

# On existence of insurer's optimal excess of loss reinsurance strategy

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## **Abstract**

The conditions of absence and existence of cedant's optimal excess of loss reinsurance strategy has been formulated in this paper for some individual risk model under premium principle based on mean of claim amount.

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## 1. Introduction.

As well known the reinsurance is the transfer of risk from a direct insurer, the *cedant*, to a second insurance carrier, the *reinsurer*. The best known examples of reinsurance are quota share, stop loss and excess of loss reinsurance (cf. [1]). The main insurer's goal of reinsurance depends on the insurer. One insurer wants to find a reinsurance policy which gives the smallest standard deviation of random claim sizes, other insurer wants to find that reinsurance policy which gives the smallest ruin probability (or largest non-ruin probability). In both cases cedant should find those optimal quota share or retention limit under which the standard deviation of random claim sizes or ruin probability are smallest. For instance, the problem of finding an optimal cedant's retention limit of purchasing reinsurance based on minimizing the standard deviation was considered in [2]. The similar problem for stop loss reinsurance was considered in [3] under premium principle based on mean and variance of the reinsurer's share of the total claim amount. An optimal quota share which maximizes the cedant's non-ruin probability was considered in [4].

In this paper we consider the excess of loss reinsurance for some individual risk model. We suppose that the cedant uses reinsurance policy in order to increase own non-ruin probability. Using properties of first two moments of cedant's share (cf. [1], [5]), it can be proved that under premium principle based on mean or variance (or standard deviation) of the claim sizes the cedant's premium income and the cedant's expected profit are increasing functions of  $r$ . Therefore, increasing own non-ruin probability, cedant decreases own premium income and expected profit.

The aim of this paper is to investigate the conditions of excess of loss reinsurance policy under which cedant's non-ruin probability may be a non-monotonic function of  $r$  on some interval  $\mathcal{I} \subset (P_0, 1)$  ( $P_0$  is a cedant's non-ruin probability prior reinsuring). Under such conditions more than one value of  $r$  correspond to each value of non-ruin probability from  $\mathcal{I}$ , or in other words under fixed value of cedant's non-ruin probability from  $\mathcal{I}$  for cedant there exists an opportunity (i.e. there exists a strategy) to chose such value of  $r$  which corresponds to larger value of cedant's expected profit.

## 2. Main results.

Let us consider a homogeneous with respect to risks portfolio of  $N$  ( $N$  is enough large number) treaties where the cedant's retention limit  $r$  is defined and it is a same for each random claim sizes  $X_i$ ,  $1 \leq i \leq N$ . Throughout this paper we will consider  $X_i$  as a non-negative random variable on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ; its distribution function we will denote by  $F_i$ .

Further we will assume that  $X_i$  are i.i.d. random variables (and then index  $i$  will be omitted) with the same continuous distribution function  $F$  that satisfies the following conditions

$$F(0) = 0, F \in \mathcal{C}^1(\mathbb{R}) \text{ and } F'(x) = f(x), x \in \mathbb{R}; \quad (1)$$

$\exists \alpha > 2, C > 0$  such that  $\forall x \geq 0$

$$1 - F(x) \leq Cx^{-\alpha}; \quad (2)$$

$$\int_0^{\infty} x^2 dF(x) < \infty. \quad (3)$$

It should be noted that the finitnness of  $\mathbb{E}X$  and  $\mathbf{Var}X$  follows from (3). For our individual risk model under excess of loss reinsuring the reinsurer pays that part of each claim amount which exceeds an agreed retention limit  $r$ , that is, it pays  $X - r$ , the cedant pays  $X^{(r)} := \min(X, r)$ . Hence, cedant's total claim amount  $S := X_1 + \dots + X_N$  will be changed to  $S^{(r)} := X_1^{(r)} + \dots + X_N^{(r)}$  after reinsuring. At the same time the cedant's premium income will also be changed: before reinsuring it was equal to

$$Np = N(1 + \theta)\mathbb{E}X,$$

where  $\theta > 0$  is a safety loading coefficient; after reinsuring cedant pays to reinsurer the amount

$$N(1 + \theta^*(r)) (\mathbb{E}X - \mathbb{E}X^{(r)}),$$

where  $\theta^*(r)$  is a safety loading coefficient defined by the following way

$$\theta^*(r) = \begin{cases} \theta^* \equiv \text{constant} > 0, & \text{under premium principle based on mean of claim amount;} \\ \beta \frac{\sqrt{\mathbf{Var} \max(0, X-r)}}{\mathbb{E} \max(0, X-r)}, & \text{under premium principle based on standard deviation} \\ & \text{of claim amount, where } \beta > 0 \text{ is defined by reinsurer.} \end{cases}$$

Hence, the cedant's premium income will be equal to

$$I(r) := N(\theta - \theta^*(r))\mathbb{E}X + N(1 + \theta^*(r))\mathbb{E}X^{(r)}.$$

Also we can determine the cedant's expected profit

$$U(r) := \mathbb{E} (I(r) - S^{(r)}) = N [(1 + \theta)\mathbb{E}X - (1 + \theta^*(r)) (\mathbb{E}X - \mathbb{E}X^{(r)}) - \mathbb{E}X^{(r)}].$$

Some properties of  $I(r)$  and  $U(r)$  are shown in the following lemma.

**Lemma 1.** *Functions  $I(r)$  and  $U(r)$  are increasing.*

*Proof.* According to the definition of limited expected value function with respect to random variable  $X$  (cf. [1], [5])

$$\mathbb{E}X^{(r)} = \int_0^r x dF(x) + r(1 - F(r)).$$

It is almost obvious that  $\mathbb{E}X^{(r)}$  is an increasing function of  $r$  (its derivative equals to  $\frac{d}{dr} (\mathbb{E}X^{(r)}) = 1 - F(r) > 0$ ,  $r > 0$ ). Under condition (1)-(3)  $\mathbb{E}X^{(r)} \longrightarrow \mathbb{E}X$ , as  $r \rightarrow +\infty$  (under  $r = +\infty$  there is no reinsurance).

Now, for the following two cases we have:

a) *premium principle based on mean of random claim amount*

$\theta^*(r) \equiv \theta^* > 0$  and

$$I(r) := N(\theta - \theta^*)\mathbb{E}X + N(1 + \theta^*)\mathbb{E}X^{(r)},$$

$$U(r) := N(\theta - \theta^*)\mathbb{E}X + N\theta^*\mathbb{E}X^{(r)}$$

are increasing functions.

b) *premium principle based on standard deviation of random claim amount*

$$\theta^*(r) = \beta \frac{\sqrt{\mathbf{Var} \max(0, X - r)}}{\mathbb{E}(X - X^{(r)})},$$

where  $\beta > 0$  is a coefficient defined by reinsurer.

$$I(r) = N(1 + \theta)\mathbb{E}X - N \left( \mathbb{E}X - \mathbb{E}X^{(r)} + \beta \sqrt{\mathbf{Var}(X - X^{(r)})} \right)$$

$$U(r) = N(1 + \theta)\mathbb{E}X - N \left( \mathbb{E}X + \beta \sqrt{\mathbf{Var}(X - X^{(r)})} \right).$$

Now, let us investigate the variance of  $X - X^{(r)}$  as a function of  $r$

$$\begin{aligned} \mathbf{Var}(X - X^{(r)}) &= \mathbb{E}(X - X^{(r)})^2 - (\mathbb{E}X - \mathbb{E}X^{(r)})^2 = \\ &= \int_r^\infty (x^2 - 2xr + r^2) dF(x) - (\mathbb{E}X - \mathbb{E}X^{(r)})^2. \end{aligned}$$

For that purpose let us investigate its derivative

$$\begin{aligned} \frac{d}{dr} (\mathbf{Var}(X - X^{(r)})) &= -r^2 f(r) - 2 \int_r^\infty x f(x) dx + 2r^2 f(r) + 2r(1 - F(r)) - r^2 f(r) + \\ &+ 2(1 - F(r)) (\mathbb{E}X - \mathbb{E}X^{(r)}) = -2F(r) (\mathbb{E}X - \mathbb{E}X^{(r)}) < 0, r > 0. \end{aligned}$$

Therefore  $\mathbf{Var}(X - X^{(r)})$  is a decreasing function of  $r$ .

Taking in account that  $\mathbb{E}X^{(r)}$  is an increasing function of  $r$ , we obtain that  $I(r)$  and  $U(r)$  are increasing function of  $r$ .  $\square$

As a reinsurer has rights to determine a coefficient  $\beta > 0$  as it wishes, it is more reasonable from the cedant's point of view to consider  $\theta^*(r)$  as a fixed number which agreed between cedant and reinsurer. Therefore, further we will consider only the case of premium principle based on mean of random claim amount.

As we noted above it is supposed that using reinsurance policy cedant wants to increase own non-ruin probability

$$P(r) = \mathbb{P} \{ S^{(r)} \leq I(r) \}.$$

Using normal approximation (for enough large number  $N$ ) we have

$$P(r) \approx \Phi \left( \frac{U(r)}{\sqrt{N \mathbf{Var} X^{(r)}}} \right) = \Phi \left( \sqrt{N} \theta^* \frac{a \mathbb{E}X + \mathbb{E}X^{(r)}}{\sqrt{\mathbf{Var} X^{(r)}}} \right),$$

where  $a = \frac{\theta - \theta^*}{\theta^*}$ ;  $\Phi$  is a standard normal distribution function.

Further we will assume that

$$U(r) > 0, r > 0, \tag{4}$$

that is

$$\mathbb{E}X^{(r)} > -a \mathbb{E}X, \tag{5}$$

Figure 1: *graphics of  $P(r)$  under  $a \geq 0$ .*

hence the condition  $a \geq -1$  should be held .

It should be noted that condition (4) is necessary for cedant's non-ruin.

In order to investigate non-ruin probability function  $P(r)$  it is enough to investigate the following function

$$\varphi(r) := \frac{(a\mathbb{E}X + \mathbb{E}X^{(r)})^2}{\mathbf{Var}X^{(r)}}.$$

Let us investigate a derivative of  $\varphi(r)$ .

$$\begin{aligned} (\mathbf{Var}X^{(r)})^2 \frac{d}{dr}\varphi(r) &= 2(1 - F(r)) (a\mathbb{E}X + \mathbb{E}X^{(r)}) \left( \mathbb{E} (X^{(r)})^2 - (\mathbb{E}X^{(r)})^2 \right) - \\ &\quad - (a\mathbb{E}X + \mathbb{E}X^{(r)})^2 (\mathbf{Var}X^{(r)})'_r. \\ (\mathbf{Var}X^{(r)})'_r &= \left( \mathbb{E} (X^{(r)})^2 \right)'_r - 2(1 - F(r))\mathbb{E}X^{(r)} = \left( \int_0^r x^2 dF(x) + r^2(1 - F(r)) \right)'_r - \\ &\quad - 2(1 - F(r))\mathbb{E}X^{(r)} = 2(1 - F(r)) (r - \mathbb{E}X^{(r)}). \end{aligned}$$

From here we have

$$\begin{aligned} (\mathbf{Var}X^{(r)})^2 \frac{d}{dr}\varphi(r) &= 2(1 - F(r)) (a\mathbb{E}X + \mathbb{E}X^{(r)}) \left( \mathbb{E} (X^{(r)})^2 - (a\mathbb{E}X - r) \mathbb{E}X^{(r)} - ra\mathbb{E}X \right) = \\ &= 2(1 - F(r)) (a\mathbb{E}X + \mathbb{E}X^{(r)}) (\mathbf{Var}X^{(r)} + (\mathbb{E}X^{(r)} - r) (a\mathbb{E}X + \mathbb{E}X^{(r)})). \end{aligned}$$

Put  $\psi(r) := \mathbb{E} (X^{(r)})^2 - (a\mathbb{E}X - r) \mathbb{E}X^{(r)} - ra\mathbb{E}X$ . Then under condition (1)-(3) function  $\psi$  satisfies the following conditions

$$\begin{cases} \psi(0+) = 0, \\ \psi(r) \longrightarrow -\infty, \text{ as } r \rightarrow +\infty. \end{cases} \quad (6)$$

Its derivative is the following function

$$\psi'(r) = 2r(1 - F(r)) + (1 - F(r))a\mathbb{E}X - r(1 - F(r)) - a\mathbb{E}X - \mathbb{E}X^{(r)} =$$

Figure 2: *graphics of  $P(r)$  under  $a \in (-1, 0)$ .*

$$= - \int_0^r x dF(x) - aF(r)\mathbb{E}X.$$

Further we consider two cases

1) *under  $a \geq 0$*

$$\psi'(r) < 0, \quad r > 0,$$

hence  $\varphi'(r) < 0, \quad r > 0$ , i.e.  $\varphi(r)$  is a decreasing function, and therefore  $P(r)$  is a decreasing function too, and in particular (see Figure 1).

- a)  $P(0) = 1$ ,
- b)  $P'(0) = -\infty$
- c)  $P(\infty) = P_0$ .

2) *under  $a \in (-1, 0)$*

In this case it follows from condition (5) that  $\exists r_1 > 0$  such that  $\forall r > r_1 \quad U(r) > 0$ . Therefore  $(r_1, \infty)$  is an admissible domain of optimal values of  $r$ .

$$\psi''(r) = -rf(r) - af(r)\mathbb{E}X > 0 \iff r < -a\mathbb{E}X,$$

$$\psi''(r) < 0 \iff r > -a\mathbb{E}X,$$

hence  $\exists \max_{r>0} \psi'(r) = \psi'(-a\mathbb{E}X) =$

$$= - \int_0^{-a\mathbb{E}X} x dF(x) - aF(-a\mathbb{E}X)\mathbb{E}X = \int_0^{-a\mathbb{E}X} F(x) dx > 0.$$

Further

$$\psi'(0) = 0 \quad \text{and} \quad \psi'(+\infty) = -(1+a)\mathbb{E}X < 0 \quad \text{for} \quad a > -1,$$

hence  $\exists r_2 > 0$  :

$$\begin{cases} \psi'(r) > 0, & \text{under } r \in (0, r_2), \\ \psi'(r) < 0, & \text{under } r > r_2. \end{cases} \quad (7)$$

Therefore, from (6)-(7) we obtain

$$\exists r_3 > 0 : \begin{cases} \psi(r) > 0, & \text{under } r \in (0, r_3), \\ \psi(r) < 0, & \text{under } r > r_3. \end{cases} \quad (8)$$

It should be noted that

$$\psi(r_1) = \mathbf{Var}X^{(r_1)} + (\mathbb{E}X^{(r_1)} - r_1) (a\mathbb{E}X + \mathbb{E}X^{(r_1)}) = \mathbf{Var}X^{(r_1)} > 0. \quad (9)$$

Hence, it follows from (8)-(9) that  $r_3 > r_1$ .

Therefore function  $P(r)$  increases on interval  $(r_1, r_3)$  from  $P(r_1)$  to  $P(r_3)$  and then decreases on interval  $(r_3, \infty)$  from  $P(r_3)$  to  $P_0 < P(r_1)$ . Let  $r_4 > 0$  such that  $P(r_4) = P(r_1)$ . For current case, when  $a \in (-1, 0)$ , the graphics of  $P(r)$  is shown on Figure 2.

Summarizing all of that obtained above, we can formulate the following theorem.

**Theorem 1.** *Let the distribution function  $F$  satisfies conditions (1)-(3). Then*

1) *Under  $a \geq 0$ , cedant's non-ruin probability is a decreasing function of  $r$  (see Figure 1). In this case only one value of retention limit corresponds to each fixed value of non-ruin probability from  $(P_0, 1)$ . In other words, taking a larger than  $P_0$  value of own non-ruin probability, cedant can find only one relevant retention limit, and therefore it has no opportunity to chose an optimal retention limit.*

2) *under  $a \in (-1, 0)$   $P(r)$  is a non-monotonic function on  $(0, \infty)$  (see Figure 2) and moreover there exists an interval  $\mathcal{I} = (P(r_1), P(r_2)) \subset (P_0, 1)$  such that for each fixed value of cedant's non-ruin probability from  $\mathcal{I}$  we have two different values of  $r$ : the first one is  $r^{(1)} \in (r_1, r_3)$  and the second one is  $r^{(2)} \in (r_3, r_4)$ . The value  $r^{(2)}$  is an optimal from cedant's point of view, as  $U(r^{(2)}) > U(r^{(1)})$ .*

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