SHADOWING BY NON UNIFORMLY HYPERBOLIC PERIODIC POINTS AND UNIFORM HYPERBOLICITY

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Abstract. We prove that, under a mild condition on the hyperbolicity of its periodic points, a map g which is topologically conjugated to a hyperbolic map(respectively, an expanding map) is also a hyperbolic map (respectively, an expanding map).

1. Introduction

Since Smale proposed the notion of *uniformly hyperbolic dynamical system*, the theory and results obtained by dynamicists around the world have described many of its features, from the structural and measure-theoretical points of view.

Nevertheless, the study of conditions for a non uniformly expanding map over subset to be expanding restrict to this set is not well understood, regarding the few results concerning the subject. One of these results, is the remarkable theorem of Mañé [7], valid for invariant sets without critical points for interval maps. Outside this setting, not much is known and it is by itself a interesting point of research. In particular, the study of non uniform expanding rates and conditions over a given set of points and its relations with uniform expanding behavior appears in several recent papers ([5], [9] and [10]). Lets briefly describe some of this results:

We say that a *local diffeomorphism* f is non uniformly expanding (*NUE*) on a set X , if there exists $\eta < 0$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \log \sum_{j=0}^{n-1} \|Df(f^j(x))^{-1}\| \leq \eta < 0 \text{ for all } x \in X.$$

In ([5]), the authors proved that any local diffeomorphism in a compact manifold admitting non uniform expansion at a set of total probability, i.e., with full measure for any invariant measure, is in fact an expanding map. Similar results holds for diffeomorphisms.

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By Oseledets([**?**]), one knows that if μ is an invariant measure for a C^1 map f, then the number

$$\lambda(x, v) = \lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\|,$$

is defined in a set of total probability and it is called *Lyapunov exponent* at x in the direction v. In [9], the author prove that if f is a local diffeomorphism such all Lyapunov exponents are positive then it is, in fact, a expanding map and also obtained the results for diffeomorphisms admitting continuous splitting (see Theorems 7 and 13).

Here, we present an even weaker condition of hyperbolicity, requiring non uniform expansion only in the set of periodic points. As a main result, we prove that if a dynamical system is non uniformly expanding over the set of periodic points and is topologically conjugated to an expanding map then, it is itself an expanding map. We also obtain a similar result for dynamics with non uniformly hyperbolic (NUH) periodic points conjugated to an uniformly hyperbolic map.

Theorem A. *Let $g : M \to M$ be a C^2-class local diffeomorphism on a compact manifold M. Suppose that g is topologically conjugated to an expanding C^1 map f. If g is non uniformly expanding on the set $Per(g)$ of periodic points, then g is an expanding map.*

Remark 1. We observe that the condition NUE on the periodic points is not enough to assure that the map g is expanding, even if we assume that g is topologically conjugate to an expanding map. It is a standard matter that the map $z \to z^2$, defined on the circle is topologically conjugated to a map with criticalities satisfying condition NUE over the periodic points. See the figure **??**.

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In the Theorem A, due to the fact that we are dealing with maps that are local diffeomorphisms we avoid examples as in Remark 1.

For diffeomorphisms, the existence of a continuous splitting of M play a similar role. In ([10]), the authors exhibit a example of a non-hyperbolic horseshoe such that the splitting is continuous over the periodic points and all Lyapunov exponents are positive and bounded from zero. In particular, some condition of continuity of the splitting in the closure of the periodic points is necessary. In order to state our results in the invertible case, we need the following definition:

Definition 2. (Non uniformly hyperbolic set). Let $g : M \to M$ be a diffeomorphism on a compact manifold M. We say that an invariant set $S \subset M$ is a *non uniformly hyperbolic set* or, simply, *NUH*, iff:

(1) There is an $Dg-$invariant splitting $T_S M = E^{cs} \oplus E^{cu}$ The angle between $E^{cs}(p)$ and $E^{cu}(p)$ is uniformly bounded from above and below, uniformly for any $p \in S$.
(2) There exists $\eta < 0$ and an adapted Riemannian metric for which any point $p \in S$ satisfies

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Dg(g^j(p))|_{E^s(g^j(p))}\| \leq \eta$$

and

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Dg(g^{-j}(p))|_{E^u(g^{-j}(p))}\| \leq \eta$$

For the diffeomorphism case, we give two slightly different versions of the same result:

Theorem B. *Let $g : M \to M$ be a C^2 diffeomorphism on a compact manifold M. Suppose that g is topologically conjugated to a hyperbolic C^1 map f. If the set $Per(g)$ of periodic points of g is non uniformly hyperbolic (NUH), and $T_{Per(g)} = E^{cs} \oplus E^{cu}$ is a dominated splitting, then g is hyperbolic.*

Theorem C. *Let $g : M \to M$ be a C^2 diffeomorphism on a compact manifold M. Suppose that g is topologically conjugated to a hyperbolic C^1 map f. If the set $Per(g)$ of periodic points of g is non uniformly hyperbolic (NUH), and $T_{Per(g)}M = E^{cs} \oplus E^{cu}$ has a continuous extension to a splitting on $T_{\overline{Per(g)}}M$, then g is hyperbolic.*

In fact, theorem B is a consequence of theorem C.

2. The endomorphism case: non uniformly expanding periodic set

During this section, $g : M \to M$ will always be a C^2-local diffeomorphism which is topologically conjugated to a C^1 expanding endomorphism.

We recall the definition of NUE:

Definition 3. (Non uniformly expanding set). Let $g : M \to M$ be a map on a compact manifold M. We say that an invariant set $S \subset M$ is a *non uniformly expanding set* or, simply, *NUE*, iff:

There exists $\eta < 0$ and an adapted Riemannian metric for which any point $p \in S$ satisfies

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|[Dg(g^j(p))]^{-1}\| \leq \eta$$

Remark 4. In this paper, we will always focus our attention on the set of periodic points $Per(g)$ of g. Given a point $p \in Per(g)$, let us set $t = t(p) := period(p)$. In such case, the following equivalence is immediate: $S := Per(g)$ is NUE iff there exists $\varsigma < 1$ such that for each periodic point p, $\prod_{j=0}^{t(p)-1} \|[Dg(g^j(p))]^{-1}\| < \varsigma^{t(p)}$.

For the sequence, we give a simplified definition for the case of local diffeomorphisms of the notion introduced by [6]:

Definition 5. (Hyperbolic time for local diffeomorphisms) Let $z \in M$ be a regular point. We say that $k \in \mathbb{N}$ is a ς-*hyperbolic time* for z if for $i = 1, \dots, k$, holds

$$\prod_{j=1}^{i} \|[Dg|(g^{p-j}(z))]^{-1}\| \leq \varsigma^i. \tag{1}$$

Lemma 6. *Suppose that g is topologically conjugated to an expanding map f. Let x be a recurrent, regular point of g. If $Per(g)$ is NUE, then all Lyapunov exponents of x are positive.*

Proof: Let $\delta > 0$ such that, given any ball $B(z, \delta)$ the corresponding inverse branches of g are well defined diffeomorphisms. Let $\varsigma = e^\eta$, η as in definition 3, $\varsigma < \varsigma' < 1$ fixed, and let $\epsilon > 0$ such that $(\sqrt{\varsigma'})^{-1} - \epsilon > 1$. Since x is a regular point, there is $n_0 \in \mathbb{N}$ such that

$$(\varsigma_j - \epsilon)^n \cdot \|v_j\| < \|Dg^n(x) \cdot v_j\| < (\varsigma_j + \epsilon)^n \cdot \|v_j\| \forall v_j \in E_j, \forall n \geq n_0.$$

where E_j are the Lyapunov eigenspaces and $\log(\varsigma_j)$ are their respective Lyapunov exponents.

Now, by Pliss Lemma [8], there exists $n_1 > n_0$ such that any point y for which we have $\prod_{j=0}^{n-1} \|[Dg(g^j(y))]^{-1}\|^{-1} \geq \varsigma^{-n}$, for some $n \geq n_1$, then y has, at least, n_0 $\varsigma'-$hyperbolic times less than n_1.

We fix $0 < \delta' \leq \delta$ such that

$$\|[Dg^{-1}(y)\| \leq \frac{1}{\sqrt{\varsigma'}} \|Dg^{-1}(z)\|, \forall z, y; d(z, y) < \delta',$$

where g^{-1} is an inverse branch for g.

We set $0 < \delta'' < \delta'$ such that if g^{-n} is an arbitrary composition of n inverse branches for g, then $\operatorname{diam}(g^{-n}(B(z, \delta''))) < \delta'$, $\forall z \in M$, $\forall n \in \mathbb{N}$. This occurs because it is valid for the hyperbolic system f to which g is conjugated.

As x is a recurrent point, we set $n_2 \geq n_1$ a return time such that a neighborhood $V_x \subset B(x, \delta'')$ of x is taken by g^{n_2} onto $B(x, \delta'')$.

Therefore, writing $G := (g^{n_2}|_{V_x})^{-1}$, $G : B(x, \delta'') \to V_x \subset B(x, \delta'')$ has a fixed point $p \in V_x$, which is a periodic point of period n_2 for g.

By hypothesis, p is a hyperbolic periodic point for which we have

$$\prod_{j=0}^{n_2-1} \|[Dg(g^j(p))]^{-1}\|^{-1} \geq \|\varsigma^{-n_2}\| \Rightarrow \|DG(p)\| \leq \|\varsigma^{n_2}\|.$$

By our choice of n_1 and the equation above, there exists a $\varsigma'-$hyperbolic time $n_0 < n' < n_2$ for p.

Due to lemma 2.7 in [6] (see also prop. 2.23 in [2]), n' is also a $\sqrt{\varsigma'}-$hyperbolic time for x. In particular, this implies that

$$\|Dg^{n'}(x) \cdot v\| \geq \sqrt{\varsigma'}^{-n'} \|v\|, \forall v \in T_pM.$$

Therefore, $\varsigma_j \geq \sqrt{\varsigma'}^{-1} - \epsilon > 1$, $\forall j$. This means that all Lyapunov exponents of x are greater than 1.

$\square$

Our theorem A is obtained applying lemma 6 to the following result:

Theorem 7. [9] *Let $f : M \to M$ be a C^1 local diffeomorphism on a compact Riemannian manifold. If the Lyapunov exponents of every f invariant probability measure are positive, then f is uniformly expanding.*

3. The diffeomorphism case: non uniformly hyperbolic periodic set

Now, we treat the case when f is a diffeomorphism. Along this section, we suppose that the periodic set $Per(g)$ is NUH. (see definition 2, on page 2).

The following remark is the analogous of remark 4 for the diffeomorphism case:

Remark 8. We note that the set of period points $Per(g)$ is NUH iff there exists $\varsigma > 1$ such that for each periodic point p with period $t(p)$, then $\prod_{j=0}^{t(p)-1} \|[Dg|_{E^u}(g^j(p))]^{-1}\|^{-1} > \varsigma^{t(p)}$ and $\prod_{j=0}^{t(p)-1} \|Dg|_{E^s}(g^j(p))\| < \varsigma^{-t(p)}$.

Before we state and prove the next lemma let us introduce some notation. Given a periodic point $p \in M$, we denote the cone over $E^s(p)$ of width $0 < a < 1$ by

$$C_a^s(p) := \{v_s + v_u \in E^s(p) \oplus E^u(p), \text{ such that } a\|v_s\| < \|v_u\|.$$

Analogously, we define a cone over $E^u(p)$ of width a.

Now we adapt the definition of hyperbolic times to the context of diffeomorphisms (see also [1]).

Definition 9. (Hyperbolic time for stable directions) Let $0 < \lambda < 1$ and $z \in M$ be a regular point. Suppose that E is an invariant subbundle of $T_{\mathbb{S}(z)}M$, where $\mathbb{S}(z)$ is some orbit segment of z. We say that $k \in \mathbb{N}$ is a $\lambda-$*hyperbolic time* for z if for $g^{-k}(z) = y$ and $i = 1 \dots k$, holds

$$\prod_{j=0}^{i-1} \|Dg|_E(g^j(y))\| \leq \lambda^i. \tag{2}$$

An analogous definition can be done for unstable directions just exchanging g by g^{-1} in the definition above.

Lemma 10. *Suppose that g is topologically conjugated to a hyperbolic map f. Let x be a recurrent, regular point of g. Suppose that $Per(g)$ is NUH, and that the splitting $T_{Per(g)} = E^{cs} \oplus E^{cu}$ have a continuous extension to $T_{\overline{Per(g)}}M = E^1 \oplus E^2$. Then all Lyapunov exponents of x are nonzero.*

Proof: Let $\varsigma = e^{\eta}$, $\eta < 0$ as in definition 2, $\varsigma < \varsigma' < 1$ fixed, and let $\epsilon > 0$ such that $(\sqrt{\varsigma'})^{-1} - \epsilon > 1$. Since x is a regular point, there is $n_0 \in \mathbb{N}$ such that

$$(\varsigma_j - \epsilon)^n \cdot \|v_j\| < \|Dg^n(x) \cdot v_j\| < (\varsigma_j + \epsilon)^n \cdot \|v_j\| \forall v_j \in E_j, \forall n \geq n_0,$$

and

$$(\varsigma_j - \epsilon)^{-n} \cdot \|v_j\| > \|Dg^{-n}(x) \cdot v_j\| > (\varsigma_j + \epsilon)^{-n} \cdot \|v_j\| \forall v_j \in E_j, \forall n \geq n_0.$$

where E_j are the Lyapunov eigenspaces and $\log(\varsigma_j)$ are their respective Lyapunov exponents. We denote by $E^c s(x)$ (respectivelly, $E^{cu}(x)$) the space spanned by the Lyapunov eigenspaces with negative (respectivelly, positive) Lyapunov exponents. $E^0(x)$ will denote the Lyapunov eigenspace corresponding to an eventual zero Lyapunov exponent.

Let us prove that the dimension of the space $E^{cs}(x)$ corresponding to the negative Lyapunov exponents of x is equal or greater than the dimension of the stable space of any periodic point. An analogous result will obviously hold for $E^{cu}(x)$. Therefore, we conclude that $T_xM = E^{cs}(x) \oplus E^{cu}(x)$ and that all Lyapunov exponents of x are nonzero.

By taking charts, due to the uniform continuity of Dg, we can fix $0 < a < 1$ and $0 < \delta'$ such that if z is periodic,

$$\|Dg(y) \cdot v\| \leq \frac{1}{\sqrt{\varsigma'}} \|Dg|_{E^{cs}(z)}(z)\| \|v\|, \forall y \in B(z, \delta'), \forall v \in C_a^s(z).$$

Due to the continuity of E^1, we can assume a small enough such that each cone $C_a^s(z)$ contains $E^1(y)$ or all point $y \in B(z, \delta') \cap \overline{Per(g)}$.

Now, by Pliss Lemma in [8], there exists $n_1 > n_0$ such that any point $z \in Per(g)$ for which we have $\prod_{j=0}^{n-1} \|Dg|_{E^1}(g^{-n+j}(z)\| \leq \varsigma^n$, for some $n \geq n_1$, then z has, at least, n_0 $\varsigma'-$hyperbolic times less than n_1.

As x is a recurrent point (also for g^{-1}), we set $n_2 \geq n_1$ a return time for g^{-1} such that there exists one periodic point p with period n_2 that $\delta'/3-$shadows the orbit segment $\{x, g^{-1}(x), \ldots, g^{-n_2}(x)\}$.

By hypothesis, p is a hyperbolic periodic point for which we have

$$\| \prod_{j=0}^{n_2-1} Dg|_{E^{cs}}(g^j(p))\| \leq \|\varsigma^{n_2}\|$$

By our choice of n_1 and the equation above, there exists a $\varsigma'-$hyperbolic time $n_0 < n' < n_2$ for p.

Due to prop. 2.23 in [2], n' is also a $\sqrt{\varsigma'}-$hyperbolic time for x. More precisely, this means that

$$\prod_{j=0}^{n'-1} \|Dg|_{E^1(g^{-n'+j}(x))}(g^j(g^{-n'}(x)))\| \leq \sqrt{\varsigma'}^j,$$

since the space $E^1(g^{-n'+j}(x)) \subset C_a^s(g^{-n'+j}(p))$.

In particular, this implies that

$$\|Dg^{-n'}(x) \cdot v\| \geq \sqrt{\varsigma'}^{-n'} \|v\|, \forall v \in E^1(x) \tag{3}$$

This implies that the dimension of the negative Lyapunov exponents space $E^{cs}(x)$ is at least the dimension of $E^1(x)$ which equals the dimension of $E^{cs}(p)$. In fact, if we had $\dim(E^1(x)) > \dim(E^{cs}(x))$, then since $T_xM = E^{cs}(x) \oplus E^0(x) \oplus E^{cu}(x)$, the intersection $E^1(x) \cap (E^0(x) \oplus E^{cu}(x))$ would be nontrivial. That is an absurd, because no vector in $(E^0(x) \oplus E^{cu}(x)) \setminus \{0\}$ satisfies equation 3.

Applying the same arguments above to $E^{cu}(x)$ we conclude that the number of positive Lyapunov at x is at least the dimension of $E^cu(p)$ and this concludes the lemma.

□

Lemma 11. *Let $f : M \to M$ be a diffeomorphism on a compact manifold M. Let $X \subset M$ be some bi-invariant set by f. Suppose there exists some invariant dominated splitting $T_XM = E \oplus \hat{E}$, such that $\measuredangle(E, \hat{E})$ is bounded far away from zero and infinity. Then, such splitting is continuous in T_XM, and unique since we fix the dimensions of $E, \hat{E}$. Moreover, it extends uniquely and continuously to a splitting of $T_{\overline{X}}M$.*

Proof: We start by constructing an invariant dominated splitting on $T_{\overline{X}}M$ extending the one we have on $T_X M$. Let $\mathbb{O}(x)$ be an orbit contained in $\overline{X}$. Our construction will be dependent of some choices. We choose one representative of $\mathbb{O}(x)$, for example x. Let us also choose some (x_n), $x_n \in X$, $x_n \to x \in M$, as $n \to \infty$. Let $v_n^1, \ldots, v_n^s \in E(x_n)$, $\hat{v}_n^{s+1}, \ldots, \hat{v}_n^m \in \hat{E}(x_n)$ be orthonormal bases of $E(x_n)$, $\hat{E}(x_n)$, respectively. The domination property is equivalent to

$$\|Df(x_n)\sum_{j=1}^{s}\alpha_j v_n^j\| \cdot \|Df(x_n)\hat{\sum}_{i=s+1}^{m}\beta_i \hat{v}_n^i\|^{-1} \leq \lambda < 1,$$

for any convex combination $\sum_{j=1}^{s}\alpha_j v_n^j$, $\sum_{i=s+1}^{m}\beta_i v_n^i$. Replacing by some convergent subsequence, if necessary, we can suppose that $(v^1, \ldots, v^s), v^1, \ldots, v^s \in T_x M$ (resp. $(\hat{v}^{s+1}, \ldots, \hat{v}^m)$) is the limit of the sequence $(v_n^1, \ldots, v_n^s)$ (resp. of the sequence $(\hat{v}_n^{s+1}, \ldots, \hat{v}_n^m)$). Since the domination property is a closed condition,

$$\|Df(x)\sum_{j=1}^{s}\alpha_j v^j\| \cdot \|Df(x)\hat{\sum}_{i=s+1}^{m}\beta_i \hat{v}^i\|^{-1} \leq \lambda < 1,$$

holds. We write G for the Gram-Schmidt operator (which takes a linearly independent set of vectors on an orthonormal set of vectors spanning the same vector space). Now, given any iterate $y = f^k(x)$, $k \in \mathbb{Z}$, then $f^k(x_n) \to y$ and

$$G\emptyset(Df^k(x_n)v_n^1, \ldots, Df^k(x_n)v_n^s) \to G\emptyset(Df^k(x)v^1, \ldots, Df^k(x)v^s),$$

$$G\emptyset(Df^k(x_n)\hat{v}_n^{s+1}, \ldots, Df^k(x_n)\hat{v}_n^m) \to G\emptyset(Df^k(x)\hat{v}^{s+1}, \ldots, Df^k(x)\hat{v}^m),$$

as $n \to \infty$. Writing $(w_k^1, \ldots, w_k^s) := G\emptyset(Df^k(x)v^1, \ldots, Df^k(x)v^s)$ and $(\hat{w}_k^{s+1}, \ldots, \hat{w}_k^m) := G\emptyset(Df^k(x)\hat{v}^{s+1}, \ldots, Df^k(x)\hat{v}^m)$, $k \in \mathbb{Z}$, the same calculations above show that

$$T_y M = span\{w_k^1, \ldots, w_k^s\} \oplus span\{\hat{w}_k^{s+1}, \ldots, \hat{w}_k^m\} =: E(y) \oplus \hat{E}(y)$$

is a dominated splitting.

Moreover, it is clear that

$$Df(f^k(x))(span\{w_k^1, \ldots, w_k^s\}) = span\{w_{k+1}^1, \ldots, w_{k+1}^s\}$$

and

$$Df(f^k(x))(span\{\hat{w}_k^{s+1}, \ldots, \hat{w}_k^m\}) = span\{w_{k+1}^{s+1}, \ldots, \hat{w}_{k+1}^m\},$$

which implies that it is an invariant splitting.

$\square$

Lemma 12. *Suppose that g is topologically conjugated to a hyperbolic map f. Let x be a recurrent, regular point of g. Suppose that $Per(g)$ is NUH, and that the splitting $T_{Per(g)} = E^{cs} \oplus E^{cu}$ is a dominated splitting. Then all Lyapunov exponents of x are nonzero.*

Proof: The proof is quite analogous to that of lemma 10. We repeat the arguments here, anyway. Let $\varsigma = e^{\eta}$, $\eta < 0$ as in definition 2, $\varsigma < \varsigma' < 1$ fixed, and let $\epsilon > 0$ such that $(\sqrt{\varsigma'})^{-1} - \epsilon > 1$. Since x is a regular point, there is $n_0 \in \mathbb{N}$ such that

$$(\varsigma_j - \epsilon)^n \cdot \|v_j\| < \|Dg^n(x) \cdot v_j\| < (\varsigma_j + \epsilon)^n \cdot \|v_j\| \forall v_j \in E_j, \forall n \geq n_0,$$

and

$$(\varsigma_j - \epsilon)^{-n} \cdot \|v_j\| > \|Dg^{-n}(x) \cdot v_j\| > (\varsigma_j + \epsilon)^{-n} \cdot \|v_j\| \forall v_j \in E_j, \forall n \geq n_0,$$

where E_j are the Lyapunov eigenspaces and $\log(\varsigma_j)$ are their respective Lyapunov exponents.

Let us prove that the dimension of the space $E^{cs}(x)$ corresponding to the negative Lyapunov exponents of x is equal or greater than the dimension of the stable space of any periodic point. An analogous result will obviously hold for $E^{cu}(x)$. Therefore, we conclude that $T_xM = E^{cs}(x) \oplus E^{cu}(x)$ and that all Lyapunov exponents of x are nonzero.

By taking charts, due to the uniform continuity of Dg, we can fix $0 < a < 1$ and $0 < \delta'$ such that if z is periodic,

$$\|Dg(y) \cdot v\| \leq \frac{1}{\sqrt{\varsigma'}} \|Dg|_{E^{cs}(z)}(z)\| \|v\|, \forall y \in B(z, \delta'), \forall v \in C^s_a(z).$$

Due to the domination property we can assume a small enough such that each cone $C^s_a(z)$ is Dg^{-1}-invariant for all point $y \in B(z, \delta')$.

Now, by Pliss Lemma in [8], there exists $n_1 > n_0$ such that any point $z \in Per(g)$ for which we have $\prod_{j=0}^{n-1} \|Dg|_{E^1}(g^{-n+j}(z)\| \leq \varsigma^n$, for some $n \geq n_1$, then z has, at least, n_0 $\varsigma'-$hyperbolic times less than n_1.

As x is a recurrent point (also for g^{-1}), we set $n_2 \geq n_1$ a return time for g^{-1} such that there exists one periodic point p with period n_2 that $\delta'/3-$shadows the orbit segment $\{x, g^{-1}(x), \ldots, g^{-n_2}(x)\}$.

By hypothesis, p is a hyperbolic periodic point for which we have

$$\| \prod_{j=0}^{n_2-1} Dg|_{E^{cs}}(g^j(p))\| \leq \|\varsigma^{n_2}\|$$

By our choice of n_1 and the equation above, there exists a $\varsigma'-$hyperbolic time $n_0 < n' < n_2$ for p.

Due to prop. 2.23 in [2], n' is also a $\sqrt{\varsigma'}-$hyperbolic time for x. More precisely, this means that

$$\prod_{j=0}^{n'-1} \|Dg|_{Dg^j(g^{-n'}(x))\cdot E}(g^j(g^{-n'}(x)))\| \leq \sqrt{\varsigma'}^j,$$

for any space E such that $Dg^j(g^{-n'}(x)) \cdot E \subset C_a^s(g^{-n'+j}(p))$.

In particular, this implies that

$$\|Dg^{-n'}(x) \cdot v\| \geq \sqrt{\varsigma'}^{-n'}\|v\|, \forall v \in C_a^s(g^{-n'}(p)).$$

Just as in lemma 10, this implies that the dimension of the negative Lyapunov exponents space is at least the dimension of $E^{cs}(p)$. Applying the same arguments above to $E^{cu}(x)$ we conclude that the number of positive Lyapunov at x is at least the dimension of $E^c u(p)$ and this concludes the lemma.

□

Our theorem B is obtained applying lemma 10 to the following result:

Theorem 13. [9] *Let $f : M \to M$ be a C^1 diffeomorphism on a compact Riemannian manifold, with a positively invariant set Λ for which the tangent bundle has a continuous splitting $T_\Lambda M = E^{cs} \oplus E^{cu}$. If f has positive Lyapunov exponents in the E^{cu} direction and negative Lyapunov exponents in the E^{cs} direction on a set of total probability, then f is uniformly hyperbolic.*

4. On a Conjecture of A. Katok

A. Katok has conjectured that a C^{1+} system which is Hölder conjugated to an expanding map (respectivelly, a hyperbolic diffeomorphism) is also expanding (respectivelly, is also a hyperbolic diffeomorphism).

Note that, under the hypotheses of such conjecture, the periodic points of the $g : M \to M$ are hyperbolic, with uniform bounds for the eigenvalues of iterate of Dg in the period of such points. This is proven below.

First, we consider the expanding case. Let p a periodic point of period t of g. Then, $h(p)$ is a periodic point of period t of f. Let us call f^{-1} the inverse branch of f, defined on a neighborhood of the orbit of $h(p)$, for which $h(p) = \hat{p}$ is a periodic point of period t. Analogously, let us call g^{-1} be the inverse branch of g for which p is a periodic point of period t. Since f is an expanding map, there are $0 < \hat{\lambda} < 1$ and $\hat{\delta} > 0$ such that

$$d(f^{-j}(\hat{x}), f^{-j}(\hat{y})) \leq \hat{\lambda}^j d(\hat{x}, \hat{y}), \forall j \in \mathbb{N}, \forall \hat{x}, \hat{y} \in B(\hat{p}, \hat{\delta}).$$

As an immediate consequence of the C^α conjugation h there exists $\delta > 0$ such that

$$d(g^{-j}(x), g^{-j}(y)) \le (\hat{\lambda}^\alpha)^j K^{1+\alpha} d(x,y)^{(\alpha^2)}, \forall j \in \mathbb{N}, \forall x, y \in B(p,\delta)$$

and

$$d(g^{-j}(x), g^{-j}(y)) \le (\hat{\lambda}^\alpha)^j K^{1+\alpha} \delta^{\alpha^2}, \forall j \in \mathbb{N}, \forall x, y \in B(p,\delta). \qquad (4)$$

Proposition 14. *Let $B(x_0, r) \subset M$ and $G : \overline{B(x_0,r)} \to B(x_0, r)$ a class C^1 local diffeomorphism such that $G(x_0) = x_0$ and for some $0 < \lambda < 1$ and $0 < \beta < 1$*

$$d(G^n(x), G^n(y)) \le \lambda^n d(x,y)^\beta, \forall x, y \in B(x_0, r).$$

Then all eigenvalues of G are equal or less than λ.

Proof: Using charts, there is no loss of generality in supposing that M is an euclidean space and $x_0 = 0$. By contradiction, suppose there exists an invariant splitting $\mathbb{R}^m = E^s + E^c$, an adapted norm $\|x\| = \|(x_s, x_c)\| = \max\{\|x_s\|, \|x_c\|\}$ and $\sigma > \lambda$ such that

$$\|DG(0) \cdot x_s\| \le \lambda \cdot \|x_s\|, \forall x_s \in E^s,$$

$$\|DG(0) \cdot x_c\| \ge \sigma \cdot \|x_c\|, \forall x_c \in E^c.$$

Let $\epsilon > 0$ such that $\lambda + \epsilon < \sigma - \epsilon$ and take $\theta = \frac{\lambda+\epsilon}{\sigma-\epsilon}$.

Therefore, there is $\tilde{r} \le r$ such that if we write

$$G(x) = DG(0) \cdot x + \rho(x),$$

then $\|\rho(x)\| < \epsilon\|x\|$, $\forall x, \|x\| < \tilde{r}$.

We define a central cone

$$V_c := \{(x_s, x_c); \|x_s\| \le \theta \|x_c\|\}$$

By the hypothesis, there exists $\tilde{r} \le \tilde{r}$ such that $G^n(B(0,\tilde{r})) \subset B(0,\tilde{r}), \forall n \in \mathbb{N}$. So, let us iterate $x \in B(0,\tilde{r}) \cap V_c$ (we write $x^n = G^n(x)$). We obtain:

$$\|x_c^1\| \ge \sigma\|x_c^0\| - \epsilon\|x^0\| \ge (\sigma - \epsilon)\|x_c^0\|$$

and

$$\|x_s^1\| \le \lambda\|x_s^0\| + \epsilon\|x^0\| \le (\lambda + \epsilon)\|x_c^0\|.$$

This implies that

$$\|x_s^1\| \le \frac{\lambda+\epsilon}{\sigma-\epsilon}\|x_c^1\|.$$

In particular, if $x \in B(0,\tilde{r}) \cap V_c$ then $G(x) \in V_c$.

Therefore proceeding inductively, we obtain

$$\|x^n\| = \|x_c^n\| \ge (\sigma - \epsilon)^n \|x_c^0\| = (\sigma - \epsilon)^n \|x^0\|.$$

This contradicts the hipothesis, which implies that

$$\|x^n\| \le const \cdot \lambda^n, \forall n \in \mathbb{N}.$$

As $\epsilon > 0$ is arbitrary, we conclude that any eigenvalue of G is less than λ.

$\square$

However, up to now we do not know if, for example, the mild uniformity given by a Hölder conjugation, plus the conjugation itself, imply that $Per(g)$ is NUE.

As a direct consequence of the last section, we just obtained that such conjecture is valid in the case that $Per(g)$ is NUE (respectively, NUH).

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