The presence and absence of arbitrage conditions on the (B,S)– market which defined by fractional Brownian motion.

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Abstract

The problem of the presence and absence of arbitrage conditions on the three types of (B,S)– market is considered in this paper. In the first case when (B,S)– market is defined by the fractional stock, the absence of martingale measure is proved. For two others models of (B,S)– market which defined by modified fractional stock in the second case and by "homogeneous" kernel in the third case, the absence of arbitrage is proved.

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1. Introduction.

The probability space (Ω,F,P) with filtration $(F_t, t \geq 0)$ is considered. Throughout this paper we will denote this composition by $(\Omega,F,(F_t)_{t\geq 0},P)$.

Further in this paper we investigate the presence and absence of arbitrage conditions on the (B,S)– market with random stock price process $(S_t,(F_t)_{t\geq 0},P)$:

$$S_t = exp(X_t) := exp\left\{ \int_0^t \nu(s)ds + \int_0^t \mu(s)dB_s^H \right\} \tag{1}$$

and bond price process $(B_t,(F_t)_{t\geq 0},P)$: $B_t = e^{rt}, r \geq 0, t \geq 0$; where ν and μ are non-random, measurable functions, B_s^H is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$.

Definition 1. *The share that defined by (1) with bounded function μ is called fractional.*

2. Absence of martingale measure in the fractional share case.

Definition 2. *The random process $(Z_t,(F_t)_{t\geq 0},P)$ is called semimartingale if it can be presented by following*

$$Z_t = M_t + A_t \quad (2)$$

where M is a locally square integrable martingale, A is a process with integrable variation.

Further we will denote by $[Z]_t$ the quadratic variation of process Z_t. Note that for this process holds the following

$$[Z]_t = 0 \Leftrightarrow \begin{cases} A = A^c \\ M = 0 \end{cases},$$

where $A = A^c$ is a continuous process.

As it is known ([5]), (B,S)–market is an arbitrage-free market if there exists a martingale measure P^* such that $\frac{S_t}{B_t}$ is a P^*-martingale, and the absence of equivalent martingale measure is not sufficient condition of presence of arbitrage on the (B,S)–market. The following lemma shows the relation between existence of martingale measure and path property of process $\frac{S_t}{B_t}$.

Lemma 1. *If there exists a martingale measure P^*, then process X_t is a semimartingale.*

Proof. It is known ([5]), that process $U_t := \frac{S_t}{B_t} = exp(X_t - rt)$ is a P^*-martingale if and only if $U_t \cdot M_t$ is P–martingale, where $M_t = exp\left\{ Y_t - \frac{1}{2} < Y >_t \right\}$, $Y_t = \left. \frac{dP^*}{dP} \right|_{F_t}$. Therefore

$Z_t := U_t \cdot M_t = exp\left\{ Y_t - \frac{1}{2} < Y >_t - rt \right\}$ is P – martingale. Using formula Ito we have:

$$X_t + Y_t - \frac{1}{2} < Y >_t - rt = ln Z_t = ln Z_0 + \int_0^t \frac{1}{Z_s} dZ_s - \frac{1}{2} \int_0^t \frac{1}{Z_s^2} d < Z >_s ;$$

taking into consideration that $\int_0^t \frac{1}{Z_s} dZ_s$ is a local martingale and $\int_0^t \frac{1}{Z_s^2} d < Z >_s$ is a process with bounded variation we obtain that process $X_t + Y_t - \frac{1}{2} < Y >_t - rt$ is semimartingale, hence it follows that X_t is semimartingale.

□

Corollary 1 *If process X_t is not a semimartingale, then there doesn't exist equivalent martingale measure for process U_t.*

Before investigating process X_t that defined by fractional integral $\int_0^t \mu(s) dB_s^H$ we remind the definition of fractional integral ([3]).

For $H > \frac{1}{2}$ let Γ denote the integral operator

$$\Gamma f(t) := H(2H-1) \int_0^\infty f(s) \left| s - t \right|^{2H-2} ds ,$$

and defined the inner product

$$< f, g >_\Gamma := < f, \Gamma g > = H(2H-1)\int_0^\infty\int_0^\infty f(s)g(t)|s-t|^{2H-2}\,ds dt,$$

where $< \circ >$ denotes the usual inner product of $L_2([0,\infty])$.

Denote by L_2^Γ the space of equivalence classes of measurable functions f such that $< f, f >_\Gamma < \infty$. Now, it is easy to check that the association $B_t^H \mapsto 1_{[0,t)}$ can be extended to an isometry between the Gaussian space generated by random variables $\{B_t^H, t \ge 0\}$, as the smallest closed linear subspace of $L_2(\Omega, F, P)$ containing them, and the function space L_2^Γ. For $f \in L_2^\Gamma$, the integral $\int_0^\infty f(s)dB_s^H$ can now be defined as the image of f in this isometry.

***Theorem 1**. Let μ is bounded, measurable function on the real axis. Then process $R_t = \int_0^t \mu(s)dB_s^H$ is not a semimartingale.*

Proof. Obvious R_t is not a continuous process with bounded variation. Therefore from ***Definition 2*** follows that it is enough to prove the vanishing of quadratic variation $[R]_t$ for our theorem.

Let $c := \sup_{s\in(-\infty;+\infty)} |\mu(s)| < \infty$. $\forall n \ge 1$ $\lambda = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ is finite partition of segment $[0,t]$, $t > 0$ with partition diameter $|\lambda|$. Then

$$E\left|\sum_{k=0}^{n-1}(R_{t_{k+1}} - R_{t_k})^2 - 0\right| \le \sum_{k=0}^{n-1} E\left|\int_{t_k}^{t_{k+1}}\mu(s)dB_s^H\right|^2 \le c^2 H(2H-1)\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}}|s-t|^{2H-2}ds\,dt =$$

$$= c^2 H\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\left(|t_{k+1}-t|^{2H-1} - |t_k - t|^{2H-1}\right)dt = c^2\sum_{k=0}^{n-1}|t_{k+1}-t_k|^{2H} \le$$

$$\le c^2|\lambda|^{2H-1}\sum_{k=0}^{n-1}|t_{k+1}-t_k| = c^2\, t\,|\lambda|^{2H-1} \to 0,\ |\lambda| \to 0$$

□

***Corollary 2.** From **Lemma 1** and **Theorem 1** follows that there doesn't exist equivalent martingale measure in the fractional share case.*

Further we will show that for fractional share (1) with $\nu(t) \equiv r$ the self-financing portfolio $\pi(t)$ that represents the amount invested in the stock and allows arbitrage opportunity on the (B,S)–market can be constructed.

The *discounted capital gains process* associated with the portfolio $\pi(t)$ is defined to be

$$G_t := \int_0^t \pi(s)(B_s)^{-1}[\mu(s)\,dB_s^H + (\nu(s)-r)\,ds] = \int_0^t \pi(s)(B_s)^{-1}\mu(s)\,dB_s^H. \quad (3)$$

The economic justification of (3) is that the capital gain from holding the stock between time s and $s+ds$ is

(#Shares owned)(Price increase in stock) – (potential gain from bond), or

$$\frac{\pi(s)}{S_s}dS_s - \frac{\pi(s)}{B_s}dB_s; \quad (4)$$

$(B_s)^{-1}$ is the discounting factor and integration from 0 to t yields the discounted capital gain (3).

Definition 3. *The portfolio π is called an arbitrage opportunity if its discounted gains process satisfies the following three conditions:*

1) $P\{ G_0 = 0 \} = 1$;

2) $P\{ G_1 \geq 0 \} = 1$;

3) $P\{ G_1 > 0 \} > 0$.

Theorem 2. *The portfolio* $\pi(t) = 2S_t \left(exp\left(\int_0^t \mu(s) dB_s^H \right) - 1 \right)$ *is an arbitrage opportunity.*

Proof. In the first order it should be noted that $S_t = exp\left(rt + \int_0^t \mu(s) dB_s^H \right)$ satisfies the following equation

$$dS_t = S_t(r\,dt + \mu(t) dB_t^H),$$

which is the same as $S_t = 1 + r\int_0^t S_u\, du + \int_0^t S_u\, \mu(u) dB_u^H$, in particularly under $r = 0$ we have

$$exp\left\{ \int_0^t \mu(u) dB_u^H \right\} - 1 = \int_0^t \mu(u) exp\left\{ \int_0^u \mu(s) dB_s^H \right\} dB_u^H . \qquad (5)$$

Now, using (5) we calculate G_t:

$$G_t = \int_0^t \pi(s)(B_s)^{-1} \mu(s) dB_s^H = \int_0^t 2exp\left(\int_0^s \mu(u) dB_u^H + rs \right) exp(-rs) \left(exp\left(\int_0^s \mu(u) dB_u^H \right) - 1 \right) \mu(s)\, dB_s^H =$$

$$= 2\int_0^t \mu(s) exp\left(2\int_0^s \mu(u) dB_u^H \right) dB_s^H - 2\int_0^t \mu(s) exp\left(\int_0^s \mu(u) dB_u^H \right) dB_s^H = \left(exp\left(\int_0^t 2\mu(s) dB_s^H \right) - 1 \right) -$$

$$-2\left(exp\left(\int_0^t \mu(s) dB_s^H \right) - 1 \right) = \left(exp\left(\int_0^t \mu(s) dB_s^H \right) - 1 \right)^2 .$$

Hence, it is easy to check that G_t satisfies conditions 1) – 3) of ***Definition 3***.

□

3. Absence of arbitrage in the modified fractional model.

Let modify share (1) by such way that the process X_t will be a semimartingale.

Let

$$X_t := \int_0^t K(t,s) a(s) ds + \int_0^t K(t,s) b(s) dB_s^H , \qquad (6)$$

where a and b non-random, measurable functions, $K(t,s) = \begin{cases} 0, & s > t, \\ s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H}, & 0 < s < t \end{cases}$ is a kernel defined in [3].

We will search a i b such that X_t will be a semimartingale, and in particularly it will be presented by the following formula

$$X_t := \int_0^t \alpha(s)ds + \int_0^t \beta(s)dW_s, \quad (W_s = B_s^{\frac{1}{2}}) \tag{7}$$

Definition 4. *The function f is absolutely continuous on the segment $[c,d] \subset (-\infty,\infty)$ if $\exists \varphi \in L_1([c,d])$: $f(x) = f(c) + \int_0^x \varphi(t)dt,\ x \in [c,d]$. Notation: $f \in AC([c,d])$* [1].

Theorem 3. *The following statements are correct:*

1) Let $\forall B > 0$ $\quad s^{\frac{1}{2}-H} a(s) \in AC([0,B])$ then there exists a derivative

$$\frac{d}{dt}\int_0^t (t-u)^{\frac{1}{2}-H} u^{\frac{1}{2}-H} a(u)du, \quad t > 0 \tag{8}$$

and integral $\int_0^t (t-u)^{\frac{1}{2}-H} u^{\frac{1}{2}-H} a(u)du$ can be presented by $\int_0^t \alpha(u)du$, where $\alpha(t) = \frac{d}{dt}\int_0^t (t-u)^{\frac{1}{2}-H} u^{\frac{1}{2}-H} a(u)du, \quad t > 0$.

2) *Let $\forall t \in (-\infty,\infty)$ $\quad b(t) = c \equiv const$,*

then under $\beta(u) = c_H\, c\, u^{\frac{1}{2}-H}$, $c_H = \sqrt{\dfrac{2H\,\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}$, the following equality holds

$$\int_0^t (t-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H} b(s)dB_s^H = \int_0^t \beta(s)dW_s, \quad t > 0 \tag{9}$$

Proof. 1) Let $\forall B > 0$ $\quad s^{\frac{1}{2}-H} a(s) \in AC([0,B])$.

This is necessary and sufficient condition of existence of Abel equation solution ([4]):

$$\frac{1}{\Gamma(\frac{3}{2}-H)}\int_0^t \frac{\Gamma(\frac{3}{2}-H)s^{\frac{1}{2}-H}a(s)}{(t-s)^{H-\frac{1}{2}}}ds = \int_0^t \alpha(s)ds, t > 0 \tag{10}$$

and this solution is equal $\alpha(t) = \frac{d}{dt}\int_0^t (t-u)^{\frac{1}{2}-H} u^{\frac{1}{2}-H} a(u)du, \quad t > 0$.

2) In this point of our proof we will find conditions for β i b such that the following equality holds

$$\int_0^t s^{\frac{1}{2}-H} b(s)dB_s^H = \int_0^t (t-s)^{H-\frac{1}{2}} \beta(s)dW_s, \quad t > 0 \tag{11}$$

Let $\Phi(t) := \dfrac{1}{t^{\frac{1}{2}-H} b(t)}$, $\ t > 0$. We apply simultaneously to both sides of (11) the following transformation:

$$A_1 := \int_0^t \Phi'(s)\left(\int_0^s u^{\frac{1}{2}-H} b(u)dB_u^H\right)ds = \int_0^t \Phi'(s)\left(\int_0^s (s-u)^{H-\frac{1}{2}} \beta(u)dW_u\right)ds =: A_2 .$$

Hence

$$A_1 = \Phi(t)\int_0^t u^{\frac{1}{2}-H} b(u)dB_u^H - B_t^H = \Phi(t)\int_0^t (t-u)^{H-\frac{1}{2}} \beta(u)dW_u - \int_0^t z(t,u)dW_u ,$$

where the kernel

$$z(t,u) = (H-\frac{1}{2})c_H u^{\frac{1}{2}-H} \int_u^t v^{H-\frac{1}{2}} (v-u)^{H-\frac{3}{2}} dv$$

was defined in [3].

$$A_2 = \Phi(t)\int_0^t (t-u)^{H-\frac{1}{2}} \beta(u)dW_u - (H-\frac{1}{2})\int_0^t \Phi(s)\int_0^s (s-u)^{H-\frac{3}{2}} \beta(u)dW_u \, ds =$$

$$= \Phi(t)\int_0^t (t-u)^{H-\frac{1}{2}} \beta(u)dW_u - (H-\frac{1}{2})\int_0^t \beta(u)\int_u^t (s-u)^{H-\frac{3}{2}} \Phi(s)\, ds\, dW_u$$

And therefore: $(H-\frac{1}{2})\int_0^t \beta(u)\int_u^t (s-u)^{H-\frac{3}{2}} \Phi(s)ds = z(t,u)$,

i.e. $\beta(u) = c_H b(t)u^{\frac{1}{2}-H} = c_H c u^{\frac{1}{2}-H}$, $u>0$.

□

Remark. It turns out that the derivative of $I(t) := \int_0^t K(t,\, s)\, c(s)\, ds$ can not be defined everywhere on the segment $[0,\infty)$ for some functions $c \in C([0,\infty])$ that $c' \notin C([0,\infty])$, i.e. the condition (8) is essential. We show this in the following lemma.

Lemma 2. *Let* $c(s) = \begin{cases} s+(t_0-t_1)^{1-r}-t_1, & s\in[0,t_1) \\ (t_0-s)^{1-r}, & s\in[t_1,t_0] \\ -(s-t_0)^{1-r}, & s>t_0 \end{cases}$ *, where* $t_0>t_1>0$, $r\in\left(\frac{3}{2}-H,\,1\right), H\in\left(\frac{1}{2},1\right)$.

Then the derivative of function $I(t)$ *there doesn't exist in* $t=t_0$.

Proof. Note that

$$I(t) = \int_0^t (t-s)^{\frac{1}{2}-H} s^{\frac{1}{2}-H} c(s)ds = t^{2-2H}\int_0^1 (1-u)^{\frac{1}{2}-H} u^{\frac{1}{2}-H} c(tu)du =: t^{2-2H} I_1(t).$$

Now let calculate the derivative of function $I_1(t)$ in $t=t_0$.

$$\frac{I_1(t_0+h)-I_1(t_0)}{h} = \int_0^1 (1-u)^{\frac{1}{2}-H} u^{\frac{1}{2}-H} u\, \frac{c(t_0u+hu)-c(t_0u)}{hu} du .$$

Under $h\to 0$ for $\theta := \frac{t_1}{t_0} \in (0,1)$ we have:

$\lim\limits_{h\to 0}\dfrac{I_1(t_0+h)-I_1(t_0)}{h}=\int_0^\theta(1-u)^{\frac{1}{2}-H}u^{\frac{3}{2}-H}du-(1-r)t_0^{-r}\int_\theta^1(1-u)^{\frac{1}{2}-H-r}u^{\frac{3}{2}-H}du$, where the first term is equal $B_\theta(\frac{5}{2}-H,\frac{3}{2}-H)$, and the second term is infinite, because

$$\left|\int_\theta^1(1-u)^{\frac{1}{2}-H-r}u^{\frac{3}{2}-H}du\right|\geq\theta^{\frac{3}{2}-H}\left|\left(-\frac{(1-u)^{\frac{3}{2}-H-r}}{\frac{1}{2}-H-r}\right)\Bigg|_\theta^1\right|=\infty.$$

Hence, $\lim\limits_{h\to 0}\dfrac{I_1(t_0+h)-I_1(t_0)}{h}=-\infty$.

□

4. Absence of arbitrage in model with "homogeneous" kernel.

Let consider the case when X_t is presented by the following formula

$$X_t=V_h^c(t):=\int_0^t h(t-s)c(s)dW_s,\ t>0 \tag{12}$$

According to the special look of kernel h it should be called a homogeneous. Under $c\equiv 1$ the process $V_h^c(t)$ was considered in [2].

In the next theorem the semimartingale condition of $V_h^c(t)$ was formulated. This is sufficient condition of absence of arbitrage on the (B,S)-market according the paragraph 2.

Theorem 4. *1) Let the following condition holds*

$$\int_0^t(h'(t-u)c(u))^2du<\infty,\ t\geq 0 \tag{13}$$

Then $V_h^c(t)$ is a semimartingale.

2) If $V_h^c(t)$ is a semimartingale and c – nondecreasing function, then condition (13) holds.

Proof. 1) Note that $h(t)=h(0)+\int_0^t h'(u)du$, hence, using stochastic Fubini theorem ([6]) we obtain

$$V_h^c(t)=\int_0^t h(t-s)c(s)dW_s=h(0)\int_0^t c(s)dW_s+\int_0^t\left(\int_0^{t-s}h'(u)du\right)c(s)dW_s=h(0)\int_0^t c(s)dW_s+$$

$$+\int_0^t\int_s^t h'(v-s)c(s)dv\,dW_s=h(0)\int_0^t c(s)dW_s+\int_0^t\int_0^v h'(v-s)c(s)\,dW_s\,dv=h(0)\int_0^t c(s)dW_s+\int_0^t V_{h'}^c(v)\,dv,$$

i.e. $V_h^c(t)$ is a semimartingale.

2) Let $V_h^c(t)$ is a semimartingale. Then, according to the ***Definition 2***, it can be presented by the following

$V_h^c(t)=M_t+A_t$, where M is a locally square integrable martingale, A is a process with integrable variation, and the following inequalities :

$$\forall\, 0<s<t \qquad E(\underset{[s,t]}{Var} A) \geq E\left| E\left(V_h^c(t)-V_h^c(s) \middle| F_s\right)\right| \geq L\cdot\left(\int_0^s \left(h(t-u)-h(s-u)\right)^2 c^2(u)du\right)^{\frac{1}{2}} =$$

$$= L\cdot\left(\int_0^s \left(h(t-s-u)-h(u)\right)^2 c^2(s-u)du\right)^{\frac{1}{2}}, \quad L>0.$$

Therefore the semimartingale property of process $V_h^c(t)$ can be written by the following way:

$\Sigma(t):=\sup_{\lambda\in\Lambda_t}\sum_{\lambda}\left(\int_0^{t_i}\left(h(t_{i+1}-t_i+u)-h(u)\right)^2 c^2(t_i-u)du\right)^{\frac{1}{2}} \leq \frac{1}{L}\, E\left(\underset{[0,t]}{Var} A\right)<\infty$, where Λ_t – the set of finite partitions of segment $[0,\, t]$. Now, for uniformly partition $\lambda_n, n\geq 1$ of $[0,\, t]$ with partition diameter $|\lambda_n|=\frac{t}{n}$, using monotoneness of c, we obtain:

$\forall\theta\in(0,t)$

$$\Sigma(t)\geq\sum_{i=0}^{n-1}\left(\int_0^{t_i}\left(h(t_{i+1}-t_i+u)-h(u)\right)^2 c^2(t_i-u)du\right)^{\frac{1}{2}} \geq$$

$$\geq \sum_{\substack{0\leq i\leq n-1\\ i:t_i>\theta}}\left(\int_0^{t_i}\left(h(t_{i+1}-t_i+u)-h(u)\right)^2 c^2(t_i-u)du\right)^{\frac{1}{2}} \geq \left[\frac{t-\theta}{|\lambda_n|}\right]\left(\int_0^{\theta}\left(h(|\lambda_n|+u)-h(u)\right)^2 c^2(\theta-u)du\right)^{\frac{1}{2}}.$$

Hence

$$\infty > \lim_{|\lambda_n|\to 0}\int_0^{\theta}\left(\frac{h(|\lambda_n|+u)-h(u)}{|\lambda_n|}\right)^2 c^2(\theta-u)du = \int_0^{\theta}(h'(u)c(\theta-u))^2 du =$$

$$=\int_0^{\theta}(h'(\theta-u)c(u))^2 du.$$

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