

Thermodynamical Formalism for Open Classes of Potentials and Non-uniformly Hyperbolic Maps

Krerley Oliveira and Marcelo Viana *

May 18, 2006

Abstract

We develop a Ruelle-Perron-Frobenius transfer operator approach to the ergodic theory of a large class of non-uniformly expanding transformations on compact manifolds. For Hölder continuous potentials not too far from constant, we prove that the transfer operator has a strictly positive Hölder continuous eigenfunction, and use this fact to show that there is exactly one equilibrium state. Moreover, the equilibrium state is a non-lacunary Gibbs measure, a non-uniform version of the classical notion of Gibbs measure that we introduce here.

1 Introduction

The theory of equilibrium states of uniformly hyperbolic (Axiom A) dynamical systems, developed by Sinai, Ruelle, Bowen in the early seventies, is a major achievement in ergodic theory, and a spectacular example of ideas from statistical physics applied in the realm of smooth dynamics. Besides its intrinsic beauty, this theory yields a surprisingly complete picture of the behavior of such systems, at the statistical level: finitely many invariant physical (or SRB) probability measures, which describe the asymptotic time averages of Lebesgue almost every point.

The strategy, initiated by Sinai [Sin72] in the case of Anosov diffeomorphisms, and carried out in full generality by Ruelle and Bowen [Bo75, BR75, Ru76], may be briefly outlined as follows. Uniformly hyperbolic systems admit finite generating Markov partitions. Via the itinerary relative to such a partition, points in phase space are identified with configurations of a one-dimensional lattice gas. The Gibbs distributions of the gas correspond to the equilibrium states of the dynamical system.

Extension of this approach and of the conclusions beyond the Axiom A context involves some fundamental difficulties, even restricted to non-uniformly

*K.O. was partially supported by CNPq and Fapeal, and M.V. partially supported by Faperj, Brazil.

hyperbolic systems, that is, such that almost every point admits an asymptotically hyperbolic splitting of the tangent space (in other words: no zero Lyapunov exponent). For one thing, generating Markov partitions are not known to exist in general. Even when they do exist, Markov partitions usually have infinitely many atoms; this leads to considering gases with infinitely many states, a difficult subject, not yet well understood.

There has also been substantial recent progress concerning physical measures. In particular, Alves, Bonatti, Viana [ABV00, BV00] proved existence and uniqueness of SRB measure for certain large classes of non-uniformly hyperbolic maps. One important difficulty in this context lies in the very definition of non-uniform hyperbolicity: [ABV00] assume that *Lebesgue* almost every point has only non-zero Lyapunov exponents, but it is not clear how this kind of condition could be useful when considering more general potentials, since most equilibrium states should be singular relative to the Lebesgue measure.

In [O103] the first author overcame such difficulties and proved the existence of equilibrium states for open sets of non-uniformly expanding maps and of continuous potentials. Roughly, the map should be expanding on most of phase space, with possibly contracting behavior on the complement. Concerning the potential ϕ , it is assumed that its oscillation $\max \phi - \min \phi$ is not too large. This ensures, *a priori*, that a large set of candidates to equilibrium states accord a (uniformly) small weight to the possibly contracting regions. Using this fact, we could find a true equilibrium state within that set of candidates. This approach was then extended in [AMO04] to obtain similar results in the context of random non-uniformly expanding maps.

Important contributions have been given recently by several other authors: Bruin, Keller [BK98], Denker, Urbanski [DU92, Urb98], Wang, Young [WY01] for interval maps, rational functions of the sphere, and Hénon-like maps; and Buzzi, Maume, Sarig [Buz99, BMD02, BS02, Sar03] and Yuri [Yur99, Yur00, Yur03], for countable Markov shifts and for piecewise expanding maps, to mention just a few of the most recent works. Several of these papers, and particularly [DU92, Sar03, Yur99, Yur00, Yur03], consider systems with neutral periodic points, a setting of non-hyperbolic dynamics which has attracted a great deal of attention over the last years.

The results in the present paper are similar in flavor to those of [O103] and [OV06], but they improve that work in several important ways, even if they do not supersede it. To begin with, our hypotheses on the dynamical system are milder and much more natural. In fact, they are quite close to conditions in the Appendix of [ABV00]. In addition, we develop a powerful approach, based on the Ruelle-Perron-Frobenius operator, which provides a better understanding of the equilibrium states: for instance, we are able to prove that they are non-lacunary Gibbs states, a non-uniform variation of the classical notion, closely related to the weak Gibbs states in [Yur99]. Most important, this new approach allows us to prove, for the first time, uniqueness of the equilibrium state for every Hölder continuous potential whose oscillation is not too large; in our previous work, uniqueness was obtained only in the case of constant potentials (measures of maximal entropy).

Precise statements of these conditions and conclusions follow.

2 Setting and statements

Let $f : M \rightarrow M$ be a continuous transformation on a compact space M , and $\phi : M \rightarrow \mathbb{R}$ be a continuous function.

Definition 2.1. An f -invariant measure is an *equilibrium state* of f for the potential ϕ if it maximizes the functional

$$\eta \mapsto h_\eta(f) + \int \phi d\eta$$

among all f -invariant probabilities η .

By the variational principle [Wa82], the supremum of this functional over the invariant probabilities coincides with the *pressure* $P(f, \phi)$ of f for ϕ .

2.1 The class of maps and potentials

Throughout this paper, we consider $f : M \rightarrow M$ a C^1 local diffeomorphism on a compact Riemannian manifold, satisfying

(H1) There exist pairwise disjoint connected open sets with finite inner diameter

$$\mathcal{R} = \{R_1, \dots, R_q, R_{q+1}, \dots, R_{q+p}\}$$

such that $\cup_i \bar{R}_i = M$ and every $f|_{\bar{R}_i}$ is injective, and there exist constants $\delta_0 > 0$ and $\sigma_1 > q$ such that

1. f is expanding at every $x \in \bar{R}_{q+1} \cup \dots \cup \bar{R}_{q+p}$: $\|Df(x)^{-1}\| \leq \sigma_1^{-1}$.
2. f is never too contracting: $\|Df(x)^{-1}\| \leq 1 + \delta_0$ for every $x \in M$.
3. \mathcal{R} is a transitive Markov partition: the image $f(\bar{R}_i)$ of every atom is a union of atoms \bar{R}_j , and there exists N such for every $i = 1, \dots, p + q$ we have $f^N(R_i) = M$.

Let us emphasize that *this partition is not required to be generating*: the diameters of cylinders of length n need not decrease to zero when $n \rightarrow \infty$.

We associate to each such transformation a certain positive number $c_0(f)$. Since the definition is somewhat technical, we postpone it for a while. But we advance that c_0 depends only on δ_0, σ_1, q, p , and it converges to $\log q$ when either δ_0 goes to zero or σ_1 goes to infinity. As f is a local diffeomorphism and M is a compact connected manifold, the number $k \in \mathbb{N}$ of pre-images is the same for every point. We assume that $k > q$ and the potential $\phi : M \rightarrow \mathbb{R}$ satisfies

(H2) ϕ is Hölder continuous and $\max \phi - \min \phi < \log k - c_0(f)$

The inequality holds, for instance, if the oscillation of ϕ is less than $\log k$ and δ_0 is small enough.

2.2 Non-lacunary Gibbs measures

For each $n \geq 1$, let $[i_0, \dots, i_{n-1}]$ be the *cylinder* of points admitting i_0, \dots, i_{n-1} as (length- n) *itinerary* relative to \mathcal{R} , that is

$$[i_0, \dots, i_{n-1}] = \{y \in M; f^j(y) \in \bar{R}_{i_j} \text{ for } j = 0, \dots, n-1\}.$$

Let \mathcal{R}^n be the partition of M into length- n cylinders, and $R^n(x) \in \mathcal{R}^n$ denote some atom that contains a given point $x \in M$. We write $S_n\phi(x) = \sum_{j=0}^{n-1} \phi(f^j(x))$ for each $x \in M$ and $n \in \mathbb{N}$.

Definition 2.2. A probability η (not necessarily invariant) is a *Gibbs measure* of f for ϕ if there exist $P \in \mathbb{R}$ and $K > 0$ such that

$$K^{-1} \leq \frac{\eta(R^n(x))}{\exp(S_n\phi(x) - nP)} \leq K \quad (1)$$

for every $x \in M$ and $n \geq 1$. More generally, η is a *non-lacunary Gibbs measure* of f for ϕ if for η -almost every x there exists a non-lacunary sequence of values of $n \geq 1$ for which (1) is satisfied.

An integer sequence $n_j \in \mathbb{N}$ is called *non-lacunary* if it is increasing and n_{j+1}/n_j converges to 1. In the proofs, this will correspond to the sequence of hyperbolic times [Alv00, ABV00]:

Definition 2.3. We say that $n \in \mathbb{N}$ is a *hyperbolic time* for $x \in M$ if

$$\prod_{k=0}^{j-1} \|Df(f^{n-k}(x))^{-1}\| \leq e^{-2cj} \quad \text{for every } 1 \leq j \leq n.$$

We say that n is a hyperbolic time for a cylinder $R^n \in \mathcal{R}^n$ if n is a hyperbolic time for every $x \in R^n$. We denote by \mathcal{R}_h^n the set of the cylinders $R^n \in \mathcal{R}^n$ for which n is a hyperbolic time.

For the definition the constant c is arbitrary (we should speak of c -hyperbolic times), but we shall fix it once and for all, as indicated in relation (6) below. We denote by H the set of $x \in M$ with *infinitely many hyperbolic times*, that is, such that $R^n(x) \in \mathcal{R}_h^n$ for infinitely many values $n_1 < \dots < n_j < \dots$ of n .

Remark 2.4. If η is a non-lacunary Gibbs measure then for η -almost every point $x \in M$ there exists a sequence $K_n = K_n(x)$ such that $\lim_{n \rightarrow \infty} \frac{1}{n} \log K_n = 0$ and

$$K_n^{-1} \leq \frac{\eta(R^n(x))}{\exp(S_n\phi(x) - nP)} \leq K_n \quad (2)$$

for every $x \in M$ and $n \geq 1$. A proof is given at the end of this section. This property is close to the notion of *weak Gibbs measure* in [Yur99]: Yuri requires condition (2) at all points and K_n independent of x .

Theorem A. *Assume f and ϕ satisfy hypotheses (H1) and (H2). Then f has some non-lacunary Gibbs measure ν for ϕ , with $\nu(H) = 1$ and $\text{supp } \nu = \bar{H}$.*

Observe that the measure ν is usually not invariant. For instance, when $\phi = -\log |\det Df|$ our construction yields $\nu = \text{Lebesgue measure}$.

Theorem B. *Assume f and ϕ satisfy hypotheses (H1) and (H2). Then f admits some invariant non-lacunary Gibbs measure μ for ϕ , absolutely continuous relative to ν with density bounded from zero and infinity. Moreover, μ is ergodic, $\mu(\partial\mathcal{R}) = 0$ and all its Lyapunov exponents are positive.*

The Ruelle-Perron-Frobenius operator $\mathcal{L}_\phi : C(M) \rightarrow C(M)$ of $f : M \rightarrow M$ is defined on the space $C(M)$ of continuous functions $g : M \rightarrow \mathbb{R}$ by

$$\mathcal{L}_\phi g(x) = \sum_{f(y)=x} e^{\phi(y)} g(y).$$

The next result provides another construction of invariant non-lacunary Gibbs measures, with additional information on the regularity of the density.

Theorem C. *Assume f and ϕ satisfy hypotheses (H1) and (H2). Then, there exists a constant P and a Hölder continuous function $h : M \rightarrow (0, \infty)$ such that $\mathcal{L}_\phi h = e^P h$ and so $\mu = h\nu$ is an invariant non-lacunary Gibbs measure. Moreover, The constant P is uniquely determined and coincides with the spectral radius of \mathcal{L}_ϕ .*

2.3 Equilibrium states

Finally, from the previous conclusions we deduce our main result:

Theorem D. *Assume f and ϕ satisfy hypotheses (H1) and (H2). Then, $P = P(\phi, f)$ in Theorem C, and the corresponding measure μ is the unique equilibrium state of f for ϕ .*

A few comments are in order concerning the hypothesis (H1) and (H2) and the its relation with the support of the measure ν . Firstly, $\text{supp } \nu$ needs not coincide with the whole M . For instance, it is easy to see that our hypotheses are compatible with the presence of periodic attractors:

Example 2.5. Let $f_0 : \mathbb{T}^d \rightarrow \mathbb{T}^d$ be a linear expanding map. Fix some Markov partition \mathcal{R} for f_0 and some $R_1 \in \mathcal{R}$ containing a fixed (or periodic) point p . Then deform f_0 on a small neighborhood of p inside R_1 in such a way that p becomes an attractor for the new transformation f . By construction, outside R_1 we have $f = f_0$ and so f is uniformly expanding. Clearly, \mathcal{R} remains a Markov partition for f . It is also clear that we may take the deformation in such a way that f is never too contracting in R_1 . This means that (H1) holds in this case. Taking $\phi = 0$ we also have condition (H2), so that our results apply: we obtain an ergodic invariant measure μ that μ maximizes the entropy of f on its support. Clearly, H and $\text{supp } \mu$ are disjoint from the basin of the periodic attractor.

However, the basin of a periodic attractor carries no entropy and it is probably the case that the measure μ maximizes entropy of f on the whole ambient space M . Thus, it remains open whether Theorem D would be false without the additional assumption.

On the other hand, robust classes of systems for which ν is supported on the whole ambient space do exist. For instance, if $\phi = -\log |\det Df|$ satisfies (H2) then there can be no periodic attractors: as we shall see in relation (5), the Jacobian of f must be larger than 1 at every point. In fact, in this case ν is the Lebesgue measure and H has full Lebesgue measure. Here is one case where $\phi = -\log |\det Df|$ does satisfy (H2):

Example 2.6. Let $f_0 : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a linear expanding map with real eigenvalues σ_1, σ_2 . Notice that $k = \sigma_1\sigma_2$ is the number of pre-images of any point $y \in \mathbb{T}^2$. Assume that $f_0(x_1, x_2) = (\sigma_1x_1, \sigma_2x_2)$ in appropriate local coordinates (x_1, x_2) close to the fixed point 0. Let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be a C^∞ even function such that $\eta(0) = 1$, η is non-increasing on $(0, \infty)$, and $\eta(x) = 0$ for $x \geq 1$. Let $\varepsilon > 0$ be some small number and $s_i \in [0, \sigma_i]$ for $i = 1, 2$. Define

$$f(x_1, x_2) = ([\sigma_1 - s_1\eta(x_1/\varepsilon)\eta(x_2/\varepsilon)]x_1, [\sigma_2 - s_2\eta(x_1/\varepsilon)\eta(x_2/\varepsilon)]x_2).$$

Assume ε has been chosen small enough so that $f = f_0$ outside some Markov rectangle R_1 of f_0 containing the fixed point $p = 0$ in its interior. The Jacobian of f is given by

$$\begin{aligned} \det Df(x_1, x_2) &= (\sigma_1 - s_1\eta(x_1/\varepsilon)\eta(x_2/\varepsilon))(\sigma_2 - s_2\eta(x_1/\varepsilon)\eta(x_2/\varepsilon)) \\ &\quad - (\sigma_1 - s_1\eta(x_1/\varepsilon)\eta(x_2/\varepsilon))s_2\eta(x_1/\varepsilon)\eta'(x_2/\varepsilon)(x_2/\varepsilon) \\ &\quad - (\sigma_2 - s_2\eta(x_1/\varepsilon)\eta(x_2/\varepsilon))s_1\eta'(x_1/\varepsilon)(x_1/\varepsilon)\eta(x_2/\varepsilon). \end{aligned}$$

We consider $0 \leq s_i \leq \sigma_i$ for $i = 1, 2$. Recall that $|\eta(x)| \leq 1$ and $\eta'(x)x \leq 0$, and the equalities hold at $x = 0$. It follows that $\det Df \geq (\sigma_1 - s_1)(\sigma_2 - s_2)$ and the equality holds at the fixed point 0. Thus, condition (H2) for the function $\phi = -\log |\det Df|$ may be re-written as

$$\log(\sigma_1\sigma_2) - \log(\sigma_1 - s_1)(\sigma_2 - s_2) < \log k - c_0(f) \Leftrightarrow \log \det Df(0) > c_0(f).$$

Therefore, our assumptions hold for ϕ as long as we choose s_1 and s_2 such that $\det Df(0) > 1$, and then take δ_0 small enough in (H1).

In [O103], the author construct open classes of maps in a very related setting such potentials satisfying (H2) have equilibrium measures. Moreover, he proved that *all* equilibrium measures have only positive Lyapounov exponents. See [O103] for details. In [OV06], the authors extends this approach under topologically mixing condition and a clear assumption about the derivative of f , and prove uniqueness, at least for the maximal entropy measure and characterize this measure as a Gibbs state

Another importante remark concern the necessity of some control on the potential ϕ . In absence of the hypothesis (H2), the uniqueness may fail either

if we assume (H1). In fact, we may construct an example of a transformation $f : [0, 1] \rightarrow [0, 1]$ of the interval with a neutral fixed point x_0 in $(\frac{1}{3}, \frac{2}{3})$ performing a deformation of the map $x \rightarrow 3x \bmod 1$ close to $x_0 = \frac{1}{2}$. If we procede carefully, we may get all conditions in (H1) with $\delta_0 = 0$, $p = 2$, $q = 1$, $R_1 = (\frac{1}{3}, \frac{2}{3})$, $R_2 = (0, \frac{1}{3})$ and $R_3 = (\frac{2}{3}, 1)$. If the angle that f meets the diagonal is carefully choose, that there exists a invariant measure μ absolutely continuous with respect to the Lebesgue measure. It is not difficult to see using by Pesin formula and Ruelle inequality, that μ is an equilibrium measure for the potential $\phi(x) = -\log |Df(x)|$. On the other hand, since $h_{\delta_{x_0}}(f) = 0$ and $Df(x_0) = 1$, δ_{x_0} is another equilibrium measure for ϕ .

To complete our statements, we are left to define $c_0(f)$. We begin with the following combinatorial lemma, that counts the number of itineraries with frequent visits to the region $\mathcal{B} = R_1 \cup \dots \cup R_q$ where condition (H1) allows the map to be non-expanding.

Lemma 2.7. *Given $\gamma \in (0, 1)$, let $I_{\gamma, n}$ be the*

$$\{(i_0, \dots, i_{n-1}) \in \{1, \dots, q, q+1, \dots, q+p\}^n ; \#\{0 \leq j < n ; i_j \leq q\} > \gamma n\} \quad (3)$$

and define $c_\gamma = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \#I_{\gamma, n}$. Then c_γ goes to $\log q$ when $\gamma \rightarrow 1$.

Proof. This is an easy consequence of Stirling's formula. In fact,

$$\#I_{\gamma, n} \leq \sum_{r \geq \gamma n} \binom{n}{r} p^{n-r} q^r.$$

By Stirling's formula, there exists a universal constant $B > 0$ such that

$$r \geq \frac{kn}{k+1} \quad \Rightarrow \quad \binom{n}{r} \leq B \left(1 + \frac{1}{k}\right) (1+k)^{\frac{1}{k}}^r \leq B \left(1 + \frac{1}{k}\right) (1+k)^{\frac{1}{k}}^n.$$

This implies that if $\gamma \geq \frac{k}{k+1}$ then

$$c_\gamma \leq \log \left[\left(1 + \frac{1}{k}\right) (1+k)^{\frac{1}{k}} p^{\frac{1}{k+1}} q \right].$$

Now just note that the term on the right goes to $\log q$ when $k \rightarrow \infty$. □

Note that c_γ depends only on γ and p, q . Fix $0 < \gamma < 1$ such that

$$(1 + \delta_0)^\gamma \sigma_1^{-(1-\gamma)} < 1 \quad (4)$$

and then take any $c_0(f) > c_\gamma$. As we announced, $c_0(f)$ depends only on δ_0, σ_1, p, q and can be choose arbitrarily close to $\log q$ when either $\delta_0 \rightarrow 0$ or $\sigma_1 \rightarrow \infty$.

To conclude this introduction, let us give the

Proof of Remark 2.4. By assumption, for almost every $x \in M$ there exists an increasing sequence $n_i \in \mathbb{N}$ such that $\varepsilon_i = (n_{i+1} - n_i)/n_i$ converges to zero and

$$K^{-1} \leq \frac{\eta(R^n(x))}{\exp(S_n\phi(x) - nP)} \leq K$$

whenever $n = n_i$. Given $n \geq 1$ (it is no restriction to suppose n larger than the first hyperbolic time), let $i = i(n)$ be such that $n_i \leq n < n_{i+1}$. Then

$$\eta(R^{n_{i+1}}(x)) \leq \eta(R^n(x)) \leq \eta(R^{n_i}(x)).$$

Moreover,

$$|S_n\phi(x) - S_{n_i}\phi(x)| \leq (n - n_i) \max|\phi| \leq \varepsilon_i n \max|\phi|$$

and analogously for $|S_n\phi(x) - S_{n_{i+1}}\phi(x)|$. It follows that

$$K^{-1} e^{-\varepsilon_i n (\max|\phi| + P)} \leq \frac{\eta(R^n(x))}{\exp(S_n\phi(x) - nP)} \leq K e^{\varepsilon_i n (\max|\phi| + P)}.$$

Define $K_n = K \exp[\varepsilon_i n (\max|\phi| + P)]$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log K_n = \lim_{n \rightarrow \infty} \frac{1}{n} \log K + \varepsilon_{i(n)} (\max|\phi| + P) = 0,$$

and so the proof is complete. \square

Acknowledgements: The authors are grateful to J. Bochi, C. Matheus, V. Baladi and A. Arbieto for useful conversations, and to IMPA for a fine scientific environment.

3 An expanding reference measure

As a first step in our arguments we are going to construct a, not necessarily invariant, reference probability ν which is expanding, meaning that almost every point has infinitely many hyperbolic times, and is a non-lacunary Gibbs measure. This probability is found as an eigenvector of the dual transfer operator \mathcal{L}_ϕ^* defined in the cone \mathcal{M} of finite positive Borel measures of M by

$$\int g d(\mathcal{L}_\phi^* \eta) = \int (\mathcal{L}_\phi g) d\eta, \quad \text{for every continuous } g : M \rightarrow \mathbb{R}.$$

3.1 Eigenvectors of the dual transfer operator

The *Jacobian* of a measure μ with respect to f is the (essentially unique) function $J_\mu f$ satisfying

$$\mu(f(A)) = \int_A J_\mu f d\mu.$$

for any measurable set A such that $f|_A$ is injective. In other words, the Jacobian is defined by $J_\mu f = d(f_*\mu)/d\mu$. Jacobians need not exist, in general.

Lemma 3.1. *Suppose ν is any Borel probability which is an eigenvector for \mathcal{L}_ϕ^* : there exists $\lambda > 0$ such that $\mathcal{L}_\phi^* \nu = \lambda \nu$. Then, the Jacobian of ν with respect to f is $J_\nu f = \lambda e^{-\phi}$.*

Proof. Let A be any measurable set such that $f|_A$ is injective. Take a sequence $\{g_n\} \in C(M)$ such that $g_n \rightarrow \chi_A$ at ν -almost every point and $\sup |g_n| \leq 2$ for all n . Then,

$$\mathcal{L}_\phi(e^{-\phi} g_n)(x) = \sum_{f(y)=x} e^{\phi(y)} e^{-\phi(y)} g_n(y) = \sum_{f(y)=x} g_n(y).$$

The last expression converges to $\chi_{f(A)}(x)$ at ν -almost every point. Hence, by the dominated convergence theorem,

$$\int \lambda e^{-\phi} g_n d\nu = \int e^{-\phi} g_n d(\mathcal{L}_\phi^* \nu) = \int \mathcal{L}_\phi(e^{-\phi} g_n) d\nu \rightarrow \nu(f(A)).$$

Since the left hand side also converges to $\int_A \lambda e^{-\phi} d\nu$, we conclude that

$$\nu(f(A)) = \int_A \lambda e^{-\phi} d\nu,$$

which proves the lemma. \square

Lemma 3.2. *The spectral radius λ of the operator \mathcal{L}_ϕ is an eigenvalue for \mathcal{L}_ϕ^* and λ is bigger than $e^{\max \phi + c_0(f)}$.*

Proof. Observing that the spectral radius of \mathcal{L}_ϕ is equal to the spectral radius of \mathcal{L}_ϕ^* , and the dual cone of the positive functions is the cone of positive Borel measures, the first part is a quite standard fact from functional analysis and we just cite [DE] for the interested reader. For the inequality, we have

$$\mathcal{L}_\phi 1(x) = \sum_{f(y)=x} e^{\phi(y)} \geq \deg(f) e^{\min \phi} = e^{\log k + \min \phi} > e^{\max \phi + c_0(f)},$$

by hypothesis (H2). It follows that the eigenvalue $\lambda = \int \mathcal{L}_\phi 1 d\nu > e^{\max \phi + c_0(f)}$, as claimed. \square

In all that follows, we consider ν an eigenmeasure of \mathcal{L}_ϕ^* as in Lemma 3.2.

Combining lemmas 3.1 and 3.2, we get $J_\nu f = \lambda e^{-\phi} > e^{c_0(f)}$. Hence, we may find $\kappa > c_0(f)$ such that

$$J_\nu f > e^\kappa > q. \tag{5}$$

This property allow us to prove that ν -almost every point spends at most a fraction γ of time inside \mathcal{B} :

Proposition 3.3.

$$\nu\left(\left\{x \in M; \limsup_{n \rightarrow \infty} \frac{1}{n} \#\{0 \leq j \leq n-1; f^j(x) \in \mathcal{B}\} \leq \gamma\right\}\right) = 1.$$

Proof. Consider any cylinder $R^n \in \mathcal{R}^n$. Since f is injective on each element of \mathcal{R} , we have that f^n is injective on R^n and by the definition of Jacobian and the inequality (5)

$$1 \geq \nu(f^n(R^n)) = \int_{R^n} J_\nu f^n d\nu = \int_{R^n} \prod_{j=0}^{n-1} (J_\nu f \circ f^j) d\nu \geq e^{\kappa n} \nu(R^n).$$

This proves that $\nu(R^n) \leq e^{-\kappa n}$. Now let $A(n)$ be the union of all $R^n \in I_{\gamma, n}$, that is, all length- n cylinders $R^n = [i_0, \dots, i_{n-1}]$ such that

$$\frac{1}{n} \#\{0 \leq j \leq n-1; i_j \leq q\} > \gamma.$$

We are using the terminology of Lemma 2.7. The cardinality of $I_{\gamma, n}$ grows at exponential rate c_γ with n . Recall that $c_0(f) = c_\gamma$ and we have taken $\kappa > c_0(f)$. So, this implies that, for some $C > 0$,

$$\nu(A(n)) = \sum_{R^n \in I_{\gamma, n}} \nu(R^n) \leq C e^{c_\gamma n} e^{-\kappa n}$$

decays exponentially fast with n . Then, by Borel-Cantelli lemma, for every x in a full ν -measure set and for every sufficiently large n , we have that $x \notin A(n)$. In other words, the cylinder $R^n(x) = [i_0, \dots, i_{n-1}]$ has

$$\frac{1}{n} \#\{0 \leq j \leq n-1; i_j \leq q\} \leq \gamma,$$

for all sufficiently large n . As $i_j \leq q$ means, precisely, that $f^j(x) \in \mathcal{B}$, this proves the proposition. \square

3.2 Hyperbolic times

This notion was introduced by Alves [Alv00] and further developed in [ABV00]. In principle, the exponent c in Definition 2.3 is arbitrary, but for our applications we always consider it to be such that

$$(1 + \delta_0)^\gamma \sigma_1^{-(1-\gamma)} \leq e^{-4c} < 0. \quad (6)$$

Compare (4). The following simple property will be useful in the sequel:

Remark 3.4. If n is a hyperbolic time for x and m is a hyperbolic time for $f^n(x)$ then $m + n$ is a hyperbolic time for x .

The following is a key property of hyperbolic times, proved in [ABV00]:

Lemma 3.5. *There exists $\delta > 0$ depending only on f and c , such that given any hyperbolic time $n \geq 1$ for a point $x \in M$, and given any $1 \leq j \leq n$, the inverse branch $f_{x, n}^{-j}$ of f^j that sends $f^n(x)$ to $f^{n-j}(x)$ is defined on the whole ball of radius δ around $f^n(x)$, and satisfies*

$$d(f_{x, n}^{-j}(z), f_{x, n}^{-j}(w)) \leq e^{-jc} d(z, w)$$

for every z, w in the ball $B(f^n(x), \delta)$.

The same arguments, together with the observation that we may cover every atom R_i by finitely many balls of radius

Lemma 3.6. *There exists $C > 0$ depending of \mathcal{R} and δ such if $R^n \in \mathcal{R}_h^n$ for every $x, y \in R^n$ then*

$$d(f^{n-j}(x), f^{n-j}(y)) \leq Ce^{-jc}d(f^n(x), f^n(y))$$

Proof. It is clear using that we may cover any element R_1, \dots, R_{p+q} by a finite number of balls with radius δ , observing that any point of R^n have hyperbolic time n and the inner diameter of the rectangles of \mathcal{R} is finite. \square

Corollary 3.7. *Given any α -Hölder continuous function $\phi : M \rightarrow \mathbb{R}$, there exists a constant A such that for every $R^n \in \mathcal{R}_h^n$ and for every $x, y \in R^n$ then*

$$|S_n\phi(x) - S_n\phi(y)| \leq Ad(f^n(x), f^n(y))^\alpha.$$

Proof. Since ϕ is α -Hölder continuous we have that

$$|S_n\phi(x) - S_n\phi(y)| \leq \sum_{i=0}^{n-1} |\phi(f^i(x)) - \phi(f^i(y))| \leq \sum_{i=0}^{n-1} d(f^i(x), f^i(y))^\alpha.$$

By Lemma 3.6 we have that $d(f^{n-j}(x), f^{n-j}(y)) \leq Ce^{-jc}d(f^n(x), f^n(y))$ for every $j = 0, \dots, n-1$. Putting these together:

$$|S_n\phi(x) - S_n\phi(y)| \leq \sum_{i=0}^{n-1} C^\alpha e^{-\alpha ic} d(f^n(x), f^n(y))^\alpha \leq Ad(f^n(x), f^n(y))^\alpha.$$

This completes the proof. \square

Corollary 3.8. *There exists K such if $R^n \in \mathcal{R}_h^n$ and $x, y \in R^n$ then*

$$K^{-1} \leq \frac{J_\nu f^n(x)}{J_\nu f^n(y)} \leq K$$

Proof. Observe that $J_\nu f^n(x) = \lambda^n e^{-S_n\phi(x)}$. By Corollary 3.7 we have that

$$\frac{J_\nu f^n(x)}{J_\nu f^n(y)} = e^{S_n\phi(y) - S_n\phi(x)} \leq e^{Ad(f^n(x), f^n(y))}$$

Then we just need to choose $K = e^{AD}$, where D is the supremum of the inner diameter of the rectangles of \mathcal{R} . \square

3.3 Expanding measures

Recall that H denotes the set of points $x \in M$ with infinitely many hyperbolic times. We call a probability η an *expanding measure* if $\eta(H) = 1$.

Our next goal is to prove that if ν is as in Section 3.1, then it is an expanding measure and the first hyperbolic time is ν -integrable. This will follow from the recurrence property in Proposition 3.3. The first step is

Lemma 3.9. *Denote $a_k = \log \sup_{x \in R_k} \|Df(x)^{-1}\|$ for each $k = 1, \dots, p + q$. Then, for ν -almost every point $x \in M$ there exists an itinerary i_0, \dots, i_j, \dots of x such*

$$\limsup \frac{1}{n} \sum_{j=0}^{n-1} a_{i_j} \leq -4c < 0.$$

Proof. This is a direct consequence of the choice of γ . Indeed, observe that $a_k < \sigma_1^{-1}$ for every $k = q + 1, \dots, p + q$ and $a_i \leq (1 + \delta_0)$ for $i = 1, \dots, q$. Besides, by Lemma 3.3 we have that almost every point x spends at most a proportion γ of the total time in \mathcal{B} . On the other hand, by equation (4) we have that

$$\log(1 + \delta_0)^\gamma \sigma_1^{-(1-\gamma)} \leq -4c$$

Then, using the inequalities above

$$\limsup \frac{1}{n} \sum_{j=0}^{n-1} a_{i_j} \leq \log(1 + \delta_0)^\gamma \sigma_1^{-(1-\gamma)} \leq -4c < 0$$

as claimed. □

We use the following result of Pliss; see for instance [ABV00] for a proof.

Lemma 3.10. *Given $A \geq c_2 > c_1 > 0$, let $d_0 = \frac{c_2 - c_1}{A - c_1}$. If a_1, \dots, b_n are real numbers such that $b_i \leq A$ and*

$$\sum_{i=1}^n b_i \geq c_2 n$$

then there are integer numbers $l > d_0 n$ and $1 < n_1 < \dots < n_l \leq n$ so that, for every $0 \leq k \leq n_i$ and $i = 1, \dots, l$:

$$\sum_{j=k+1}^{n_i} b_j \geq c_1(n_i - k).$$

We denote by H_n the set of points $x \in M$ whose first hyperbolic time $n_1(x)$ is exactly n .

Proposition 3.11. *If ν is a probability satisfying the conclusion of Proposition 3.3 then ν is an expanding measure. Moreover, $\int n_1(x) d\nu(x) < \infty$ and the*

sequence of hyperbolic times has positive density at infinity: there exists $\theta > 0$ such that

$$\frac{1}{n} \#\{0 \leq j \leq n-1; R^j(x) \in \mathcal{R}_h^j\} \geq \theta,$$

for every large enough n and ν -almost every $x \in M$.

Proof. Denote by $A(n)$ the union of all cylinders $R^n = [i_0, \dots, i_{n-1}]$ such that

$$\frac{1}{n} \#\{0 \leq j \leq n-1; i_j \leq q\} > \gamma$$

and let $L(n) = \bigcap_{i=1}^{n-1} (H_i)^c$. By Lemma 3.3 we have that the sequence $\nu(A(n))$ decreases exponentially with n . Take any $x \in A(n)^c$. Then, if (i_0, \dots, i_{n-1}) denotes the itinerary of x ,

$$\frac{1}{n} \sum_{j=0}^{n-1} a_{i_j} \leq \log(1 + \delta_0)^\gamma \sigma_1^{-(1-\gamma)} \leq -4c < 0.$$

Taking $A = \sup_{x \in M} -\log \|Df(x)^{-1}\|$, $c_1 = 2c$, $c_2 = 3c$ and $b_j = -a_{i_j}$ in the previous lemma, we may conclude that there exists $n_1 < \dots < n_l \leq n-1$ such that $R^{n_i}(x) \in \mathcal{R}_h^{n_i}$ and $l > \theta$, for $\theta = c/(A-2c)$. It follows that $L(n) \subset A(n)$ and that $\nu(L(n))$ decreases exponentially with n , since the same occurs with $\nu(A(n))$. This proves the lemma, since $\int n_1 d\nu = \sum_{i=0}^{\infty} \nu(L(n)) < +\infty$. □

It follows from the proposition above that $\{H_n\}_{n \in \mathbb{N}}$ is a partition of M mod 0, and

$$\int n_1 d\nu = \sum_{n=1}^{\infty} n \nu(H_n).$$

3.4 Proof of Theorem A

Recall that we write $P = \log \lambda$.

Lemma 3.12. *There exists $K > 0$ such that for every n and $x \in R^n \in \mathcal{R}_h^n$,*

$$K^{-1} \leq \frac{\nu(R^n)}{\exp(S_n \phi(x) - Pn)} \leq K.$$

Proof. By Lemma 3.1, the Jacobian of f^n is given by $J_\nu f^n = e^{nP - S_n \phi}$. Hence, if $R^n = [i_0, \dots, i_{n-1}]$,

$$\nu(f(R_{i_{n-1}})) = \nu(f^n(R^n)) = \int_{R^n} J_\nu f^n d\nu = \int_{R^n} e^{nP - S_n \phi(x)} d\nu(x).$$

By Corollary 3.8, there exists K_1 not depending on n such for every $x, y \in R^n$

$$K_1^{-1}J_\nu f^n(y) \leq J_\nu f^n(x) \leq K_1 J_\nu f^n(y).$$

It follows that

$$K_1^{-1}\nu(f(R_{i_{n-1}})) \leq \frac{\nu(R^n)}{e^{-Pn+S_n\phi(x)}} \leq K_1\nu(f(R_{i_{n-1}}))$$

for any $x \in R^n$. We now observe that $\nu(f(R_i)) > 0$ for every $i = 1, \dots, p+q$. Indeed, consider any i fixed. By the third condition in hypothesis (H1), there exists $N \in \mathbb{N}$ such that $f^N(R_i) = M$ and, consequently, has total ν -measure. Now, we may decompose $f(R_i)$ into finitely many subsets such that f^{n-1} is injective on each one of them. So, using the fact that ν has a Jacobian, it follows that $\nu(f(R_i)) > 0$ is also positive, as claimed. To finish the proof, just take $K = \max\{K_1 \sup \nu(f(R_i)), K_1 \sup \nu(f(R_i))^{-1}\}$. \square

Now, to prove that ν is a non-lacunary Gibbs measure, we only have to show that the sequence of hyperbolic times is non-lacunary for ν -almost every point. We postpone the proof to Section 4, see Remark 4.8. Apart from that, the proof of Theorem A is completed by

Lemma 3.13. *The support of ν coincides with the closure of H .*

Proof. Observe that by lemma 3.11 we have that $\nu(H) = 1$ and this immediately imply that the support of ν is contained in \bar{H} . Conversely, take any $x \in \bar{H}$ and observe that by the definition of H , there exists a sequence $n_i = n_i(x) \in \mathbb{N}$ such the cylinders R^{n_i} are hyperbolic. By previous lemma, $\nu(R^{n_i}(x)) > \exp(S_n\phi(x) - Pn)K^{-1} > 0$. Since the diameters of $R^{n_i}(x)$ converge to zero, we have that any neighborhood of x has positive ν measure and, thus, \bar{H} is contained in support of ν . \square

4 Proof of Theorem B

The main tool to prove Theorem B is a uniformly expanding map $F : H \rightarrow H$, defined as the first hyperbolic time iterate of f . We prove that F admits an invariant probability μ_F absolutely continuous with respect to ν and with density bounded from zero and from infinity. Using the fact that the first hyperbolic time n_1 is ν -integrable, we produce from μ_F an f -invariant probability μ . Using also the property of ν in Lemma 3.12 we deduce that μ is a non-lacunary Gibbs measure, and an equilibrium state of f for ϕ . Let us detail this outline of the proof.

4.1 An induced map

Recall that H is the set of points $x \in M$ with infinitely many hyperbolic times: there are $n_1(x) < \dots < n_j(x) < \dots$ such that each $n_j(x)$ is a hyperbolic time

for every point in the cylinder $R^{n_j(x)}(x)$. The precise definition of the induced map F is

$$F : H \rightarrow H, \quad F(x) = f^{n_1(x)}(x). \quad (7)$$

Note that F sends each cylinder $R^{n_1(x)}(x)$ injectively onto its image. More generally, for each $k \geq 1$ and $x \in H$ there is a largest cylinder containing x such that F^k is injective on that cylinder. We denote by F_x^{-k} the corresponding inverse branch of F^k .

The next lemma is a reformulation of Lemma 3.5, and shows that F is a uniformly expanding map:

Lemma 4.1. *Given any inverse branch F_x^{-1} of F , we have*

$$d(F_x^{-1}(y), F_x^{-1}(z)) \leq e^{-cn_1(x)} d(y, z)$$

for every y, z in the domain of F_x^{-1} .

Proof. Let us write $G = F_x^{-k}$. Observe that, since $n_1(x)$ is a hyperbolic time for $G(y)$ and $G(z)$, this lemma follows directly from Lemma 3.6. \square

Lemma 4.2. *There exists $K > 0$ depending only on f such for every $k \geq 1$, every inverse branch F_x^{-k} , and every measurable subsets A, B of the domain of F_x^{-k} ,*

$$K^{-1} \frac{\nu(A)}{\nu(B)} \leq \frac{\nu(F_x^{-k}(A))}{\nu(F_x^{-k}(B))} \leq K \frac{\nu(A)}{\nu(B)} \quad (8)$$

Proof. As in the previous lemma, let us write $G = F_x^{-k}$. Observe that

$$\frac{\nu(A)}{\nu(B)} = \frac{\int_{G(A)} J_\nu F^k d\nu}{\int_{G(B)} J_\nu F^k d\nu}$$

By Corollary 3.8, there is $K > 0$ such that

$$K^{-1} \leq \frac{J_\nu F^k(y)}{J_\nu F^k(z)} \leq K$$

for every y, z on the image of $G = F_x^{-k}$. The conclusion of the lemma is an immediate consequence. \square

Corollary 4.3. *Every accumulation point μ_F of the sequence*

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} F_\star^i \nu$$

is an F -invariant probability absolutely continuous with respect to ν , with density $h = d\mu_F/d\nu$ bounded from zero and from infinity.

Proof. It follows from (8) that $C^{-1}\nu(A) \leq F_*^k\nu(A) \leq C\nu(A)$, for some constant C . Indeed, observe that $\nu(F^{-k}(A))$ is the sum of the terms $\nu(G(A))$ over all inverse branches $G = G_k : R_{i_{n_k-1}} \rightarrow [i_0, \dots, i_{n_k-1}]$ of F^k and the same for $B = R_{i_{n_k-1}}$. As in the proof of Lemma 3.12, considering $C_3 = \inf_{i=1, \dots, p+q} \nu(R_i) > 0$, summing over all inverse branches and observing the inequalities 8:

$$K^{-1}\nu(A) \leq \frac{\nu(F^{-k}(A))}{\nu(F^{-k}(B))} \leq KC_3^{-1}\nu(A),$$

which implies that $C^{-1}\nu(A) \leq \nu(F^{-k}(A)) \leq C\nu(A)$, for $C = KC_3^{-1}$. From this fact, follows immediately that $C^{-1}\nu(A) \leq \mu_n(A) \leq C\nu(A)$ and from a well-known fact from measure theory follows that any accumulation point μ_F of μ_n satisfy same inequalities. These implies that μ_F is absolutely continuous with respect to ν and its density $h = \frac{d\mu_F}{d\nu}$ is bounded from below and above in H by uniform constants. \square

Lemma 4.4. *For every probability μ_F as in Corollary 4.3, there exists $K_1 > 0$ such that for every n and every $x \in R^n \in \mathcal{R}_h^n$,*

$$K_1^{-1} \leq \frac{\mu_F(R^n)}{\exp(S_n\phi(x) - Pn)} \leq K_1.$$

Proof. This is a direct consequence of Lemma 3.12 and the fact that the density h is bounded from zero and from infinity. \square

4.2 Invariant non-lacunary Gibbs measures

Recall that H_n is the set of points whose first hyperbolic time is equal to n and $H_0 = M$. We define a measure μ_f in M by

$$\mu_f(A) = \sum_{n=0}^{\infty} \sum_{i>n} \mu_F(f^{-n}(A) \cap H_i), \quad (9)$$

for every measurable set $A \subset M$. To finish the Theorem B we need to prove that

Lemma 4.5. *The measure μ_f is finite, f -invariant, absolutely continuous with respect to ν , with density bounded from zero and infinity. In particular, μ is expanding with integrable first hyperbolic time and there exists $K_2 > 0$ such that for every n and every $x \in R^n \in \mathcal{R}_h^n$,*

$$K_2^{-1} \leq \frac{\mu_f(R^n)}{\exp(S_n\phi(x) - Pn)} \leq K_2.$$

Proof. Invariance: if A is any Borel set, by the definition of μ_f

$$\mu_f(f^{-1}(A)) = \sum_{n=0}^{\infty} \sum_{i>n} \mu_F(f^{-(n+1)}(A) \cap H_i) = \sum_{n=1}^{\infty} \sum_{i \geq n} \mu_F(f^{-n}(A) \cap H_i).$$

We may write the above expression as

$$\mu_f(f^{-1}(A)) = \sum_{n=1}^{\infty} \sum_{i>n} \mu_F(f^{-n}(A) \cap H_i) + \sum_{n=1}^{\infty} \mu_F(f^{-n}(A) \cap H_n).$$

Observe that since $\{H_i\}$ is a partition mod 0 of M :

$$\mu_F(A) = \sum_{i \geq 1} \mu_F(A \cap H_i)$$

and using that μ_F is F -invariant:

$$\mu_F(A) = \mu_F(F^{-1}(A)) = \sum_{n=1}^{\infty} \mu_F(f^{-n}(A) \cap H_n)$$

Then,

$$\mu_f(f^{-1}(A)) = \sum_{n=1}^{\infty} \sum_{i>n} \mu_F(f^{-n}(A) \cap H_i) + \sum_{i=1}^{\infty} \mu_F(A \cap H_i) = \mu_f(A).$$

and this proves that μ_f is f -invariant.

Absolute continuity: Observe that if $\nu(A) = 0$ then $\nu(f^{-k}(A)) = 0$, for every $k \geq 0$. Indeed, if $\nu(f^{-k}(A)) > 0$ we can choose a subset $B \subset f^{-k}(A)$ such f^k restrict to B is injective and $\nu(B) > 0$. Follows from the jacobian formula that $\nu(f^k(B)) = \int_B J_\nu f^k d\nu > 0$, which is a contradiction, since $f^k(B) \subset A$. Hence, if $\nu(A) = 0$ we have that $\mu_F(f^{-k}(A)) = 0$ for every $k \geq 0$ and it imply that $\mu_f(A) = 0$, as we want to show. This proves that μ_f is absolutely continuous with respect to ν .

Bounds for the density: In order to bound the density of μ_f , observe that it follows from the definition of μ_f that

$$\mu_F(A) \leq \mu_f(A) = \sum_{n=0}^{\infty} \mu_F(f^{-n}(A) \cap L_n),$$

where $L_n = \bigcup_{i>n} H_i$. Observe that the number of the cylinders B of length n such $B \subset L_n$ is less then $e^{c_0 n}$, by Lemma 2.7. Besides, given any of these $B \subset L_n$ we have that:

$$\nu(A) \geq \nu(f^n(f^{-n}(A) \cap B)) \geq e^{\kappa n} \nu(f^{-n}(A) \cap B)$$

Summing over all $B \subset L_n$:

$$\nu(f^{-n}(A) \cap L_n) \leq e^{(-\kappa+c_0)n} \nu(A).$$

Observing that $\kappa > c_0$ and that $\mu_F = h\nu$ with h bounded from below and from above, we have for some constant K that

$$K^{-1} \nu(A) \leq \mu_f(A) = \sum_{n=0}^{\infty} \mu_F(f^{-n}(A) \cap L_n) \leq K \nu(A).$$

Finiteness: Observe, by Lemma 3.11, that $\int n_1 d\nu < \infty$. Since the density of μ_f with respect to ν is bounded from above and $\mu_f(M) = \int n_1(x) d\mu_f(x)$, we have that $\mu_f(M) < +\infty$.

Ergodicity: Let \tilde{A} be a f -invariant set with positive μ_f measure. Since H is a full measure set for μ_f , we may consider the invariant set $A = \tilde{A} \cap H$. Since $\mu_f(A) > 0$, we may pick up a point $x \in A$ of density for the measure μ_f . Thus, given $\epsilon > 0$, there exists $r > 0$ such for every open set B containing x with diameter less than r , then

$$\frac{\mu_f(B \cap A)}{\mu_f(B)} > 1 - \epsilon.$$

In particular, since $x \in H$, there exists $n_1, n_2, \dots \rightarrow \infty$ hyperbolic times for x . Observe that by Lemma 3.6 the diameter of the cylinders $\mathcal{R}^{n_i}(x)$ converge to zero. Thus, for n_i bigger enough, the diameter of $\mathcal{R}^{n_i}(x)$ is less than r and

$$\frac{\mu_f(\mathcal{R}^{n_i}(x) \cap A)}{\mu_f(\mathcal{R}^{n_i}(x))} > 1 - \epsilon.$$

If $R_{k_i} = f^{n_i}(\mathcal{R}^{n_i}(x))$, applying f^{n_i} and Corollary 3.8:

$$\frac{\mu_f(R_{k_i} \cap A)}{\mu_f(R_{k_i})} > 1 - \epsilon.$$

As ϵ is arbitrary and $\{R_1, \dots, R_{p+q}\}$ is a finite set, then making ϵ goes to zero, there exists some k such $\mu_f(A \cap R_k) = \mu_f(R_k)$. By hypothesis (H1), $f^N(R_k) = M$ which imply by the invariance of A that

$$\mu_f(A) \geq \mu_f(f^N(A \cap R_k)) = \mu_f(A \cap f^N(R_k)) = \mu_f(M),$$

as we want to prove.

Now let μ be the normalization of μ_f , that is,

$$\mu(A) = \frac{\mu_f(A)}{\mu_f(M)}. \quad (10)$$

We prove now that $\mu(\partial\mathcal{R}) = 0$. In fact, consider the f -invariant set A defined by:

$$A = \bigcup_{i=0}^{+\infty} f^{-i}(\partial\mathcal{R}),$$

and suppose by contradiction that $\mu(A) = 1$. Observe that $f(\partial\mathcal{R}) \subset \partial\mathcal{R}$ and if $x \in A$ then there exists some $n = n(x)$ such that $f^n(x) \in \partial\mathcal{R}$. Thus, if x is a recurrent point for f then x must belongs to $\partial\mathcal{R}$.

Using the recurrence theorem, follows that $\mu(\partial\mathcal{R}) = 1$, since μ is f -invariant and almost every point with respect to μ is recurrent. Now, observe that for every $i = 1, \dots, p+q$ we have that $\mu(R_i) > 0$, since by hypothesis (H1) we have $f^N(R_i) = M$ for some N . Thus, it is a contradiction with $\mu(\partial\mathcal{R}) = 1$, since $\partial\mathcal{R} \cap R_i = \emptyset$. This prove the lemma. \square

To complete the proof of Theorem B, we only need to show that μ is a non-lacunary Gibbs measure for f . Now, the statements in Lemma 4.5 remain true for μ , up to changing the constant K_2 . Moreover, in view of Lemma 4.5, it suffices to check that the sequences of hyperbolic times are non-lacunary at μ -almost every point. We use the following general fact:

Lemma 4.6. *If η is an invariant expanding measure with integrable first hyperbolic time then the sequence of hyperbolic points is non-lacunary at η -almost every point.*

Proof. Let D be the set of points for which the sequence of hyperbolic times is non-lacunary. For each $\theta > 0$, define $L_\theta(n) = \{x \in M; n_1(x) \geq \theta n\}$. On the one hand,

$$\theta \sum_{n=1}^{\infty} \eta(L_\theta(n)) = \theta \sum_{n_1=1}^{\infty} \sum_{n \leq n_1/\theta} \eta(H_{n_1}) \leq \sum_{n_1=1}^{\infty} n_1 \eta(H_{n_1}) = \int n_1 d\eta < +\infty.$$

On the other hand, if $x \notin D$ then there exists a rational number $\theta > 0$, and there are infinitely values of i such that $n_{i+1} \geq (1 + \theta)n_i$. By Remark 3.4, the latter implies that $n_1(f^{n_i}(x)) \geq n_{i+1} - n_i \geq \theta n_i$. Hence, x belongs to the set

$$L = \bigcup_{\theta \in \mathbb{Q}} \bigcap_{k=0}^{\infty} \bigcup_{n \geq k} f^{-n}(L_\theta(n)).$$

η is invariant, $\eta(f^{-n}(L_\theta(n))) = \eta(L_\theta(n))$ for all n . We have already seen that $\sum_{n=1}^{\infty} \eta(L_\theta(n))$ is finite. Then, by the Borel-Cantelli lemma, $\eta(L) = 0$. Since we have shown that $D^c \subset L$, this completes the proof. \square

Corollary 4.7. *Any probability μ as in (9)-(10) is an f -invariant non-lacunary Gibbs measure.*

Proof. In Lemma 4.5 we proved the defining inequality at hyperbolic times and in Corollary 4.6 we checked that these constitute non-lacunary sequences μ -almost everywhere. \square

Remark 4.8. Since μ and ν are equivalent measures, we also get that hyperbolic times are non-lacunary ν -almost everywhere. In view of Lemma 3.12, it follows that ν is a non-lacunary Gibbs measure.

We have completed the proofs of Theorems A and B.

5 Proof of Theorem C

The strategy to prove Theorem C is to consider a sequence of linear operators $\mathcal{T}_{\phi,n}$ which take into account only “good” pre-images, namely, those for which n is a hyperbolic time. This ensures equicontinuity of the sequence $\mathcal{T}_{\phi,n}1$, and that allows us to find the function h as a Cesaro accumulation point of that sequence.

5.1 Constructing an eigenfunction

Note that n th iterate of the transfer operator \mathcal{L}_ϕ is given by

$$\mathcal{L}_\phi^n g(x) = \sum_{f^n(y)=x} e^{S_n\phi(y)} g(y).$$

We define $\mathcal{T}_{\phi,n} : C(M) \rightarrow C(M)$ by

$$\mathcal{T}_{\phi,n} g(x) = \sum_{f^n(y)=x, R^n(y) \in \mathcal{R}_h^n} e^{S_n\phi(y)} g(y).$$

That is, the sum is restricted to those $y \in f^{-n}(x)$ such that n is a hyperbolic time for the atom $R^n(y)$. In what follows we denote

$$T_n(x) = \mathcal{T}_{\phi,n} 1(x) = \sum_{f^n(y)=x, R^n(y) \in \mathcal{R}_h^n} e^{S_n\phi(y)} \quad \text{and} \quad Z_n = \sum_{R^n \in \mathcal{R}_h^n} e^{S_n\phi(R^n)},$$

where $S_n\phi(R^n) = \max\{S_n\phi(y) : y \in R^n\}$. It is clear that $T_n(x) \leq Z_n$. There are two reasons why the inequality may be strict. One is that the definition of Z_n is in terms of the maximum of ϕ . The other is that x needs not have pre-images in all atoms $R^n \in \mathcal{R}_h^n$.

Lemma 5.1. *There exists a constant $K_3 > 0$ such that for every $n \geq 1$*

$$\lambda^{-n} Z_n \leq K_3 \quad \text{and} \quad K_3^{-1} \leq \frac{1}{n} \sum_{j=0}^{n-1} \lambda^{-j} Z_j \leq K_3.$$

Proof. Let μ be the non-lacunary Gibbs state constructed in Theorem B. We have

$$K^{-1} \leq \frac{\mu(R^n)}{\exp(S_n\phi(y) - nP)} \leq K$$

or, equivalently,

$$K^{-1} \mu(R^n) \leq \lambda^{-n} e^{S_n\phi(y)} \leq K \mu(R^n) \tag{11}$$

for every $y \in R^n \in \mathcal{R}_h^n$. Taking the maximum and then summing over all $R^n \in \mathcal{R}_h^n$ we get

$$K^{-1} \mu(B_n) \leq \lambda^{-n} Z_n \leq K \mu(B_n) \leq K,$$

where B_n is the union of all $R^n \in \mathcal{R}_h^n$. The upper inequalities in the statement are immediate consequences, taking $K_3 \geq K$. We also get that

$$K^{-1} \frac{1}{n} \sum_{j=0}^{n-1} \mu(B_j) \leq \frac{1}{n} \sum_{j=0}^{n-1} \lambda^{-j} Z_j. \tag{12}$$

So, to prove the lower inequality and finish the proof of the lemma, we should verify that $n^{-1} \sum_{j=0}^{n-1} \mu(B_j)$ is uniformly bounded away from zero.

For that purpose, write

$$\frac{1}{n} \sum_{j=0}^{n-1} \mu(B_j) = \int \int \mathcal{X}_{B_j}(x) d\mu(x) dm_n(j).$$

where m_n is the normalized counting measure on the set $\{1, \dots, n\}$. Since the hyperbolic times have density $\geq \theta > 0$ at infinity, we have

$$\sum_{j=0}^{n-1} \mathcal{X}_{B_j}(x) > \theta n$$

for μ -almost every x and every large $n = n(x)$. Then, if n is large, the set X of points which satisfy the above inequality has measure μ bigger than $\frac{1}{2}$. By Fubini's theorem, it follows that

$$\frac{1}{n} \sum_{j=0}^{n-1} \mu(B_j) = \int \int \mathcal{X}_{B_j}(x) dm_n(j) d\mu(x) \geq \int_X \theta d\mu(x) \geq \frac{\theta}{2} > 0.$$

The lower inequality follows from this inequality and (12), taking $K_3 \geq K/2\theta$ in the statement. \square

Corollary 5.2. *The sequence $\lambda^{-n}T_n$ is uniformly bounded.*

Proof. Just note that $\lambda^{-n}T_n(x) \leq \lambda^{-n}Z_n < K_3$. \square

Lemma 5.3. *There exists a constant $K_4 > 0$ such*

$$\left| \frac{T_n(x_1)}{T_n(x_2)} - 1 \right| \leq K_4 d(x_1, x_2)^\alpha$$

for every x_1 and x_2 in the image $f(R_i)$ of the same atom of \mathcal{R} .

Proof. Let R^n be any atom in \mathcal{R}_h^n such that $f^{n-1}(R^n) = R_i$. Then both x_1 and x_2 have (unique) pre-images under f^n inside R^n , which we denote y_1 and y_2 . By Corollary 3.7, there is a uniform constant A such that

$$|S_n \phi(y_1) - S_n \phi(y_2)| \leq Ad(x_1, x_2)^\alpha.$$

Then,

$$e^{-Ad(x_1, x_2)^\alpha} \leq \frac{T_n(x_1)}{T_n(x_2)} = \frac{\sum_{R^n} e^{S_n \phi(y_1)}}{\sum_{R^n} e^{S_n \phi(y_2)}} \leq e^{Ad(x_1, x_2)^\alpha}$$

Now it suffices to observe that, for all x_1 and x_2 ,

$$|e^{\pm Ad(x_1, x_2)^\alpha} - 1| \leq K_4 d(x_1, x_2)^\alpha$$

if K_4 is chosen sufficiently large with respect to A and the diameter of M . \square

Corollary 5.4. *The sequence $\lambda^{-n}T_n$ is equicontinuous on M .*

Proof. By lemmas 5.1 and 5.3, given x_1 and x_2 in the same $f(R_i)$,

$$|\lambda^{-n}T_n(x_1) - \lambda^{-n}T_n(x_2)| \leq \lambda^{-n}|T_n(x_2)|K_4d(x_1, x_2)^\alpha \leq K_3K_4d(x_1, x_2)^\alpha.$$

To conclude the equicontinuity, it suffices to take δ as the Lebesgue number of the cover $\{f(R_i)\}$ and observe that given $x, y \in M$ such $d(x, y) \leq \delta$ then x, y lies in some $f(R_i)$, $R_i \in \mathcal{R}$. \square

Observe that, by Lemma 5.3 and Corollary 5.2 we have that $|\lambda^{-n}T_n(x) - \lambda^{-n}T_n(y)| \leq K$ for every $x, y \in R_i$. Thus, the value of $\lambda^{-n}T_n(x)$ is comparable with $\lambda^{-n}Z_n$. More precisely:

Remark 5.5. There exists $K_5 > 0$ such that $K_5\lambda^{-n}Z_n(\phi) \leq \lambda^{-n}T_n(x)$ for every $n \geq 1$.

By Corollaries 5.2 and 5.4, we may apply the theorem of Ascoli-Arzelà to conclude that

$$\frac{1}{n} \sum_{i=0}^{n-1} \lambda^{-i}T_i$$

has some subsequence converging uniformly on M to a Hölder continuous function h . Moreover

Lemma 5.6. *Any accumulation function h of the sequence $n^{-1} \sum_{i=0}^{n-1} \lambda^{-i}T_i$ is bounded from zero and infinity, and an eigenvector of the transfer operator: $\mathcal{L}_\phi h = \lambda h$.*

Proof. By Corollary 5.2, h is bounded from above. For a bound from below, observe that by Lemma 5.1 and Remark 5.5, we get that $h \geq K_5K_3^{-1} > 0$, since $\frac{1}{n} \sum_{i=0}^{n-1} \lambda^{-i}T_i \geq K_5K_3^{-1}$. In order to prove that $\mathcal{L}_\phi h = \lambda h$, first we prove that $\frac{1}{n} \sum_{i=0}^{n-1} \lambda^{-i} \mathcal{L}_\phi T_i - \lambda^{-i}T_{i+1}$ converges uniformly to zero. Denote by E_n the collection of cylinders

$$E_{n+1} = \{[i_0, \dots, i_n] \in \mathcal{R}^{n+1}; [i_0, \dots, i_n] \notin \mathcal{R}_h^{n+1} \text{ and } [i_1, \dots, i_n] \in \mathcal{R}_h^n\}.$$

Observe that $\#E_{n+1} \leq \#\{[i_0, \dots, i_n] \in \mathcal{R}^{n+1}; \frac{\#\{i_j \leq q\}}{n+1} \geq \gamma\} \leq e^{c_0(n+1)}$. Hence,

$$\|\lambda^{-i} \mathcal{L}_\phi T_i - \lambda^{-i}T_{i+1}\| \leq \lambda^{-i} \sum_{R \in E_{i+1}} e^{S_i \phi(R)} \leq e^{c_0(i+1)} \lambda^{-i} e^{i \max \phi}.$$

Recalling that $\lambda > e^{\max \phi + c_0}$,

$$\|\lambda^{-i} \mathcal{L}_\phi T_i - \lambda^{-i}T_{i+1}\| \leq e^{c_0} e^{(-P(\phi) + \max(\phi) + c_0)i} < e^{c_0} \eta^i,$$

for some constant $e^{(-P(\phi) + \max(\phi) + \log c_0)} < \eta < 1$. To finish the proof observe that

$$\mathcal{L}_\phi h = \lim_k \frac{1}{n_k} \sum_{i=0}^{n_k-1} \lambda^{-i} \mathcal{L}_\phi T_i - \lambda^{-i}T_{i+1} + \frac{\lambda}{n_k} \sum_{i=0}^{n_k-1} \lambda^{-i}T_i - \frac{\lambda}{n_k} + \frac{\lambda^{-n_k}T_{n_k}}{n_k}.$$

Since $\lambda^{-n_k}T_{n_k}/n_k$ and $(1/n_k) \sum_{i=0}^{n_k-1} \lambda^{-i} \mathcal{L}_\phi T_i - \lambda^{-i}T_{i+1}$ converge to zero, the argument is complete. \square

Consider an invariant set L and let us define an auxiliary quantity $P_L(\phi)$ called *pressure of ϕ with respect to L* . For $L = H$, this quantity is related with the spectral radius of \mathcal{L}_ϕ , as we will see in Lemma 5.9. We also prove in Lemma 5.10 that under the assumption (H2), we have that $P_H(\phi) = P(\phi)$.

Fixed a cover $\mathcal{U} = U$ of H we define the number

$$M(\alpha, N, \mathcal{U}) = \inf_{\mathcal{G}} \sum_{U \in \mathcal{G}} \exp(-\alpha n(U) + S_{n(U)}(U))$$

where the infimum is taken over all finite or countable collections of cylinders \mathcal{G} of \mathcal{U} such that $n(U) = \text{length of } U \geq N$ for all $U \in \mathcal{G}$ and \mathcal{G} covers L . We also define

$$M_C(\alpha, \mathcal{U}) = \lim_{N \rightarrow +\infty} M(\alpha, N, \mathcal{U}),$$

and

$$P_L(\phi, \mathcal{U}) = \inf\{\alpha; M_C(\alpha) = 0\}.$$

With these definitions, we are able to give the *dimensional* definition of the pressure of H :

Definition 5.7. The *pressure of L* is $P_L(\phi) = \limsup_{\delta \rightarrow 0} P_H(\phi, \mathcal{U})$ where the sup above is taken over the set of covers \mathcal{U} of L with diameter less than δ .

One can see that for $L = H$ we have that $P_H(\phi) = \lim_{n \rightarrow +\infty} P_H(\phi, U_n)$ where U_n is the cover of H by cylinders $U \in \mathcal{R}_h^i$ for $i \geq n$. This is a consequence of Theorem 11.1 in [Pe97] and we refer there for further information.

Remark 5.8. Some properties of the pressure still are verified for P_H . However, the variational principle it is not true: we only can prove an *inequality* in general. If \mathcal{I}_H denote the set of the invariant measures μ such $\mu(H) = 1$, then

$$P_H(\phi) \geq \sup_{\mu \in \mathcal{I}_H} \{h_\mu(f) + \int \phi d\mu\}.$$

In particular, if some equilibrium measure belongs to the set \mathcal{I}_H , then $P_H(\phi) = P(\phi)$ and the equality above is attained. For additional comments and properties we suggest [Pe97] and [Wa82].

Lemma 5.9. *The spectral radius of \mathcal{L}_ϕ^* is equal to $e^{P_H(f, \phi)}$ and is the unique real eigenvalue of \mathcal{L}_ϕ^* bigger than $e^{c_0(f) + \max \phi}$.*

Proof. Let λ be any eigenvalue of \mathcal{L}_ϕ^* bigger than $e^{c_0(f) + \max \phi}$. Consider ν an eigenmeasure associated to λ and μ constructed as in Lemma 4.5.

By Lemma 4.5, we have that for each $U \in \mathcal{R}_h^n$:

$$K_2^{-1} \mu_f(U) \leq \exp(S_n \phi(U) - Pn) \leq \mu_f(U) K_2.$$

By Remark 3.4, if $U \in \mathcal{R}_h^i$ and $V \in \mathcal{R}_h^j$ the concatenation $W = U \cap f^{-i}(V)$ of U and V belong to \mathcal{R}_h^{i+j} . Using that \mathcal{G} covers H , it is not difficult conclude that

if $P = \log \lambda$ then $M_C(\alpha, \mathcal{U}) = +\infty$ for $\alpha < P$ and $M_C(\alpha, \mathcal{U}) = 0$ for $\alpha > P$. It is proved that $P_H(\phi, U_n) = \log \lambda$ and finish the proof of Lemma 5.9, since $P(\phi) = P_H(\phi) = \lim_{n \rightarrow +\infty} P_H(\phi, U_n) = \log \lambda$. In particular, since the spectral radius of \mathcal{L}_ϕ is an eigenvalue of \mathcal{L}^* bigger than $e^{\max \phi + c_0(f)}$, we finish the prove of the lemma. \square

Lemma 5.10. *If ϕ satisfy (H2), then $P_H(\phi) = P(\phi)$.*

Proof. By the Theorem 11.2 of [Pe97], $P(\phi) = \sup\{P_H(\phi), P_{M \setminus H}(\phi)\}$. Thus, it is sufficient to show that $P_{M \setminus H}(\phi) \leq P_H(\phi)$. In fact, observe that by the definition of $M \setminus H$, it is possible to take a cover \mathcal{U}^k of $M \setminus H$ with diameter less than or equal to $1/k$ such that any element of $U \in \mathcal{U}^k$ is a subset of an cylinder of $R(U) = [\alpha_0, \dots, \alpha_{n-1}]$ for some $n \geq k$ and $(\alpha_0, \dots, \alpha_{n-1})$ satisfying condition (3).

Thus, if we denote by \mathcal{C}_m the set of m -cylinders satisfying condition (3)

$$M_\alpha(M \setminus H, l, \mathcal{U}^k) \leq \sum_{m \geq l} \sum_{R \in \mathcal{C}_m} \exp(-\alpha n(R) + S_n(R)),$$

Observe that for every $R \in \mathcal{C}_m$ we have that for $\alpha > P_H(\phi)$ close enough to $P_H(\phi)$ we have by condition (H2) that $\exp(-\alpha m + S_m(R)) < e^{-\beta m}$, for some $c_0(f) < \beta < \kappa$. On the other hand, $\#\mathcal{C}_m \leq e^{c_0 m}$ for m bigger enough. Considering all together, for l big enough:

$$M_\alpha(M \setminus H, l, \mathcal{U}^k) \leq \sum_{m \geq l} e^{(-\beta + c_0)m} < \infty.$$

Taking $l \rightarrow \infty$ and $k \rightarrow 0$, this proves that $P_{M \setminus H}(\phi) < P_H(\phi)$. \square

6 Proof of Theorem D

In this section, we apply Theorems B and C to prove the existence and uniqueness of equilibrium states for any Hölder potential satisfying (H2). The equilibrium state is absolutely continuous with respect to the eigenmeasure of \mathcal{L}_ϕ^* associated to the eigenvalue $P = P(\phi, f)$. Its density is the function h obtained in Theorem C.

We start with the following

Lemma 6.1. *Choose h as in Lemma 5.6 with $\int h d\nu = 1$. Then $\mu = h\nu$ is an invariant non-lacunary Gibbs probability measure.*

Proof. The non-lacunary Gibbs property is a direct consequence of this property for ν and the fact that $h > 0$. The invariance is quite standard and may be found in [Bo75]. \square

Lemma 6.2. *Given any expanding measure η , any measurable set $E \subset M$, and any $\varepsilon > 0$, there exist hyperbolic cylinders C_1, \dots, C_k such that*

$$\eta(E \Delta \bigcup_{i=1}^k C_i) \leq \varepsilon.$$

Proof. Let $K_1 \subset E$ and $K_2 \subset E^c$ be compact sets such $\eta(E \Delta K_1) \leq \frac{\varepsilon}{3}$ and $\eta(E^c \Delta K_2) \leq \frac{\varepsilon}{3}$ and consider $r = \text{dist}(K_1, K_2)$. Observe that if n is a hyperbolic time for C then $\text{diam}(C) \leq Ke^{-cn}$. Then, if $C \in \mathcal{R}_h^n$ and $n > n_0$ is big enough we have that $\text{diam}(C) \leq Ke^{-cn} \leq r$. Since $\eta(H) = 1$, we can choose $C_i \in \mathcal{R}_h^{n_i}$, $i = 1, \dots, k$ such every $n_i > n_0$ and

$$\eta(K_1 - \bigcup_{i=1}^k C_i) \leq \frac{\varepsilon}{3}.$$

Observing that $C_i \cap K_2 = \emptyset$ for every $i = 1, \dots, k$, we have

$$\eta(E \Delta \bigcup_{i=1}^k C_i) \leq \eta(E - K_1) + \frac{\varepsilon}{3} + \eta(E^c - K_2) \leq \varepsilon.$$

This completes the proof. \square

Corollary 6.3. *Any two non-lacunary Gibbs measures associated to a constant P which the non-lacunar sequence is the sequence of hyperbolic times are equivalent.*

Proof. Consider two non-lacunary Gibbs measures ν_1 and ν_2 . Observe that for each $R \in \mathcal{R}_h^n$ we have that

$$K^{-1}e^{S_n\phi(x)-nP} \leq \nu_i(R) \leq Ke^{S_n\phi(x)-nP},$$

for $i = 1, 2$. This imply that $K^{-2}\nu_2(R) \leq \nu_1(R) \leq K^2\nu_2(R)$. By Lemma 6.2, we have the above inequality for every Borel set. \square

The proof of Theorem D has two main steps: first we prove that any equilibrium state is a non-lacunary Gibbs measure; then we show that there exists a unique invariant non-lacunary Gibbs measure.

Let ν be the eigenmeasure of \mathcal{L}_ϕ^* and h be the eigenfunction of \mathcal{L}_ϕ associated to eigenvalue $\lambda > \max \phi + c_0(f)$ that we constructed in the previous sections. Define

$$g : M \rightarrow (0, \infty), \quad g(x) = \lambda^{-1}e^{\phi(x)} \frac{h(x)}{h(f(x))}.$$

Recall that $h > 0$. In what follows we write $h^{-1} = 1/h$. Observe also that, for every $x \in M$,

$$\sum_{f(y)=x} g(y) = \frac{\sum_{f(y)=x} e^{\phi(y)} h(y)}{\lambda h(x)} = \frac{\mathcal{L}_\phi h(x)}{\lambda h(x)} = 1 \quad (13)$$

The next proposition is a variation of a result in [BS02]. In the proof we use the following elementary fact from Calculus:

Lemma 6.4. *Let $p_i, x_i (i = 1, 2, \dots, n)$ be real numbers such $p_i > 0, x_i > 0$, and $\sum_{i=1}^n p_i = 1$. Then $\sum_{i=1}^n p_i \log x_i \leq \log(\sum_{i=1}^n p_i x_i)$ and the equality holds if and only if the x_i are all equal.*

Proposition 6.5. *Any probability μ as in (9)-(10) is μ is an equilibrium state for ϕ , and $P = \log \lambda$ is the pressure $P(f, \phi)$. Also, let η be any ergodic equilibrium state for ϕ . Then*

1. $h_\eta(f) + \int \log g d\eta = 0$;
2. $J_\eta f(y) = 1/g(y)$ for η -almost every $y \in M$.

Proof. Since μ is expanding, we have that \mathcal{R} is a generating partition for μ , by Lemma 6.2. From Kolmogorov-Sinai's theorem, we have that $h_\mu(f) = h_\mu(f, \mathcal{R})$. The conclusion that $h_\mu(f) + \int \phi d\mu = \log \lambda = P$ follows, straightforwardly, from Lemma 3.12.

Now, let's prove the second part and that $P(f, \phi) = \log \lambda$. Consider any probability η such $h_\eta(f) + \int \phi d\eta \geq \log \lambda$. Then,

$$h_\eta(f) + \int \log g d\eta = h_\eta(f) - \log \lambda + \int (\phi + \log h - \log h \circ f) d\eta \geq 0.$$

Using Rokhlin's formula

$$h_\eta(f) = \int \log J_\eta f d\eta.$$

For simplicity, let us write $g_\eta = 1/(J_\eta f)$. Combining the formula with the inequality above, we find

$$0 \leq \int \log \frac{g}{g_\eta} d\eta = \int \sum_{f(y)=x} g_\eta(y) \log \frac{g(y)}{g_\eta(y)} d\eta,$$

where the second equality follows from $g_\eta = 1/J_\eta f$. In view of (13), the first statement in Lemma 6.4 gives

$$0 \leq \sum_{f(y)=x} g_\eta(y) \log \frac{g(y)}{g_\eta(y)} \leq \log \left(\sum_{f(y)=x} g_\eta(y) \frac{g(y)}{g_\eta(y)} \right) = \log \left(\sum_{f(y)=x} g(y) \right) = 0$$

at η -almost every point. Since the expression on the left is non-negative, the equality must hold. Thus, we have that $h_\eta(f) + \int \phi d\eta - \log \lambda = 0$, which implies that $\log \lambda = P(f, \phi)$. Hence, using the second claim in Lemma 6.4, the values of $\log g(y)/g_\eta(y)$ must be the same for all $y \in f^{-1}(x)$. In other words, for η -almost every $x \in M$ there exists a number $c(x)$ such that

$$\frac{g(y)}{g_\eta(y)} = c(x) \quad \text{for every } y \in f^{-1}(x).$$

The assumption that η is invariant means that

$$\sum_{f(y)=x} g_\eta(x) = 1$$

for η -almost every $x \in M$. Combining this (13), we conclude that

$$c(x) = \frac{\sum_{f(y)=x} g(y)}{\sum_{f(y)=x} g_\eta(y)} = 1$$

at η -almost every point. This shows that $g(y) = g_\eta(y)$ for every y on the pre-image of a full η -measure set. By invariance, this also has full η -measure, so the proof is complete. \square

Lemma 6.6. *Let η be any equilibrium measure for ϕ . Then $\mathcal{L}_\phi(h^{-1}\eta) = \lambda(h^{-1}\eta)$.*

Proof. Given any continuous function ξ

$$\int \xi d(\mathcal{L}_\phi^* h^{-1}\eta) = \int (\mathcal{L}_\phi \xi)(x) h(x)^{-1} d\eta(x) = \int \sum_{f(y)=x} e^{\phi(y)} h(f(y))^{-1} \xi(y) d\eta(x).$$

Using the definition of g and part 2 of Lemma 6.5, we

$$e^{\phi(y)} h(f(y))^{-1} = \lambda g(y) h(y)^{-1} = \lambda g_\eta(y) h(y)^{-1}.$$

Replacing this in the previous formula,

$$\int \xi d(\mathcal{L}_\phi^*(h^{-1}\eta)) = \lambda \int \left(\sum_{f(y)=x} g_\eta(y) \xi(y) h(y)^{-1} \right) d\eta(x) = \lambda \int \xi h^{-1} d\eta.$$

Since ξ is arbitrary, this implies $\mathcal{L}_\phi^*(h^{-1}\eta) = \lambda(h^{-1}\eta)$, as claimed \square

Corollary 6.7. *Any equilibrium state η of ϕ is a non-lacunary Gibbs measure.*

Proof. Let $\nu_\eta = h^{-1}\eta$. We have just seen that ν_η is an eigenmeasure of \mathcal{L}_ϕ^* , with eigenvalue λ . So, by the arguments in Section 3, ν_η is a non-lacunary Gibbs measure. Since $\eta = h\nu_\eta$ and h is bounded from below and above, η is a non-lacunary Gibbs measure too. \square

Proof of Theorem D. Suppose that η is an ergodic equilibrium state for ϕ and let μ be a invariant non-lacunary Gibbs measure as constructed in section 4. By Corollary 6.7 we have that η is non-lacunary Gibbs. By Corollary 6.3 we also have that μ and η are equivalent: $\eta = \xi\mu$, for some μ -integrable function ξ . Since η and μ are invariant measures,

$$\eta = f^*\eta = (\xi \circ f) f^*\mu = (\xi \circ f)\mu.$$

Since the Radon-Nikodym derivative is essentially unique, we get that $\xi = \xi \circ f$ at η -almost every point. Since η is ergodic, it follows that ξ is constant. Using that $1 = \eta(M) = \int d\eta = c \int d\mu = c$, we get that $\eta = \mu$. Observe that this finishes the proof, since any ergodic component of an equilibrium measure is also an equilibrium measure. \square

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Krerley Oliveira (krerley@mat.ufal.br)
Departamento de Matemática - UFAL, Campus A.C. Simões, s/n 57072-090
Maceió, Alagoas - Brazil

Marcelo Viana (viana@impa.br)
IMPA, Est. D. Castorina 110
22460-320 Rio de Janeiro, RJ, Brazil