

Exponential formula and Girsanov theorem for mixed semilinear stochastic differential equations.

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Abstract. The existence and uniqueness conditions for solution of semilinear stochastic differential equations that contains differentials with respect to Wiener process and fractional Brownian motion are considered in this paper. Also, for such mixed Brownian – fractional Brownian semilinear stochastic differential equations the conditions of measure transformation are established.

1. Introduction

We consider the notion of stochastic integral $\int_0^t f_s dB_s^H$, where B_s^H is fractional Brownian motion (fBm) with $H > \frac{1}{2}$ ([1]), f_s is random measurable function, Hölder continuous of order H (see, for example, [2], where it is proved that such integral can be considered as a limit of Riemann-Stieltjes sums). The main subject of our consideration are semilinear stochastic differential equations (SDE) containing Wiener and fBm differentials. We establish the conditions of existence and uniqueness of solution of such equations. Similar questions were considered in [2], [3], [4] but in [2] only the existence and uniqueness of local solution were proved; in [3] the existence and uniqueness of global solution of equations containing only one differential of Hölder continuous process were considered; the equations with the same Wiener and fBm differentials were considered in [4] but semilinear case can't be included to their conditions.

In chapter 3 we establish the conditions of measure transformation for such semilinear SDE.

2. Existence and uniqueness of solutions of semilinear stochastic differential equations that contains differentials with respect to Wiener process and fBm.

Let us consider the semilinear differential equation in complete probability space (Ω, \mathcal{F}, P)

$$\begin{cases} dX_t = \sigma_1 X_t dW_t + \sigma_2 X_t dB_t^H + b(t, X_t) dt, & t \geq t_0 \\ X_{t=t_0} = X_0 \text{ is } \mathcal{F}_{t_0} \text{- measurable, random variable,} \end{cases} \quad (1)$$

where $\sigma_i \in \mathbb{R}, i = 1, 2$, B_t^H is fractional Brownian Motion (fBm) with Hurst index $H \in (\frac{1}{2}, 1)$, $W_t = B_t^{\frac{1}{2}}$ is Wiener process, and function

$$b \in \mathcal{C}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}) \quad (2)$$

satisfies Lipschitz condition on $x \in \mathbb{R}$ uniformly on $t \in [0, T]$, that is $\forall T > 0 \exists L_T > 0$:

$$\sup_{t \in [0, T]} |b(t, x_1) - b(t, x_2)| \leq L_T |x_1 - x_2| \quad (3)$$

Definition 2.1. *Solution of the equation (1) is a Hölder continuous with index H random process that has a form*

$$X_t = X_0 + \sigma_1 \int_{t_0}^t X_s dW_s + \sigma_2 \int_{t_0}^t X_s dB_s^H + \int_{t_0}^t b(X_s, s) ds, \quad t \geq t_0 \quad (4)$$

It should be noted that integrals $\int_{t_0}^t X_s dW_s, \int_{t_0}^t X_s dB_s^H$ exist if X_s is Hölder continuous with index H ([2]).

Theorem 2.2. *Let the function b satisfies (2), (3) and growth condition*

$$\forall T > 0 \exists C_T > 0 : |b(t, x)| \leq C_T (1 + |x|). \quad (5)$$

Then the equation (1) has a unique solution on $[0, +\infty)$.

Proof. Firstly let us consider an auxiliary system of partial differential equations

$$\begin{cases} \frac{\partial h}{\partial Z_j}(Y, (Z_1, Z_2)) = \sigma_j h(Y, (Z_1, Z_2)), & j = \overline{1, 2} \\ h(Y_0, \overline{Z}_0) = X_0, & \overline{Z}_0 = (W_{t_0}, B_{t_0}^H) \end{cases}$$

This system has a solution $h(Y, \overline{Z}) = (Y - Y_0 + X_0)e^{\sigma_1(Z_1 - W_{t_0}) + \sigma_2(Z_2 - B_{t_0}^H)}$, where $\overline{Z} = (Z_1, Z_2), Z_1(t) = W_t, Z_2(t) = B_t^H$

We will try to find the solution X_t of equation (1) in the following form $X_t = h(Y(t), \overline{Z}(t))$, where $Y \in \mathcal{C}'(\mathbb{R})$ with $Y(t_0) = Y_0$ that is \mathcal{F}_{t_0} -measurable random value. ■

Applying the formula Itô to the function $h(Y(t), \overline{Z}(t))$ we obtain

$$\begin{aligned} dX_t &= \sum_{i=1}^2 \frac{\partial h}{\partial Z_i}(Y(t), \overline{Z}(t)) dZ_i(t) + \frac{\partial h}{\partial Y}(Y(t), \overline{Z}(t)) Y' dt + \\ &\quad + \frac{1}{2} \sum_{i,j=1}^2 \frac{\partial^2 h}{\partial Z_i \partial Z_j}(Y(t), \overline{Z}(t)) d[Z_i, Z_j]_t = \\ &= \sum_{i=1}^2 \sigma_i h(Y(t), \overline{Z}(t)) dZ_i(t) + Y' \frac{\partial h}{\partial Y}(Y(t), \overline{Z}(t)) dt + \end{aligned}$$

$$+\frac{1}{2}\sigma_1^2 h(Y(t), \overline{Z}(t))dt.$$

Equation (1) implies the following ordinary differential equation for Y :

$$\begin{cases} Y'(t) = (c_1(t))^{-1}b(t, (Y - Y_0 + X_0) c_1(t)) - \frac{1}{2}\sigma_1^2 (Y - Y_0 + X_0) =: f(y, t) \\ Y(t_0) = Y_0, \end{cases} \quad (6)$$

where $c_1(t) = e^{\sigma_1(Z_1 - W_{t_0}) + \sigma_2(Z_2 - B_{t_0}^H)}$. Further we will assume that $\omega \in \Omega$ is fixed. Put $L_1(T) := \max_{t \in [0, T]} (c_1(t))^{-1} > 0$, $L_2(T) := \max_{t \in [0, T]} c_1(t) > 0$, $D_1 := C_T L_1(T)$, $D_2 = C_T + \frac{1}{2}\sigma_1^2$.

Then under $|Y - Y_0| \leq \beta$; $|t - t_0| \leq \alpha$:

$$M := \max_{t \in [0, T]} |f(y, t)| \leq \frac{1}{2}\sigma_1^2(\beta + |X_0|) (L_1(T) + (\beta + |X_0|)) C_T \leq D_1 + D_2(\beta + |X_0|)$$

and according to the Picard theorem there exists solution of (6) on $[t_0, t_0 + l^{(0)}]$, where

$$l^{(0)} = \min \left(\alpha, \frac{\beta}{M} \right) \geq \min \left(\alpha, \frac{\beta}{D_1 + D_2(\beta + |X_0|)} \right) =: l_0,$$

and therefore solution of (6) exists on the smaller segment $[t_0, t_0 + l_0]$. From this on segment $[t_0, t_0 + l_0]$ there exists a solution of (1).

Now we can estimate the solution of (1) in the point $t_0 + l_0$ in such a way:

$$h_{t_0+l_0} \leq |Y - Y_0 + X_0| L_2(T) \leq (\beta + |X_0|) L_2(T).$$

Further we will extend the solution h for $t \geq t_0 + l_0$. The value of solution h in the point $t_0 + l_0$ will be new initial value $X_0^{(1)} \leq (\beta + |X_0|) L_2(T)$.

Now, under $|Y(t) - Y(t_0 + l_0)| \leq \beta_1$, $|t - (t_0 + l_0)| \leq \alpha_1$, the solution of (6) exists on segment $[t_0 + l_0, t_0 + l_0 + l_1]$, where

$$l_1 = \min \left(\alpha_1, \frac{\beta_1}{D_1 + D_2(\beta_1 + (\beta + |X_0|)L_2(T))} \right)$$

Further on the n -th step of such solution extension procedure we have:

$$l_n = \min \left(\alpha_n, \frac{\beta_n}{D_1 + D_2(\beta_n + \sum_{k=0}^{n-1} \beta_{n-1-k} L_2^{k+1}(T) + |X_0|L_2(T))} \right)$$

$\alpha_0 := \alpha$, $\beta_0 := \beta$ and on $[t_0 + \sum_{i=0}^{n-1} l_i, t_0 + \sum_{i=0}^n l_i]$ there exists a solution of (6).

Now let us investigate the properties of

$$z_n =: \frac{\beta_n}{D_1 + D_2(\beta_n + \sum_{k=0}^{n-1} \beta_{n-1-k} L_2^{k+1}(T) + |X_0|L_2(T))}, \quad n \geq 0:$$

a) under $|X_0| \leq 1$ we can chose $\beta_k \equiv 1$, $k = \overline{0, n}$ and then

$$\frac{1}{D_1 + 2D_2 \frac{L_2^{n+1}(T) - 1}{L_2(T) - 1}} \leq z_n \leq \frac{1}{D_1 + D_2 \frac{L_2^{n+1}(T) - 1}{L_2(T) - 1}}.$$

b) under $|X_0| > 1$ we can chose $\beta_k \equiv |X_0|$, $k = \overline{0, n}$ and then

$$\frac{1}{D_1 + 2D_2 \frac{L_2^{n+1}(T)-1}{L_2(T)-1}} \leq z_n = \frac{1}{\frac{D_1}{|x_0|} + D_2(1 + 2 \sum_{k=1}^n L_2^k(T))} \leq \frac{1}{D_2 \frac{L_2^{n+1}(T)-1}{L_2(T)-1}}.$$

For both cases a) and b) we put $\alpha_n = \frac{1}{D_1 + 2D_2 \frac{L_2^{n+1}(T)-1}{L_2(T)-1}}$, $n \geq 0$, that is $l_n = \min(a_n, z_n) = a_n \Rightarrow \sum_{n \geq 0} l_n = S$, and then

1) under $L_2(T) \leq 1$ the series $\sum_{n \geq 0} z_n$ is divergent and therefore the series $\sum_{i \geq 0} l_i$ is also divergent, and thus there exists a finite quantity N of steps in our solution extension procedure, such that $[t_0, T] \subset [t_0, t_0 + \sum_{i=0}^N l_i]$, that is on the $[t_0, T]$ there exists a solution of (6).

2) under $L_2(T) > 1$ the series $\sum_{n \geq 0} z_n$ is convergent and moreover

$$\sum_{n \geq 0} z_n \geq \sum_{n \geq 0} \frac{1}{D_1 + 2D_2 \frac{L_2(T)^{n+1}-1}{L_2(T)-1}} =: S$$

In fact we proved that on segment $[t_0, t_0 + \frac{1}{2}S]$ there exists a bounded solution h of (1). Now we can consider a finite value of h in the point $t_0 + \frac{1}{2}S$ as a new initial value of h and for it we can prove that solution h exists on $[t_0 + \frac{1}{2}S, t_0 + S]$, because the step $\frac{1}{2}S$ of solution extension doesn't depend on initial value. Therefore we can extend solution h to whole segment $[t_0, T]$. Since T is arbitrary we obtain the solution (1) on $[0, \infty)$. Uniqueness follows from theorem 7.1.1. ([2]). In this theorem it is established that any two of solutions coincide on the common interval of definition. □

3. Girsanov theorem and measure transformation for mixed semilinear stochastic differential equations.

Consider the equation (1) and suppose that $b(t, x)$ satisfies the conditions (2), (3), (5) and can be presented as $b(t, x) = e(t, x)x$, where $e \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R})$ Further we will use the following notations: $B = B(H + 1/2, 3/2 - H)$, $B(\cdot, \cdot)$ is beta-function, $B_1 = B(H - 1/2, 3/2 - H)$, $C_1 = (2HB)^{-1}$, $C_H = \left(\frac{2H \Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)} \right)^{1/2}$, $C_2 = \frac{C_H}{2H(2-2H)^{1/2}}$, $C_0 = \frac{C_1}{C_2}$. Also, let the kernel $K(t, s) = C_1 s^{1/2-H} (t-s)^{1/2-H} I\{s \in (0, t)\}$, $\hat{e}(t, x) := e(t, x)t^{1/2-H}$.

Let the function ψ satisfies four conditions:

- 1) $\int_0^T K(t, s) |\psi_s| ds < \infty$ a.s., $t \in [0, T]$,
- 2) there exists a representation

$$\int_0^T K(t, s) \psi_s ds = \int_0^T \delta(s) ds, \quad t > 0,$$

(7)

with $\int_0^T |\delta_s| ds < \infty$ a.s., $t \in [0, T]$.

- 3) $\mathbb{E} \int_0^T s^{2H-1} \delta_s ds < \infty$,
- 4) $\mathbb{E} \exp \left\{ Z_T - \frac{1}{2} \langle Z \rangle_T \right\} = 1$, $t \in [0, T]$,

where $Z_t = \frac{1}{C_2} \int_0^t s^{H-1/2} \delta_s dW_s$ and $\langle Z \rangle_t = \frac{1}{C_2^2} \int_0^t s^{2H-1} \delta_s^2 ds$.

Now, we try to change the measure P for another probability measure Q such that $Q_T \ll P_T$, where $Q_T = Q|_{[0, T]}$, and that the drift $e(t, X_t) X_t dt$ will be annihilated under Q_T . At first, let some probability measure \tilde{Q} satisfies the assumptions

$$\frac{d\tilde{Q}_T}{dP_T} = \exp \left\{ \int_0^T \varphi_s dW_s - \frac{1}{2} \int_0^T \varphi_s^2 ds \right\}$$

and

$$\mathbb{E} \left[\exp \left\{ \int_0^T \varphi_t dW_t - \frac{1}{2} \int_0^T \varphi_t^2 dt \right\} \right] = 1, \quad (8)$$

where

$$\mathbb{E} \int_0^T |\varphi_t|^2 dt < \infty. \quad (9)$$

Then from Girsanov theorem $W_t - \int_0^t \varphi_s ds = \widehat{W}_t$ is a Wiener process under the measure \tilde{Q}_T . Also, let the measure \overline{Q} be such that $\frac{d\overline{Q}_T}{dP_T} = \exp \left\{ Z_T - \frac{1}{2} \langle Z \rangle_T \right\}$, $\overline{Q} \ll P$, then the process $\widehat{B}_t^H = B_t^H - \int_0^t \psi_s ds$ is fBm relatively to \overline{Q} (see theorem 4 from [5], also see [6], [7]). Of course, we are need in the inequality $\tilde{Q}_T = \overline{Q}_T = Q_T$, whence, in particular,

$$Z_t = \int_0^t \varphi_s dW_s. \quad (10)$$

Therefore we want to find φ_s and ψ_s in such a way that the common drift equals

$$\sigma_1 \int_0^t \varphi_s ds + \sigma_2 \int_0^t \psi_s ds = - \int_0^t e(s, X_s) ds. \quad (11)$$

In this case $\int_0^t e(s, X_s) ds + \sigma_1 \int_0^t X_s dW_s + \sigma_2 \int_0^t X_s dB_s^H = \sigma_1 \int_0^t X_s d\widehat{W}_s + \sigma_2 \int_0^t X_s d\widehat{B}_s^H$.

From (11) we obtain that

$$\sigma_1 \varphi_t + \sigma_2 \psi_t = -e(t, X_t), \quad t \in [0, T].$$

Further, we use the Abel rearrangement to the relation

$$C_1 \int_0^t (t-s)^{1/2-H} s^{1/2-H} \psi_s ds = \int_0^t \delta_s ds = C_2 \int_0^t s^{1/2-H} \varphi_s ds :$$

$$C_1 \int_0^t (t-u)^{H-3/2} \int_0^u (u-s)^{1/2-H} s^{1/2-H} \psi_s ds du = C_2 \int_0^t (t-u)^{H-3/2} \int_0^u s^{1/2-H} \varphi_s ds du,$$

or

$$C_0 B_1 \int_0^t s^{1/2-H} \psi_s ds = \int_0^t \frac{(t-u)^{H-1/2}}{H-1/2} u^{1/2-H} \varphi_u du,$$

if we differentiate, then

$$C_0 B_1 t^{1/2-H} \psi_t = \int_0^t (t-u)^{H-3/2} u^{1/2-H} \varphi_u du,$$

and

$$\sigma_1 \varphi_t + \frac{t^{H-1/2}}{C_0 B_1} \sigma_2 \int_0^t (t-u)^{H-3/2} u^{1/2-H} \varphi_u du = -e(t, X_t),$$

let further $t^{1/2-H} \varphi_t = \Theta_t$, then

$$\sigma_1 \Theta_t + \frac{\sigma_2}{C_0 B_1} \int_0^t (t-u)^{H-3/2} \Theta_u du = -\widehat{e}(t, X_t).$$

It is Volterra equation with weak singularity, and its unique solution has a form

$$\Theta_t = -\frac{\widehat{e}(t, X_t)}{\sigma_1} - \frac{1}{\sigma_1} \int_0^t \sum_{n=1}^{\infty} \rho^n \frac{(t-s)^{n(H-1/2)-1}}{\Gamma(n(H-1/2))} \widehat{e}(s, X_s) ds,$$

where $\rho = -\frac{\sigma_2}{C_0 B_1} \Gamma(H-1/2)$.

So, we proved the following result.

Theorem 3.1. Under the conditions (7), (8), (9) and (10) the equation (1) has a form

$$dX_t = \sigma_1 X_t d\widehat{W}_t + \sigma_2 X_t d\widehat{B}_t^H$$

under measure Q .

Definition 3.2. Financial market described by the equation (1) is in equilibrium on $[0, T]$ if both the kernel $\Theta_t t^{H-1/2}$ and likelihood ratio $\left. \frac{dQ}{dP} \right|_{\mathcal{F}_t}$ are the functions of t and W_t , and don't depend on the path of W_s , $s \leq t$.

This definition generalizes the usual definition of equilibrium ([8], [10]) in which path's independence of $\left. \frac{dQ}{dP} \right|_{\mathcal{F}_t}$ is declared, and the kernel equals simply $e(t, W_t)$. Note that financial market described by the equation (1) admits no arbitrage as it follows from [9].

Definition 3.3. We say that Θ_t , satisfying definition 3.2, satisfies fractal Burger's equation with index H if $g(t, W_t) := \Theta_t t^{H-1/2}$ satisfies usual Burger's equation

$$-g(s, x)g_2'(s, x) = g_1'(s, x) + \frac{1}{2}g_{22}''(s, x)$$

Evidently, fractal Burger's equation has a form

$$-s^{H-1/2}p(s, x)p_2'(s, x) = (H-1/2)s^{-1}p(s, x) + p_1'(s, x) + \frac{1}{2}p_{22}''(s, x), \quad s > 0, x \in \mathbb{R}$$

Theorem 3.4. If financial market is in equilibrium then Θ_t satisfies fractal Burger's equation.

Proof. Let

$$\int_0^t \Theta_s s^{H-1/2} dW_s - \frac{1}{2} \int_0^t \Theta_s^2 s^{2H-1} ds = G(t, W_t),$$

where $g, G \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R})$. Then

$$\int_0^t \Theta_s g(s, W_s) dW_s - \frac{1}{2} \int_0^t g^2(s, W_s) ds = G(t, W_t), \quad t \in [0, T]$$

By Itô formula,

$$G(t, W_t) = \int_0^t \left[G_1'(s, W_s) + \frac{1}{2}G_{22}''(s, W_s) \right] ds + \int_0^t G_2'(s, W_s) dW_s.$$

From here $g(s, W_s) = G_2'(s, W_s)$, $-\frac{1}{2}g^2(s, W_s) = G_1'(s, W_s) + \frac{1}{2}G_{22}''(s, W_s)$, or, simply,

$$\begin{aligned} g(s, x) &= G_2'(s, x), \\ -\frac{1}{2}g^2(s, x) &= G_1'(s, x) + \frac{1}{2}G_{22}''(s, x) \end{aligned} \quad (12)$$

Further, $g_2'(s, x) = G_{22}''(s, x)$,

$$-\frac{1}{2}g_2^2(s, x) = G_1'(s, x) + \frac{1}{2}g_2'(s, x). \quad (13)$$

Differentiate (12) in s and (13) in x :

$$\begin{aligned} g_1'(s, x) &= G_{12}''(s, x) \\ -g(s, x)g_2'(s, x) &= G_{12}''(s, x) + \frac{1}{2}g_{22}''(s, x) \end{aligned}$$

or

$$-g(s, x)g_2'(s, x) = g_1'(s, x) + \frac{1}{2}g_{22}''(s, x).$$

□

Similar problem for usual SDE with Wiener process was considered in [10].

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