{l(k),k}\}\} \cup \{T_1, T_2\}$. For each segment $[u_{i,l},\, u_{i+1,l}]$ consider the partition $\left\{\pi_{i,l}^{(m_i)},\ m_i \ge 1,\ i = 0, L(l)-1\right\}$, $\pi_{i,l}^{(m_i)} := \{u_{i,l} = u_{i,l}^{(0)} < u_{i,l}^{(1)} < ... < u_{i,l}^{(m_i)} = u_{i+1,l}\}$, $\left|\pi_{i,l}^{(m_i)}\right| \to 0,\ m_i \to \infty$. Then $\pi_l^{(m)} := \bigcup\limits_{i=0}^{L(l)-1} \pi_{i,l}^{(m_i)} \bigcup \{T_1, T_2\} = \{T_1 = u_n^{(0)} < u_n^{(1)} < ... < u_n^{(m)} = T_2\}$ is partition of $[T_1, T_2]$ with respect to variable u, and with diameter $\left|\pi_l^{(m)}\right| := \max\limits_{0 \le i \le l-1} \left|\pi_{i,l}^{(m_i)}\right|$, for which $\left|\pi_l^{(m)}\right| \to 0,\ \ m \to \infty$.

Now let estimate the difference between I_1 and I_2:

$$|I_1 - I_2| \le \left| I_1 - \sum_{k=0}^{l-1} \sum_{j=0}^{m-1} \Phi(s_l^k, u_l^{(j)}) \triangle B_{u_l^{(j)}}^H \,\triangle s_l^k \right| + \tag{6}$$

$$+ \left| I_2 - \sum_{j=0}^{m-1} \sum_{k=0}^{l-1} \Phi(s_l^k, u_l^{(j)})\, \triangle s_l^k \,\triangle B_{u_l^{(j)}}^H \right|$$

Further let estimate the first term of the inequality (6)

$$\left| I_1 - \sum_{k=0}^{l-1} \sum_{j=0}^{m-1} \Phi(s_l^k, u_l^{(j)}) \triangle B_{u_l^{(j)}}^H \,\triangle s_l^k \right| \le \left| I_1 - \sum_{k=0}^{l-1} \int\limits_{T_1}^{T_2} \Phi(s_l^k, u) dB_u^H \,\triangle s_l^k \right| +$$

$$+\sum_{k=0}^{l-1}\left|\int_{T_1}^{T_2}\Phi(s_l^k,u)dB_u^H-\sum_{j=0}^{m-1}\Phi(s_l^k,u_l^{(j)})\triangle B^H_{u_l^{(j)}}\right|\triangle s_l^k.$$

As Φ is piecewise Hölder, then according to Lemma 2

$$\left|\int_{T_1}^{T_2}\Phi(s_l^k,u)dB_u^H-\sum_{j=0}^{m-1}\Phi(s_l^k,u_l^{(j)})\triangle B^H_{u_l^{(j)}}\right|\to 0,\ m\to\infty$$

According to (5)

$$\left|I_1-\sum_{k=0}^{l-1}\int_{T_1}^{T_2}\Phi(s_l^k,u)dB_u^H\,\triangle s_l^k\right|\to 0,\quad l\to\infty.$$

For the second term of (6) we have

$$\left|I_2-\sum_{j=0}^{m-1}\sum_{k=0}^{l-1}\Phi(s_l^k,u_l^{(j)})\triangle s_l^k\triangle B^H_{u_l^{(j)}}\right|\le\left|I_2-\sum_{j=0}^{m-1}\int_{T_1}^{T_2}\Phi(s,u_l^{(j)})ds\,\triangle B^H_{u_l^{(j)}}\right|+$$

$$+\left|\sum_{j=0}^{m-1}\sum_{k=0}^{l-1}\int_{s_l^k}^{s_l^{k+1}}\Big(\Phi(s,u_l^{(j)})-\Phi(s_l^k,u_l^{(j)})\Big)\,ds\,\triangle B^H_{u_l^{(j)}}\right|=$$

$$=\left|I_2-\sum_{j=0}^{m-1}\int_{T_1}^{T_2}\Phi(s,u_l^{(j)})ds\,\triangle B^H_{u_l^{(j)}}\right|+$$

$$+\left|\sum_{k=0}^{l-1}\int_{s_l^k}^{s_l^{k+1}}\sum_{j=0}^{m-1}\Big(\Phi(s,u_l^{(j)})-\Phi(s_l^k,u_l^{(j)})\Big)\,\triangle B^H_{u_l^{(j)}}\,ds\right|.$$

$$\left|\sum_{k=0}^{l-1}\int_{s_l^k}^{s_l^{k+1}}\sum_{j=0}^{m-1}\Big(\Phi(s,u_l^{(j)})-\Phi(s_l^k,u_l^{(j)})\Big)\,\triangle B^H_{u_l^{(j)}}\,ds\right|=$$

$$=\left|\sum_{k=0}^{l-1}\sum_{i=0}^{L(l)-1}\int_{s_l^k}^{s_l^{k+1}}\sum_{j:u_l^{(j)}\in\pi_l^{(m)}\cap[u_{i,\,l},\,u_{i+1,\,l}]}\Big(\Phi(s,u_l^{(j)})-\Phi(s_l^k,u_l^{(j)})\Big)\,\triangle B^H_{u_l^{(j)}}\,ds\right|.$$

As function $\Phi(s,u)-\Phi(s_l^k,u)$ is Hölder on every segment $[u_{i,\,l},\,u_{i+1,\,l}]$, then according to the theorem 4.2.1 ([1])

$$\lim_{\left|\pi_{i,\,l}^{(m)}\right|\to 0}\sum_{j:\,u_l^{(j)}\in\pi_{i,\,l}^{(m)}}\Big(\Phi(s,\,u_l^{(j)})-\Phi(s_l^k,\,u_l^{(j)})\Big)\,\triangle B^H_{u_l^{(j)}}=\int_{u_{i,\,l}}^{u_{i+1,\,l}}\big(\Phi(s,u)-\Phi(s_l^k,u)\big)\,dB_u^H.$$

Moreover $\forall\, i = \overline{0,\, L(l)-1}$ the sequence

$$f_i^{(m)}(s, s_l^k) := \sum_{j:\, u_l^{(j)} \in \pi_{i,\,l}^{(m)}} \left(\Phi(s,\, u_l^{(j)} - \Phi(s_l^k,\, u_l^{(j)})\right) \triangle B^H_{u_l^{(j)}}$$

has integrable majorant. Really, using estimations from Corollary 20 ([2]), Lemma 1 and taking in account boundedness of Hölder norm we have

$$\left|f_{i,l}^{(m)}(s, s_l^k)\right| \le \left| f_{i,l}^{(m)}(s, s_l^k) - \int\limits_{u_{i,\,l}}^{u_{i+1,\,l}} \big(\Phi(s,u) - \Phi(s_l^k, u)\big)\, dB_u^H \right| +$$

$$+ \left| \int\limits_{u_{i,\,l}}^{u_{i+1,\,l}} \big(\Phi(s,u) - \Phi(s_l^k, u)\big)\, dB_u^H \right| \le c \left|\pi_{i,\,l}^{(m)}\right|^{\varepsilon} \left\|\Phi(s,u) - \Phi(s_l^k, u)\right\|_{\mathcal{H}^H_{[u_{i,l}, u_{i,l}]}} \times$$

$$\times \left\|B_u^H\right\|_{\mathcal{H}^{H'}_{[u_{i,l}, u_{i,l}]}} + \left| \int\limits_{u_{i,\,l}}^{u_{i+1,\,l}} \big(\Phi(s,u) - \Phi(s_l^k, u)\big)\, dB_u^H \right| \le$$

$$\le C + \left| \int\limits_{u_{i,\,l}}^{u_{i+1,\,l}} \big(\Phi(s,u) - \Phi(s_l^k, u)\big)\, dB_u^H \right|,$$

$H' < H,\ \varepsilon := H + H' - 1 > 0$

Using estimation 2) of Lemma 1 and and taking in account boundedness of Hölder norm of function Φ we obtain that $\left| \int\limits_{u_{i,\,l}}^{u_{i+1,\,l}} \big(\Phi(s,u) - \Phi(s_l^k, u)\big)\, dB_u^H \right|$ is bounded on s. Therefore we can use the Lebesgue theorem and then $\forall\, i = \overline{0,\, L(l)-1}$ we shall obtain the following Lebesgue integral:

$$\lim_{m\to\infty} \int\limits_{s_l^k}^{s_l^{k+1}} f_{i,l}^{(m)}(s, s_l^k)\, ds = \int\limits_{s_l^k}^{s_l^{k+1}} \int\limits_{u_{i,\,l}}^{u_{i+1,\,l}} \big(\Phi(s,u) - \Phi(s_l^k, u)\big)\, dB_u^H\, ds,$$

where

$$\int\limits_{u_{i,\,l}}^{u_{i+1,\,l}} \big(\Phi(s,u) - \Phi(s_l^k, u)\big)\, dB_u^H$$

is measurable and bounded function on s.

Hence

$$\sum_{k=0}^{l-1} \sum_{i=0}^{L(l)-1} \int\limits_{s_l^k}^{s_l^{k+1}} \sum_{j: u_l^{(j)} \in \pi_{i,\,l}^{(m)}} \left(\Phi(s, u_l^{(j)}) - \Phi(s_l^k, u_l^{(j)})\right) \triangle B^H_{u_l^{(j)}}\, ds \to$$

$$\to \sum_{k=0}^{l-1} \sum_{i=0}^{L(l)-1} \int\limits_{s_l^k}^{s_l^{k+1}} \int\limits_{u_{i,\,l}}^{u_{i+1,\,l}} \left(\Phi(s,u) - \Phi(s_l^k, u)\right) dB_u^H\, ds =$$

$$= \sum_{k=0}^{l-1} \int\limits_{s_l^k}^{s_l^{k+1}} \int\limits_{T_1}^{T_2} \left(\Phi(s,u) - \Phi(s_l^k, u)\right) dB_u^H\, ds =$$

$$= \int\limits_{T_1}^{T_2} \left(\int\limits_{T_1}^{T_2} \Phi(s,u) dB_u^H \right) ds - \sum_{k=0}^{l-1} \int\limits_{T_1}^{T_2} \Phi(s_l^k, u) dB_u^H \triangle s_l^k,$$

as $m \to \infty$.

Now according to condition 2) of our theorem, the integral $\int\limits_{T_1}^{T_2} \Phi(s,u) dB_u^H$ is Riemann integrable function on s, so we can use (5) and obtain

$$\left\| \int\limits_{T_1}^{T_2} \left(\int\limits_{T_1}^{T_2} \Phi(s,u) dB_u^H \right) ds - \sum_{k=0}^{l-1} \int\limits_{T_1}^{T_2} \Phi(s_l^k, u) dB_u^H \triangle s_l^k \right| \to 0, \ \text{as } l \to \infty$$

Using Lemma 2 we can obtain the following relation

$$\left| I_2 - \sum_{j=0}^{m-1} \int\limits_{T_1}^{T_2} \Phi(s, u_l^{(j)}) ds \triangle B^H_{u_l^{(j)}} \right| \to 0, \ \text{as } m \to \infty \quad \triangleleft$$

Corollary. *The proof of previous theorem is not based on adaptedness properties of function Φ with respect to $\mathcal{F}_t$. It holds not only for process B_t^H but also for any random process that satisfies Hölder conditions with index $\alpha_2 > \frac{1}{2}$.*

Now, consider one example of application of our stochastic Fubini theorem. Suppose that the stochastic "fractal-diffusion" process x_t has a form

$$x_t = B_t^H - \int\limits_0^t I(s)\, ds, \tag{7}$$

where $I(t) = \int\limits_0^t f(s)\, dB_s^H$ is stochastic integral with respect to FBM. The case when

$$x_t = \int\limits_0^t a(s, x_s)\, ds + \int\limits_0^t b(s, x_s)\, dB_t^H$$

with appropriate function a and b, can be, in some sense, reduced to the process of the form (7). According to Girsanov theorem for FBM from [3], denote

$$B = B(H + 1/2, 3/2 - H),\ C_1 = (2HB)^{-1},$$

$$C_H = \left(\frac{2H\,\Gamma(3/2-H)}{\Gamma(H+1/2)\,\Gamma(2-2H)}\right)^{1/2}, C_2 = \frac{C_H}{2H(2-2H)^{1/2}},$$

where Γ and B are Euler's integrals, and consider the kernel $K(t,s) = C_1 s^{1/2-H}(t-s)^{1/2-H} I\{s \in (0,t)\}$.

Suppose also that the function $f(t)$ is $\mathcal{F}_t$–adapted and there exists the representation

$$\int_0^t K(t,s) I(s)\,ds = \int_0^t \delta_s\,ds$$

with $\int_0^t |\delta_s|\,ds < \infty$ a.s., $t > 0$.

Then the process x_t is FBM with respect to such probability measure Q that $\frac{dQ}{dP}\mid_{\mathcal{F}_t} = \exp\left\{Z_t - \frac{1}{2}\langle Z\rangle_t\right\}$, where $Z(t) = C_2^{-1}\int_0^t s^{H-1/2}\delta_s\,dW_s$, $\{W_t, \mathcal{F}_t, t \geq 0\}$ is Wiener process.

Therefore we must differentiate the integral $\int_0^t K(t,s)\int_0^s f(u)\,dB_u^H\,ds$ and this is much more convenient to do, when we apply stochastic Fubini theorem to this integral. Similar case when $f(u)$ is non-random was considered in details in [4]. The next result permits to change the iterated integrals in our case.

Further put for $t > 0$ $J_1 := \int_0^t (t-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H}\left(\int_0^s \phi(u)\,dB_u^H\right) ds$,
$J_2 := \int_0^t \phi(u)\left(\int_u^t (t-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H}\,ds\right) dB_u^H$.

Lemma 3. *Let measurable, random function ϕ is Hölder on $[0,t]$ with index $\alpha_3 > \frac{1}{2}$. Then a.s. the equality $J_1 = J_2$ holds.*

Proof. Let represent J_1 and J_2 in such a way

$$J_1 = \int_0^t \left(\int_0^t \Phi(s,u)\,dB_u^H\right) ds,\ J_1 = \int_0^t \left(\int_0^t \Phi(s,u)\,ds\right) dB_u^H,$$

where $\Phi(s,u) = (t-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H}\phi(u)\,\mathbb{I}\{u \in [0,s]\}$. Function Φ satisfies conditions 1)–2) of stochastic Fubini theorem under $T_1 = \delta$ and $T_2 = t-\delta$, for $\delta > 0$. In particular, $\Phi(s,\cdot)$ is piecewise Hölder on $[\delta, t-\delta]$ with index α_3 and one Hölder discontinuity point $u = s$ for any $s \in [\delta, t-\delta]$. Therefore a.s. the following equality holds

$$\int_\delta^{t-\delta} (t-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H}\int_\delta^s \phi(u)\,dB_u^H\,ds = \int_\delta^{t-\delta}\phi(u)\int_u^{t-\delta}(t-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H}\,ds\,dB_u^H.$$

And this equality we can retype in such a way $J_1 - R_1 = J_2 - R_2$, where

$$R_1 = \int_0^\delta (t-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H}\int_0^s \phi(u)\,dB_u^H\,ds + \int_\delta^{t-\delta}(t-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H}\times$$

$$\times\int_0^{\delta}\phi(u)\,dB_u^H\,ds+\int_{t-\delta}^{t}(t-s)^{\frac{1}{2}-H}\,s^{\frac{1}{2}-H}\int_0^{s}\phi(u)\,dB_u^H\,ds;$$

$$R_2=\int_0^{\delta}\phi(u)\int_u^{t}(t-s)^{\frac{1}{2}-H}\,s^{\frac{1}{2}-H}\,ds\,dB_u^H+\int_{\delta}^{t-\delta}\phi(u)\times$$

$$\times\int_{t-\delta}^{t}(t-s)^{\frac{1}{2}-H}\,s^{\frac{1}{2}-H}\,ds\,dB_u^H+\int_{t-\delta}^{t}\phi(u)\int_u^{t}(t-s)^{\frac{1}{2}-H}\,s^{\frac{1}{2}-H}\,ds\,dB_u^H.$$

The process B_u^H is Hölder with index $H-\varepsilon>\frac{1}{2}$, $\varepsilon>0$, and according to condition of our lemma function ϕ is Hölder . Then, using theorem 22 ([2]) $\exists K>0:\ \left|\int_0^s\phi(u)\,dB_u^H\right|\le Ks^{H-\varepsilon}$. Thus

1) for

$$R_{11}:=\int_0^{\delta}(t-s)^{\frac{1}{2}-H}\,s^{\frac{1}{2}-H}\int_0^{s}\phi(u)\,dB_u^H\,ds$$

we have the following estimation

$$|R_{11}|\le K\int_0^{\delta}s^{\frac{1}{2}-\varepsilon}\,(t-s)^{\frac{1}{2}-H}\,ds\le K\,\delta^{\frac{1}{2}-\varepsilon}\int_0^{\delta}(t-s)^{\frac{1}{2}-H}\,ds\le$$

$$\le K\delta^{\frac{1}{2}-\varepsilon}\,\frac{t^{\frac{3}{2}-H}-(t-\delta)^{\frac{3}{2}-H}}{\frac{3}{2}-H}\to 0,\ \text{as }\delta\to 0$$

2) for $R_{12}:=\int_{\delta}^{t-\delta}(t-s)^{\frac{1}{2}-H}\,s^{\frac{1}{2}-H}\int_0^{\delta}\phi(u)\,dB_u^H\,ds$ the following estimation holds

$$R_{12}\le K\delta^{H-\varepsilon}\,\delta^{2-2H}\,(t-2\delta)\to 0,\ \text{as }\delta\to 0$$

3) for $R_{13}:=\int_{t-\delta}^{t}(t-s)^{\frac{1}{2}-H}\,s^{\frac{1}{2}-H}\int_0^{s}\phi(u)\,dB_u^H\,ds$ we have

$$R_{13}\le K\,t^{\frac{1}{2}-\varepsilon}\int_{t-\delta}^{t}(t-s)^{\frac{1}{2}-H}\,ds=Kt^{\frac{1}{2}-\varepsilon}\,\frac{\delta^{\frac{3}{2}-H}}{\frac{3}{2}-H}\to 0,\ \text{as }\delta\to 0.$$

It should be noted that function $\int_u^t(t-s)^{\frac{1}{2}-H}\,s^{\frac{1}{2}-H}\,ds$ is Hölder with index $\frac{3}{2}-H$ on $[0,\delta]$ and on $[t-\delta,t]$, where $0<\delta<t$

Really

a) for $\{u,v\}\subset[0,\delta]$

$$\left|\int_u^{v}(t-s)^{\frac{1}{2}-H}\,s^{\frac{1}{2}-H}\,ds\right|\le$$

$$\le (t-\delta)^{1/2-H}\left|v^{3/2-H}-u^{3/2-H}\right| \le (t-\delta)^{1/2-H}|v-u|^{3/2-H}$$

b) for $\{u,v\}\subset[t-\delta,t]$

$$\left|\int\limits_u^v (t-s)^{\frac{1}{2}-H}\, s^{\frac{1}{2}-H}\, ds\right| \le$$

$$\le (t-\delta)^{1/2-H}\left|(t-u)^{3/2-H}-(t-v)^{3/2-H}\right| \le (t-\delta)^{1/2-H}|v-u|^{3/2-H}$$

Then function $\phi(u)\int\limits_u^t (t-s)^{\frac{1}{2}-H}\, s^{\frac{1}{2}-H}\, ds$ is Hölder on $[0,\delta]$ and on $[t-\delta,t]$ with index $\alpha_4 := \min\{\alpha_3, 3/2-H\} > 1/2$.

Now using theorem 22 ([2]) we obtain the following estimations

$$R_{21} := \left|\int\limits_0^\delta \phi(u)\int\limits_u^t (t-s)^{\frac{1}{2}-H}\, s^{\frac{1}{2}-H}\, ds\, dB_u^H\right| \le K\delta^{H-\varepsilon}$$

$$R_{22} := \left|\int\limits_\delta^{t-\delta} \phi(u)\int\limits_{t-\delta}^t (t-s)^{\frac{1}{2}-H}\, s^{\frac{1}{2}-H}\, ds\, dB_u^H\right| \le K(t-2\delta)^{H-\varepsilon}(t-\delta)^{1/2-H}\frac{\delta^{\frac{3}{2}-H}}{\frac{3}{2}-H}$$

$$R_{23} := \left|\int\limits_{t-\delta}^t \phi(u)\int\limits_u^t (t-s)^{\frac{1}{2}-H}\, s^{\frac{1}{2}-H}\, ds\, dB_u^H\right| \le K\delta^{H-\varepsilon}$$

Now, under $\delta\to 0$ we obtain $J_1 = J_2$ and proof is over. ◁

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